Value of Information

Approximate Computations for Value of Information Analysis

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Plan for course

Time	Topic
Lecture 1	Introduction and motivating examples
	Elementary decision analysis and the value of information
	Multivariate statistical modeling, dependence, graphs
	Value of information analysis for dependent models
Lecture 2	Re-cap of VOI and statistical dependence
	Spatial statistics, spatial design of experiments
	Value of information analysis in spatial decision situations
	Examples of value of information analysis in Earth sciences
Lecture 3	Computational aspects of VOI analysis, approximate calculations
	Sequential information gathering
	Examples from Earth sciences

Every day: Small exercises.

Bayesian model

- All the currently available information about variables:

$$p(\mathbf{x})$$

- New data (and the data gathering scheme) is represented by a likelihood model:

$$p(\mathbf{y} \mid \mathbf{x})$$

- If we collect data, the model is updated to the posterior, conditional on the new observations:

$$p(\mathbf{x} \mid \mathbf{y}) = \frac{p(\mathbf{y} \mid \mathbf{x})\, p(\mathbf{x})}{p(\mathbf{y})}, \qquad p(\mathbf{y}) = \sum_{\mathbf{x}} p(\mathbf{y} \mid \mathbf{x})\, p(\mathbf{x})$$

Information gathering

	Perfect	Imperfect
Total	Exact observations are gathered for all locations. $\boldsymbol{y} = \boldsymbol{x}$	Noisy observations are gathered for all locations. $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\varepsilon}$
Partial	Exact observations are gathered at some locations. $\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}},\quad \mathbb{K}$ subset	Noisy observations are gathered at some locations $\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}} + \boldsymbol{\varepsilon}_{\mathbb{K}},\quad \mathbb{K}$ subset

Value of information (VOI)

Prior value:

$$PV = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\}$$

Posterior value:

$$PoV\left(\boldsymbol{y}\right) = \int \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x}, \boldsymbol{a} \right) \mid \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

VOI = Expected posterior value – Prior value

$$VOI\left(\boldsymbol{y}\right) = PoV\left(\boldsymbol{y}\right) - PV$$

$\boldsymbol{x}$ - Uncertainties

$\boldsymbol{a}$ - Alternatives

$v\left(\boldsymbol{x}, \boldsymbol{a}\right)$ - Value function

$\boldsymbol{y}$ - Data

Decoupling – values are sums

	Assumption: Decision Flexibility	**Assumption: Value Function**
Low decision flexibility; Decoupled value	Alternatives are easily enumerated $a \in A$	Total value is a sum of value at every unit $v(\mathbf{x},a)=\sum_j v(x_j,a)$
High decision flexibility; Decoupled value	None $\mathbf{a} \in \mathbf{A}$	Total value is a sum of value at every unit $v(\mathbf{x},\mathbf{a})=\sum_j v(x_j,a_j)$
Low decision flexibility; Coupled value	Alternatives are easily enumerated $a \in A$	None $v(\mathbf{x},a)$
High decision flexibility; Coupled value	None $\mathbf{a} \in \mathbf{A}$	None $v(\mathbf{x},\mathbf{a})$

Profit is sum of timber volumes from units.

Computation - Formula for VOI

$$PV = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{a}) p\left(\boldsymbol{x} \right) d\boldsymbol{x} \right\}$$

$$PoV\left(\boldsymbol{y} \right) = \int \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x}, \boldsymbol{a} \right) | \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y} \right) d\boldsymbol{y}$$

Main challenge.

Techniques – Computing the VOI

$$PV = \max_{\boldsymbol{a} \in A} \left\{ E\left(v(\boldsymbol{x},\boldsymbol{a})\right) \right\} = \max_{\boldsymbol{a} \in A} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x},\boldsymbol{a}) p\left(\boldsymbol{x}\right) d\boldsymbol{x} \right\}$$

$$PoV\left(\boldsymbol{y}\right) = \int \max_{\boldsymbol{a} \in A} \left\{ E\left(v\left(\boldsymbol{x},\boldsymbol{a}\right) | \boldsymbol{y}\right) \right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

Inner integral.

Outer integral.

Computational techniques :

- Fully analytically tractable for special cases, like two-actions, **Gaussian, linear models**.
- Various approximations and Monte Carlo approaches usually applicable.
- Should avoid double Monte Carlo (inner and outer). Too time consuming.

Partly analytical, Monte Carlo for outer

$$PV = \max\left\{0, E\left(v\left(\boldsymbol{x}, a=1\right)\right)\right\}$$

Inner integral solved.

$$PoV\left(\boldsymbol{y}\right) = \int \max\left\{0, f\left(\boldsymbol{y}\right)\right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

$$= \frac{1}{B}\sum_{b=1}^{B} \max\left\{0, f\left(\boldsymbol{y}^{b}\right)\right\}$$

Use sampling.

$$f\left(\boldsymbol{y}\right) = E\left(v\left(\boldsymbol{x}, a=1\right) \mid \boldsymbol{y}\right),$$

$$\boldsymbol{y}^{b} \sim p\left(\boldsymbol{y}\right), \qquad b = 1, \ldots, B.$$

Approximate computation

Outer expectation: $\boldsymbol{y}$

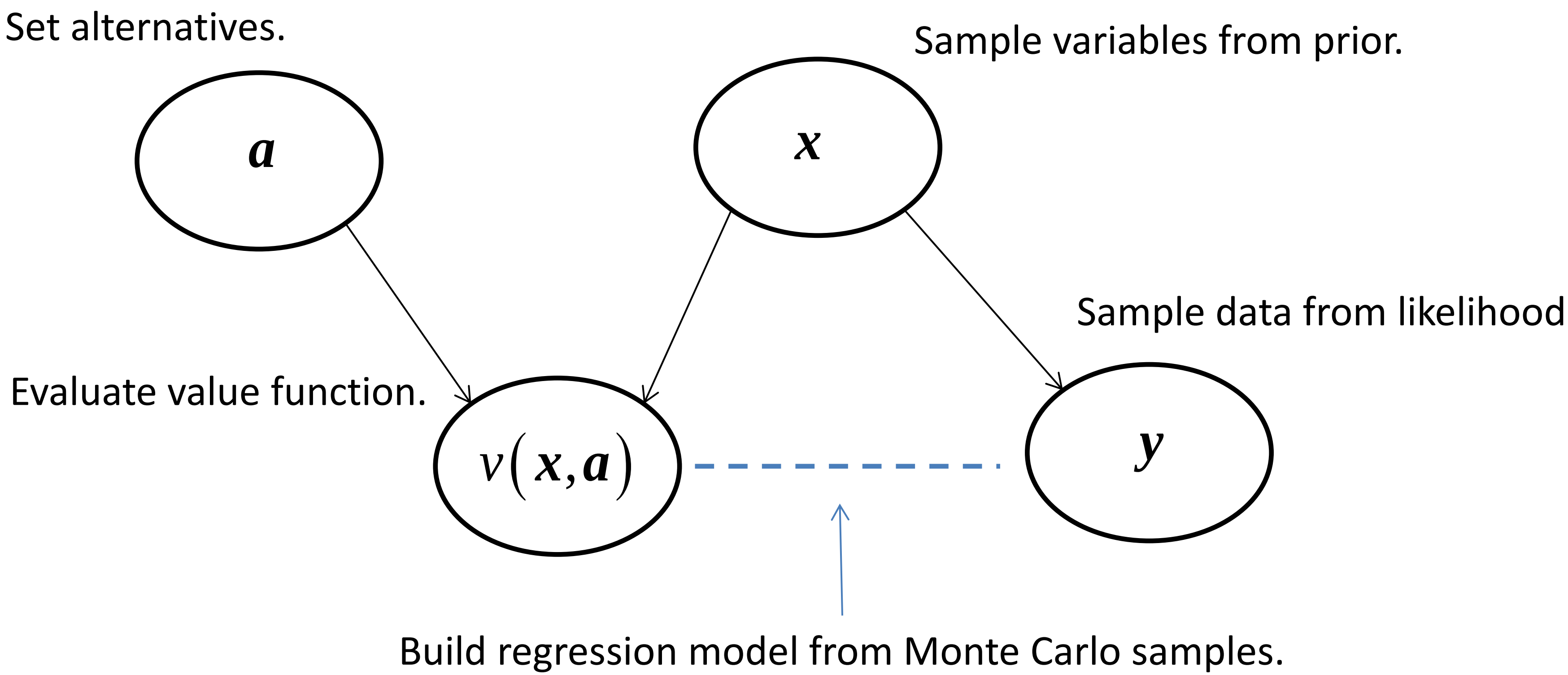
$$PoV(\boldsymbol{y}) = \sum_{\boldsymbol{y}} \max_{\boldsymbol{a} \in A} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \mid \boldsymbol{y}\right) \right\} p(\boldsymbol{y})$$

Inner expectation: $\boldsymbol{x} \mid \boldsymbol{y}$

$$VOI(\boldsymbol{y}) = PoV(\boldsymbol{y}) - PV$$

- **Suggest Monte Carlo (outer) and regression approximation (inner).**

Simulation-regression illustration



Simulation-regression algorithm

Outer expectation

$$PoV(\boldsymbol{y}) = \sum_{\boldsymbol{y}} \max_{\boldsymbol{a}\in\boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x},\boldsymbol{a}\right) \mid \boldsymbol{y}\right)\right\} p\left(\boldsymbol{y}\right)$$

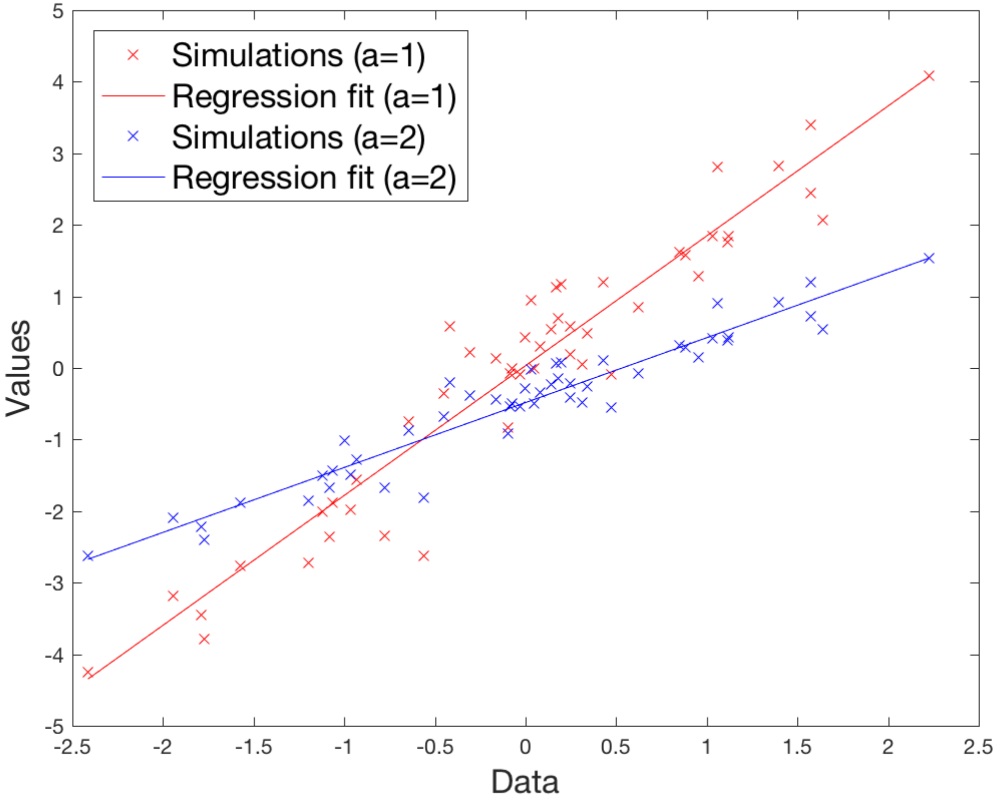
Inner expectation

1. Simulate uncertainties: $\boldsymbol{x}^b \sim p(\boldsymbol{x}), \quad b = 1,\ldots,B$
2. Compute values, for all alternatives: $v_{\boldsymbol{a}}^b = v\left(\boldsymbol{x}^b, \boldsymbol{a}\right), \quad b = 1,\ldots,B, \quad \boldsymbol{a} \in \boldsymbol{A}$
3. Simulate data: $\boldsymbol{y}^b \sim p\left(\boldsymbol{y} \mid \boldsymbol{x}^b\right), \quad b = 1,\ldots,B$
4. Regress samples to fit conditional mean: $\hat{E}\left(v_a \mid \boldsymbol{y}\right)$

$$PoV(\boldsymbol{y}) \approx \frac{1}{B} \sum_{b=1}^{B} \max_{\boldsymbol{a}\in\boldsymbol{A}} \left\{ \hat{E}\left(v_{\boldsymbol{a}} \mid \boldsymbol{y}^b\right)\right\}$$

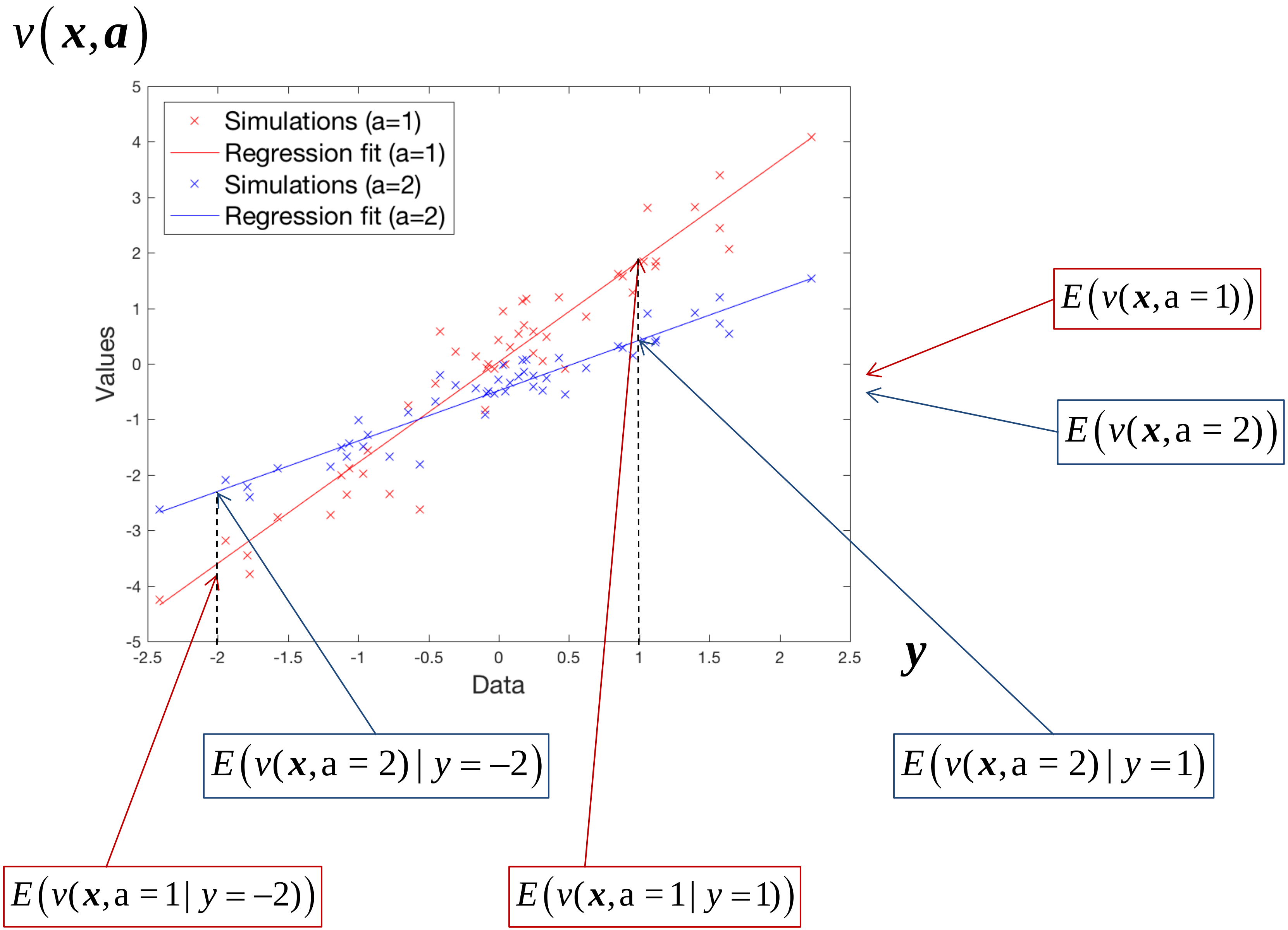
Illustration - fit regression model to samples

$v(\boldsymbol{x}, \boldsymbol{a})$



$\boldsymbol{y}$

Illustration - fit regression model to samples

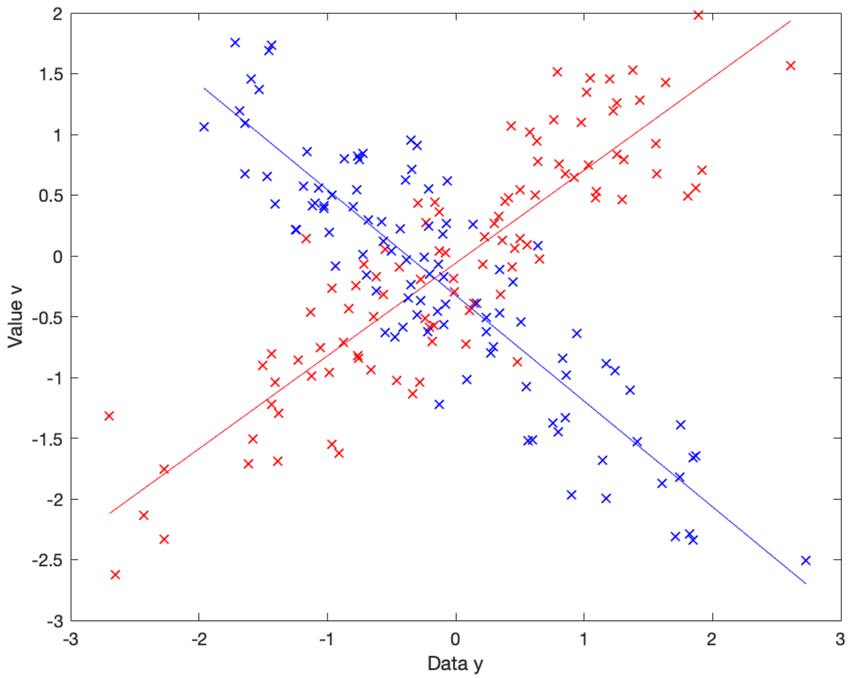


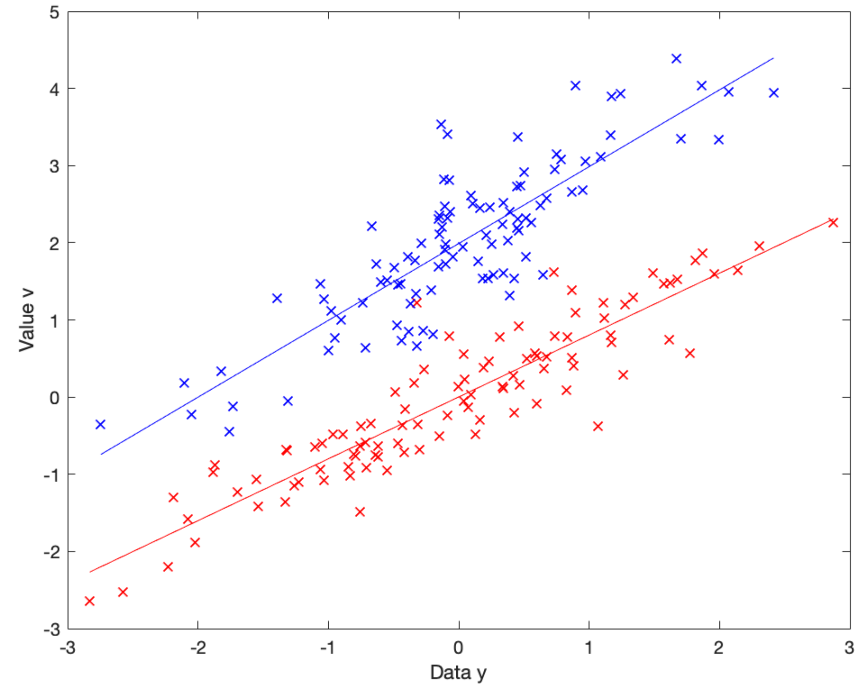
Choice of regression method

- Linear regression
- Principal component regression
- Neural networks
- K-nearest neighbors
- And many others

- Cross-validation to check model fit. Look at residuals

Exercise - two different cases



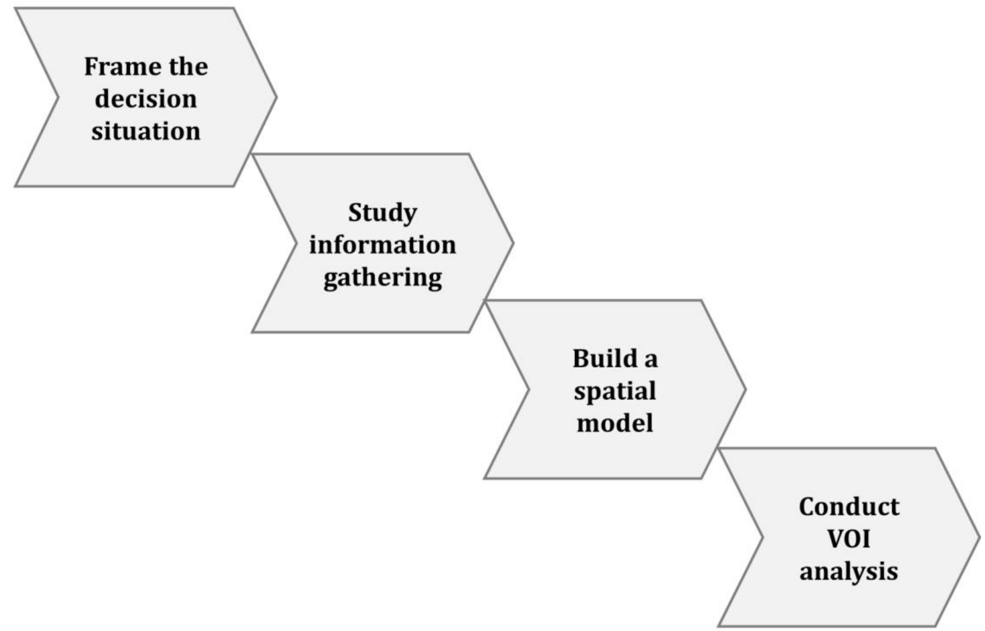


In both displays:
Alternative 1,
Alternative 2

for which of these two cases is the VOI largest.

Reservoir dogs - petroleum example

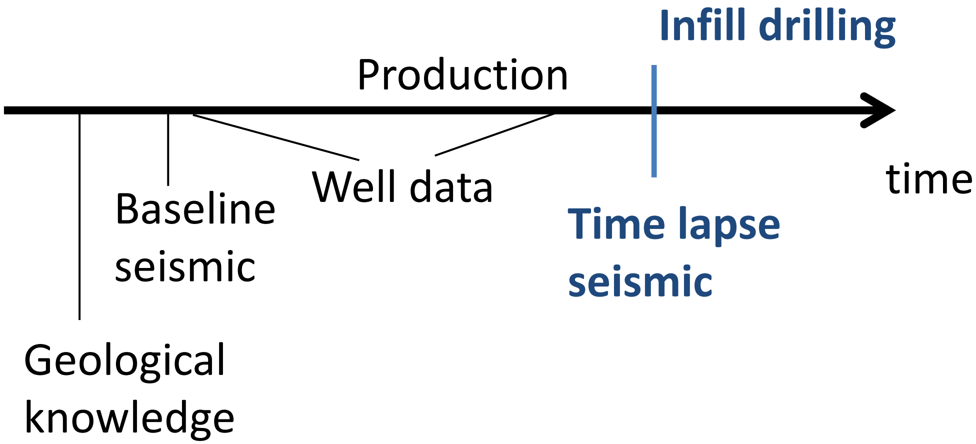
- Decisions about drilling alternatives.
- Seismic information.
- Model is a represented by multiple realizations, building on prior knowledge.
- VOI analysis done by a simulation-regression approach.

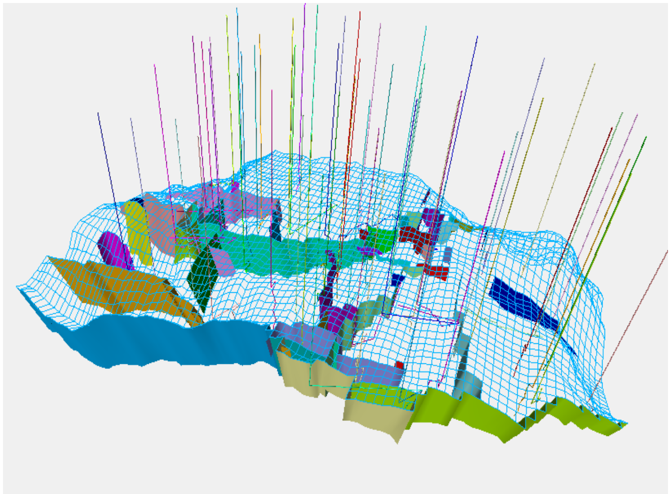


Key questions:

- Decisions about infill drilling for improved oil recovery.
  - Uncertainty, heterogeneity and dependence make this choice difficult.

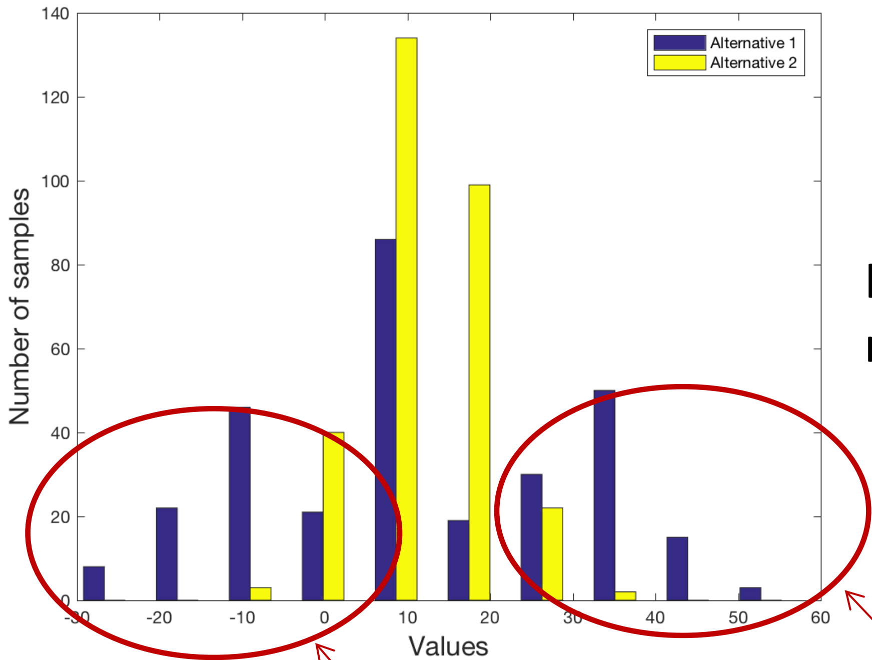
- Data gathering decisions about time-lapse seismic data.
  - Which kind of data are likely to be valuable? How much data is enough?





Wells drilled at the Gullfaks field, North Sea.

Illustration of values and data influence



Infill drilling (Alternative 1) can give more value, but can also mean loss.

If data indicate reservoir variables corresponding to these small values -> avoid infill drilling!

If data indicate reservoir variables corresponding to these high values -> do infill drilling!

… such data would lead to better decisions in this situation.

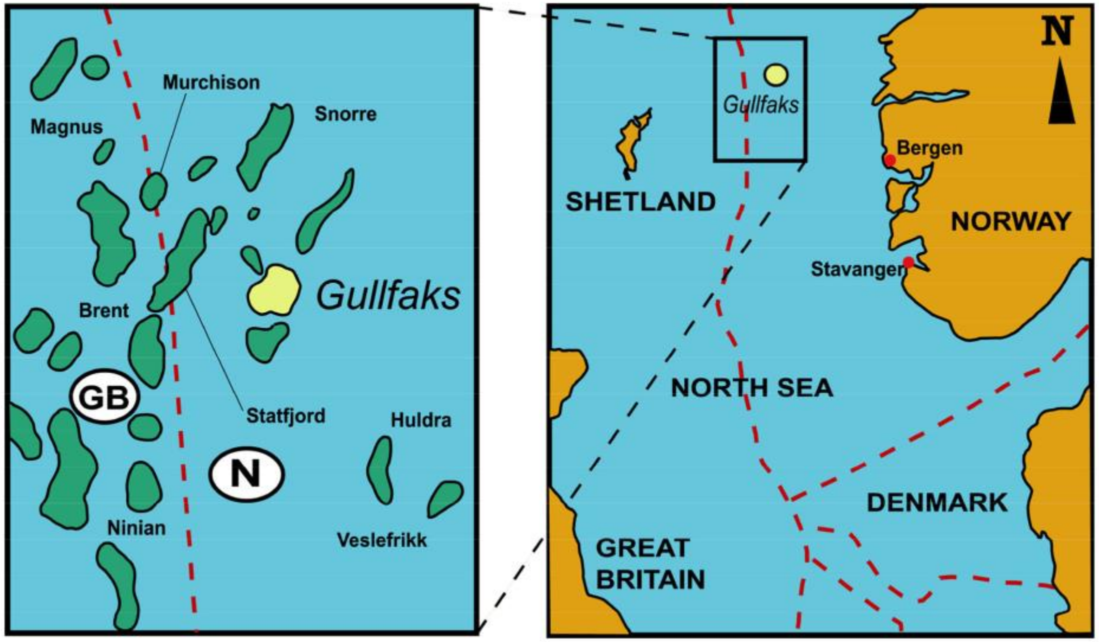
Information gathering and VOI

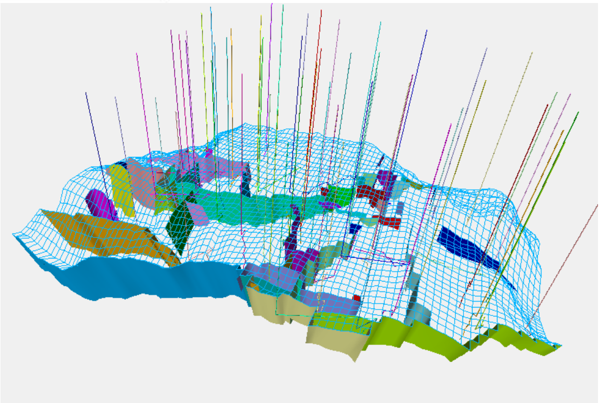
VOI is interpretable as follows:

- **Is VOI larger than price of time-lapse seismic experiment?**
- Is VOI larger for seismic acquisition design A or B ?
- Is VOI larger for seismic processing type I or II ?

VOI = Expected posterior value – Prior value

Gullfaks case



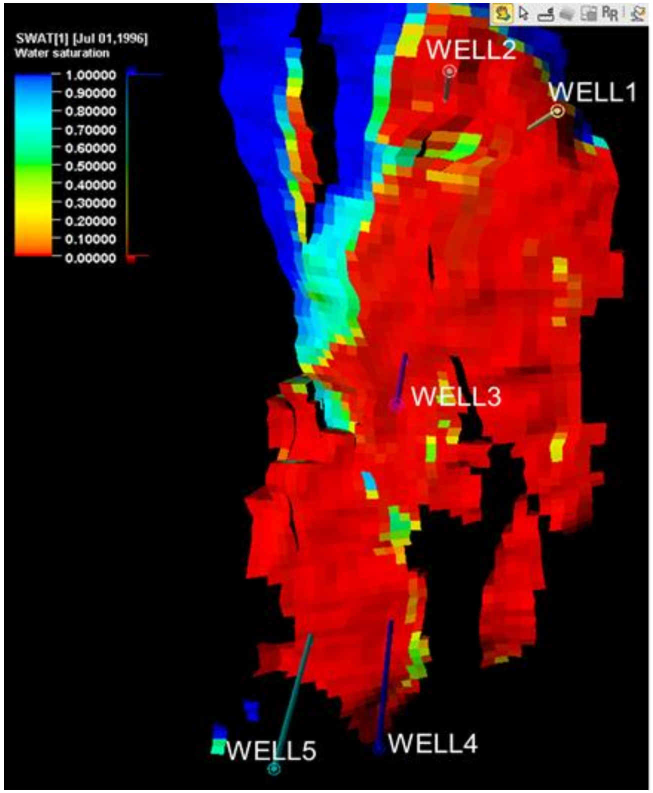


Wells drilled in this part of Gullfaks.

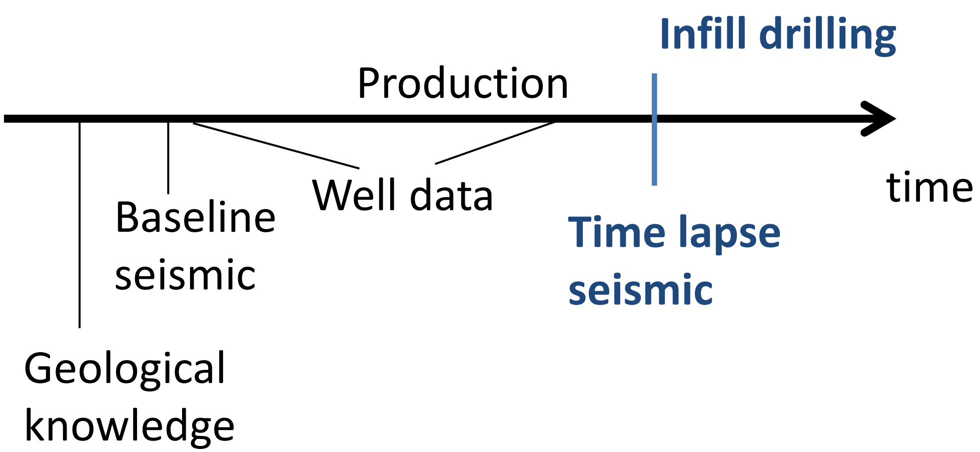
Gullfaks case (infill drilling and time lapse)

Time-lapse seismic has shown useful at Gullfaks. But no formal VOI analysis was conducted up-front.

We consider this case in retrospect.



5 decision alternatives.

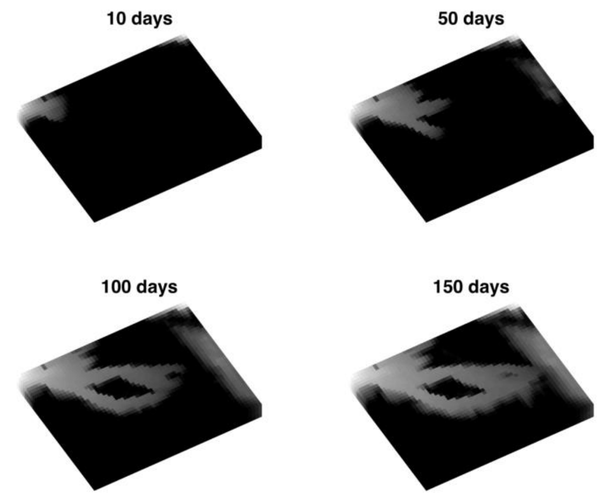


Prior - Reservoir uncertainty

Uncertainties: saturation, pressure, porosity, permeability and fault transmissibilities. (Conditioned on existing data.)

Prior is $p(\boldsymbol{x})$.

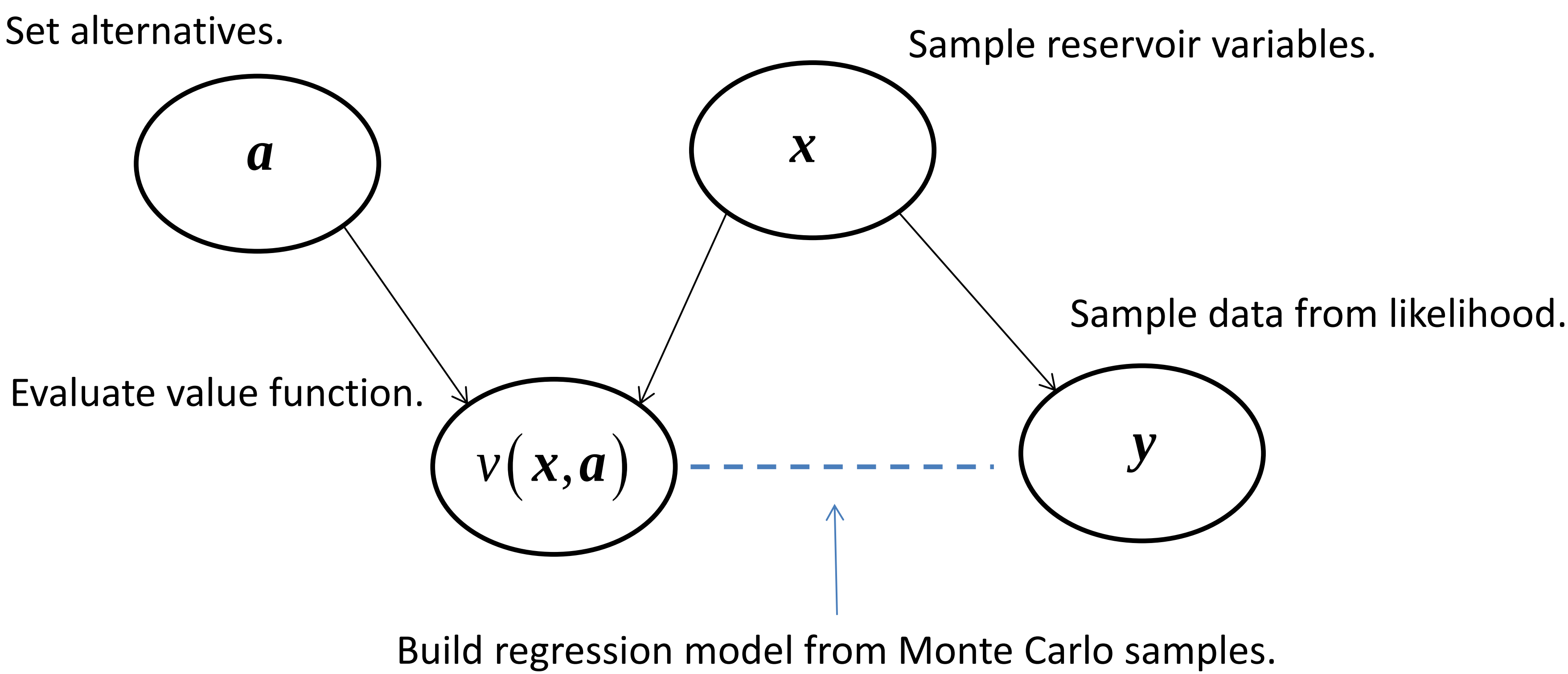
This distribution of reservoir variables is represented by multiple Monte Carlo realizations from the prior distribution.



Sample 1

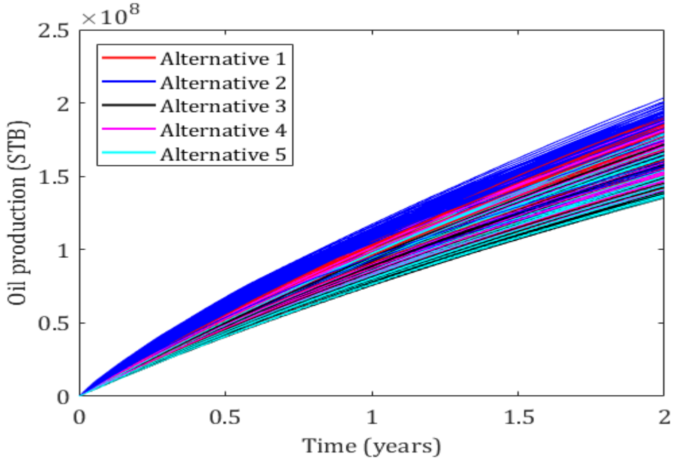
………… Sample 1000

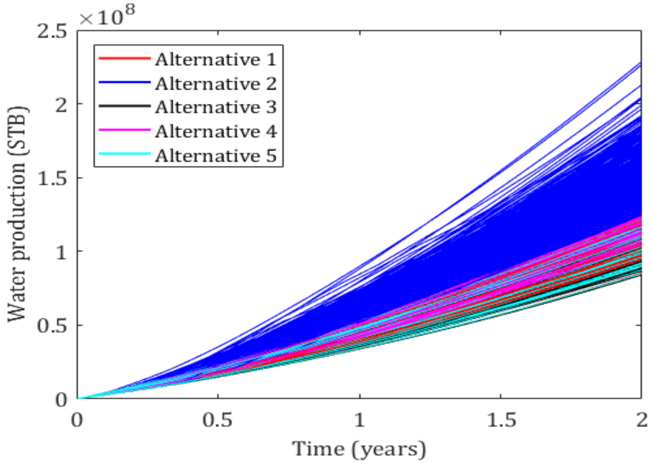
Simulation-regression illustration



Gullfaks case (values)

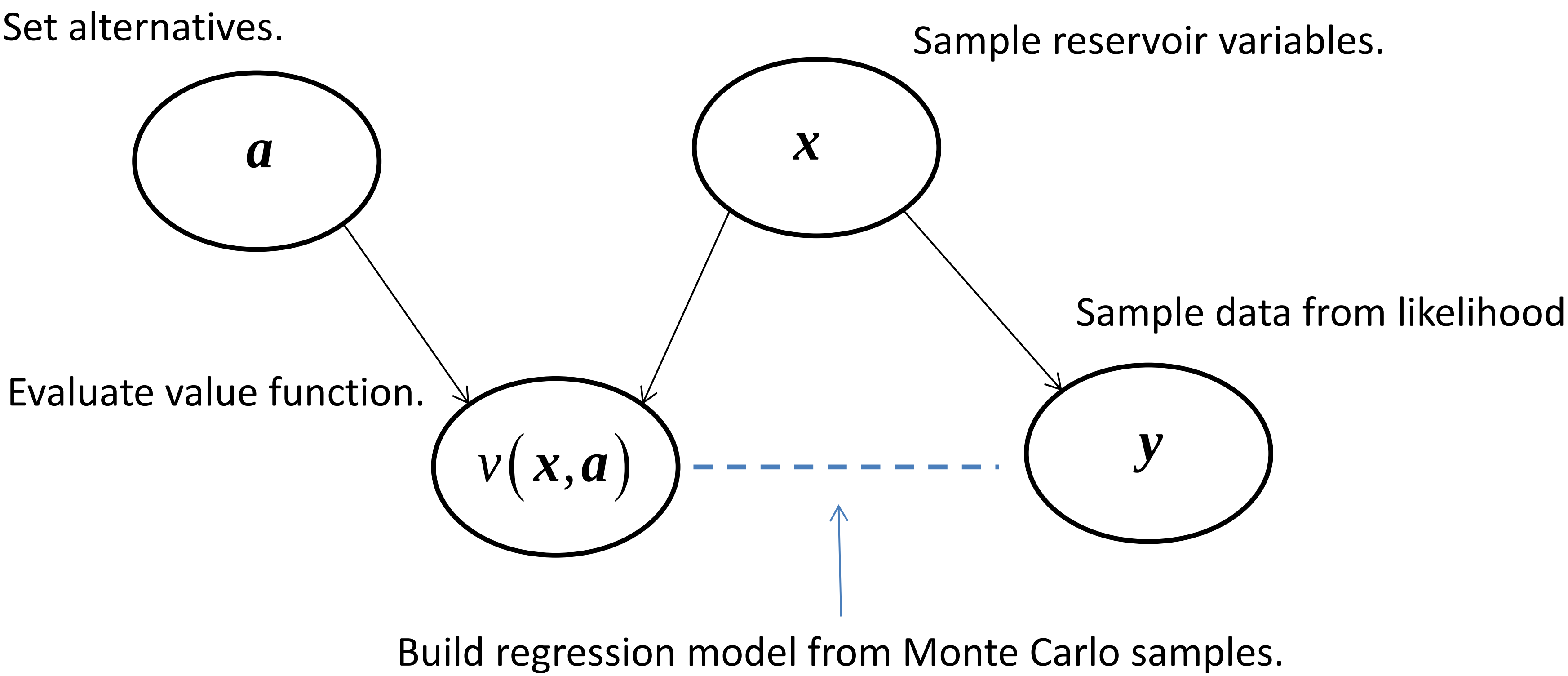
Future production for 5 different infill drilling alternatives.
- for each realization, all alternatives are «produced».



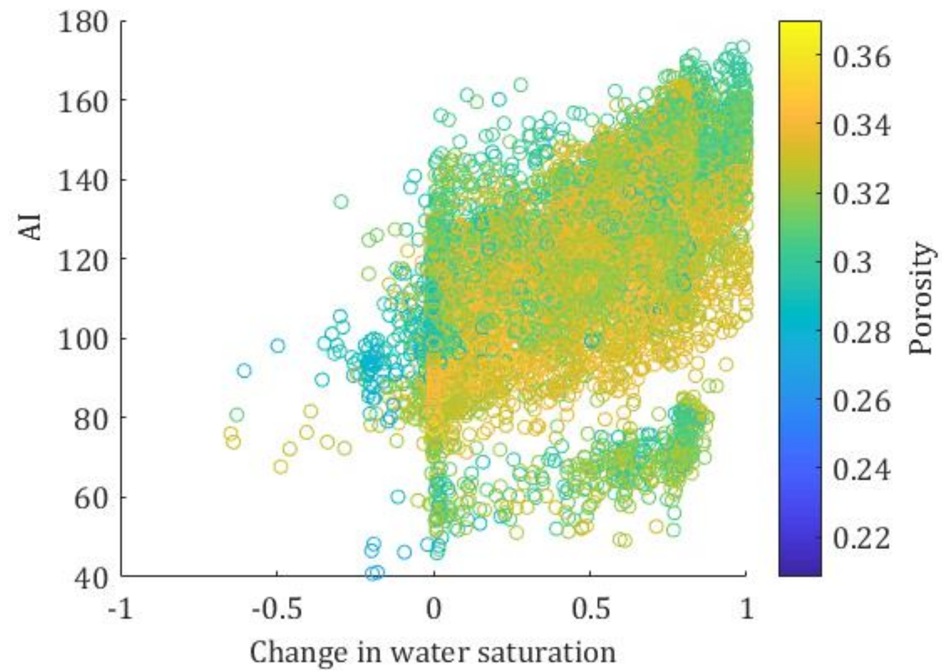


$$v\left(\boldsymbol{x}^b,\boldsymbol{a}\right)=\int\frac{q_o\left(t,\boldsymbol{x}^b,\boldsymbol{a}\right)r_o-q_w\left(t,\boldsymbol{x}^b,\boldsymbol{a}\right)r_w}{\left(1+\alpha\right)^t}dt-C_{drill}\left(\boldsymbol{a}\right),\qquad b=1,\ldots,1000$$

Simulation-regression illustration



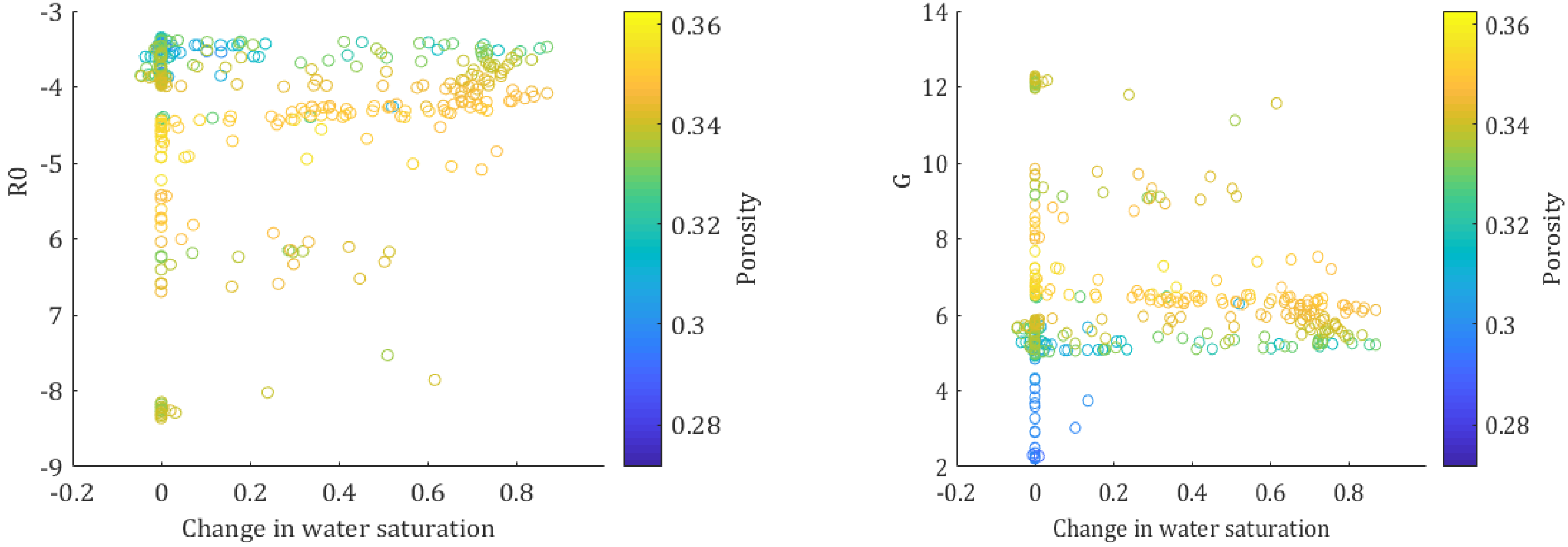
Gullfaks case (likelihood of AI data)



Synthetic time-lapse seismic (acoustic impedance (AI) proessing):
Use rock physics relations connecting reservoir properties to AI.

Simulations indicate some information about saturation from AI for this case.

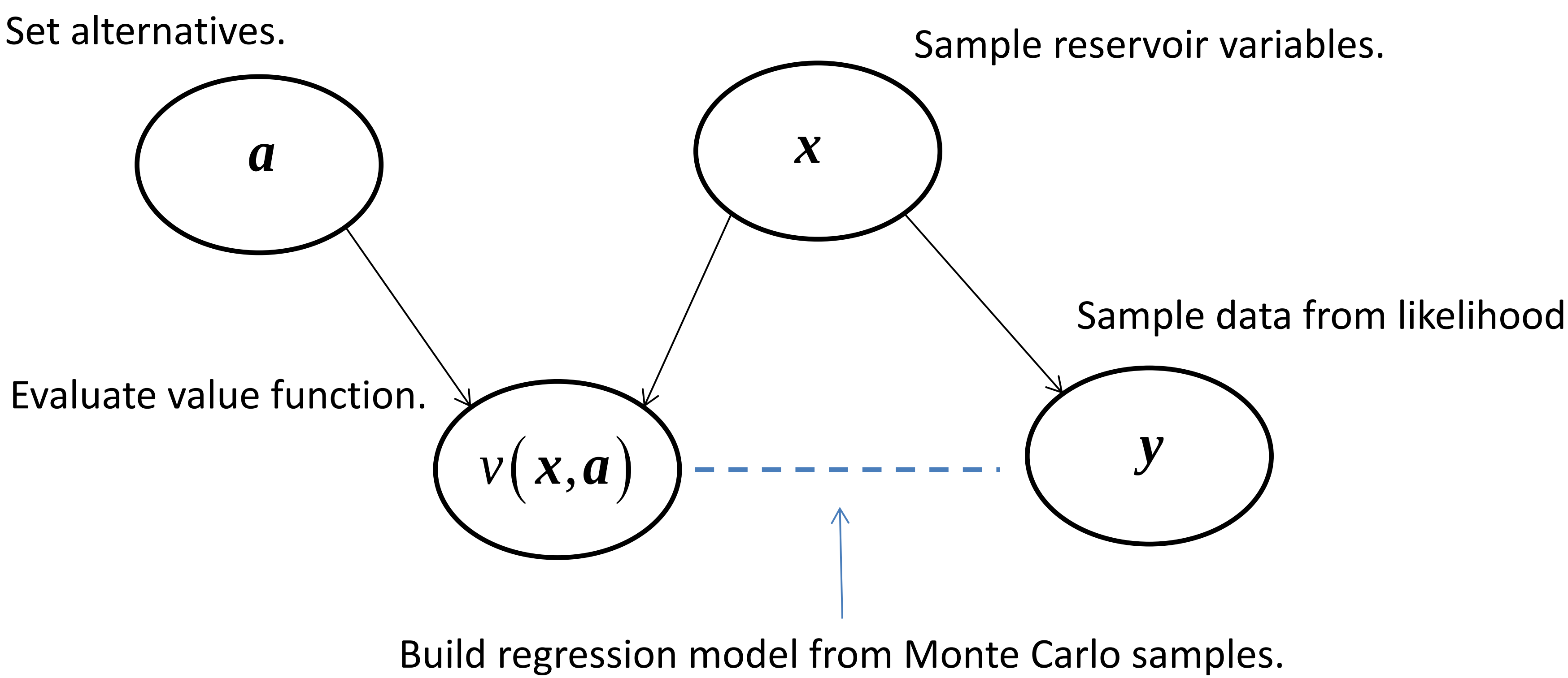
Gullfaks case (likelihood of R0,G data)



Synthetic time-lapse seismic (processing more angle information (R0,G)):
Use rock physics relations connecting reservoir properties to (R0,G).

Simulations indicate limited information about saturation from (R0, G).

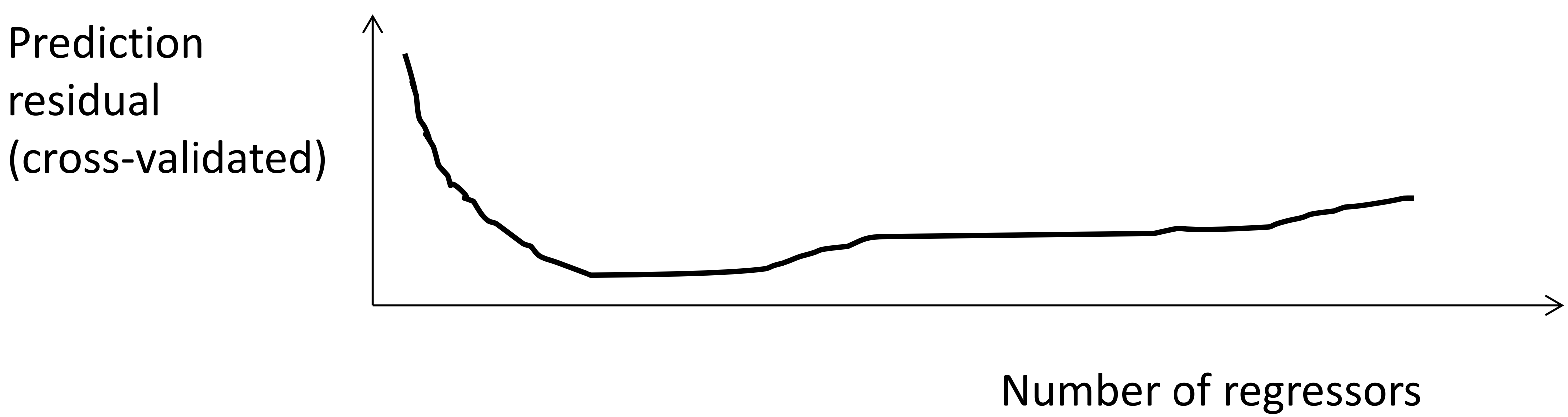
Simulation-regression illustration



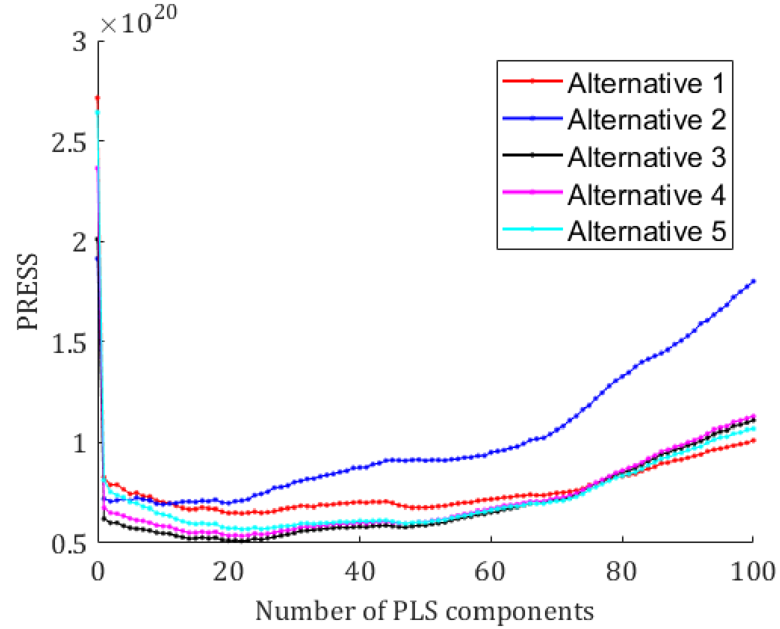
Large data: Partial least squares regression

- Partial least squares (PLS) regression is used for regression values on large seismic data set.
- Cross-validation to find optimal number of linear combinations.
- PLS is similar to Principle component regression (PCR).

  (PLS focuses on explaining covariance instead of variance.)



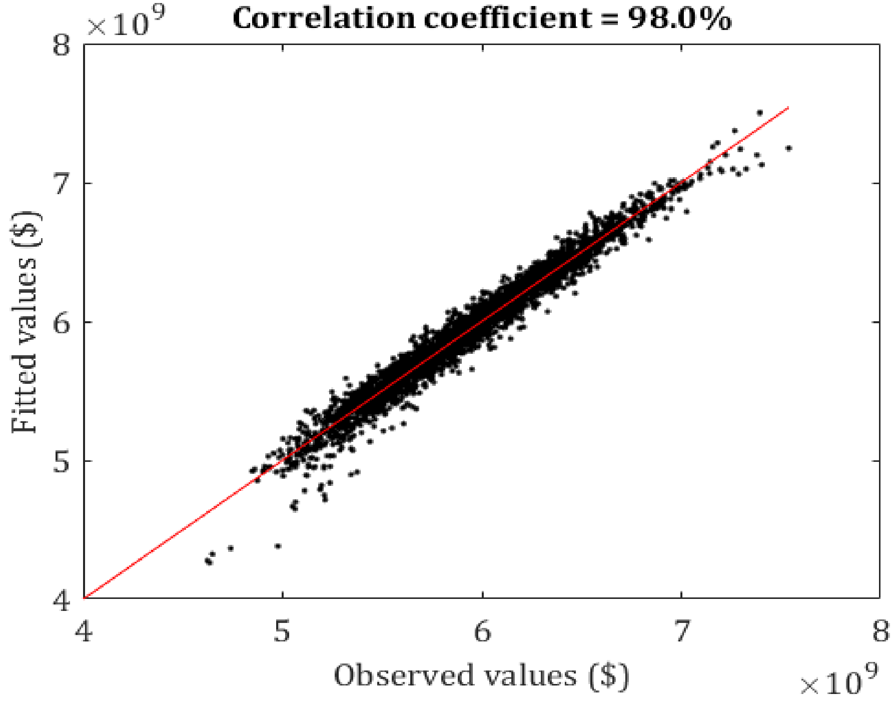
Gullfaks case (PLS for expected values)



$E\left(v(\boldsymbol{x},\boldsymbol{a} \mid \boldsymbol{y})\right)$

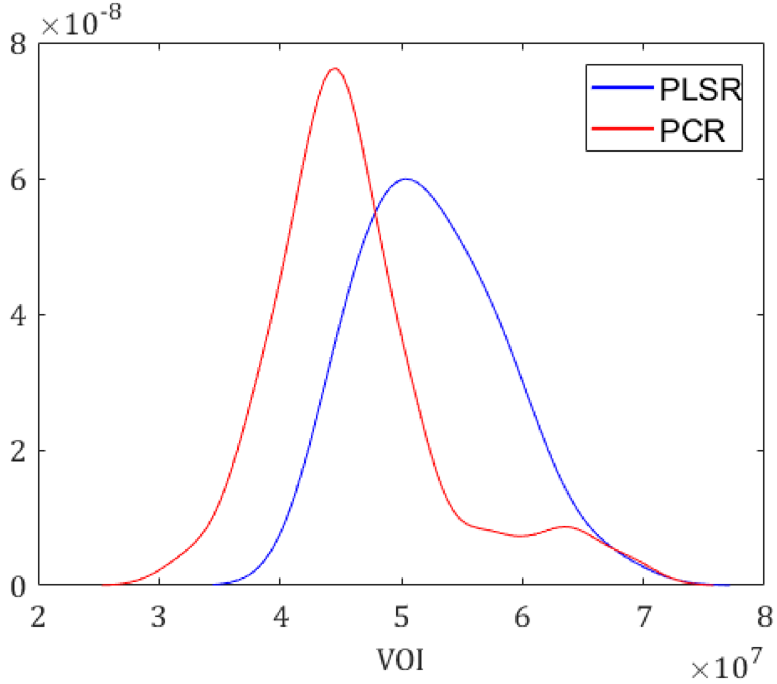
Fit regression model from Monte Carlo samples.
12 regressor components in the PLS regression.

Gullfaks case (predictive power)

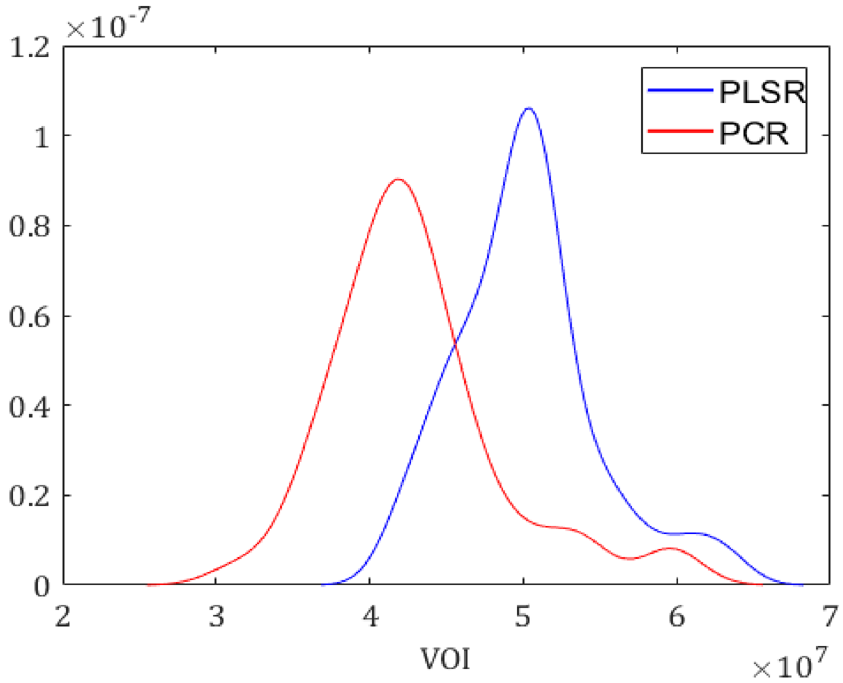


Fit of regression models is reasonable (based on AI data here).

Gullfaks case (VOI results)



Acoustic impedance (AI)



Angle information, (R0,G)

VOI of time-lapse data is about $50 million.
No big differences in VOI of processing methods
(but the price of these likely differ).

(Bootstrap used to get distribution.)

Wrap up example

- The type of simulation and regression would be very case specific. And residual plots should be used to check performance.

- If there are lots of alternatives, some kind of clustering of alternatives should be used.

- VOI approximation is difficult to check, but bootstrap (or bagging) can be used to study uncertainty, and to do sensitivity over different regression models.

Formula for VOI

$$PV = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{a}) p\left(\boldsymbol{x} \right) d\boldsymbol{x} \right\}$$

$$PoV\left(\boldsymbol{y} \right) = \int \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x}, \boldsymbol{a} \right) | \, \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y} \right) d\boldsymbol{y}$$

$$VOI\left(\boldsymbol{y} \right) = PoV\left(\boldsymbol{y} \right) - PV.$$

The analysis is usually done for **static decisions** and **static data gathering schemes**:

- We make the one-time decisions here and now.
- We can only collect the data here and now.

Sequential decisions or **sequential tests** can give benefits over this situation.

Information gathering

	Perfect	Imperfect
Total	Exact observations are gathered for all variables. $$\boldsymbol{y} = \boldsymbol{x}$$	Noisy observations are gathered for all variables. $$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\varepsilon}$$
Partial	Exact observations are gathered at some variables. $$\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}}, \quad \mathbb{K} \text{ subset}$$	Noisy observations are gathered at some variables. $$\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}} + \boldsymbol{\varepsilon}_{\mathbb{K}}, \quad \mathbb{K} \text{ subset}$$

Could also have **sequential** (adaptive) information gathering.

Sequential information gathering

Decision maker has the opportunity of dynamic testing, where one can stop testing, or continue testing, depending on the currently available data. The sequential order of tests and the number of tests also depend on the data.

$$PoV_{seqtest}\left(\boldsymbol{y}_1\right) = \int \max \left\{ \begin{array}{l} \max_{j \neq 1} \left\{ CV\left(j \mid 1\right) \right\}, \\ \sum_{i=1}^{n} \max_{a_i} \left\{ E\left(v\left(x_i, a_i\right) \mid \boldsymbol{y}_1\right) \right\} \end{array} \right\} p(\boldsymbol{y}_1) d\boldsymbol{y}_1$$

Continue testing.

Stop testing.

$$CV\left(j \mid 1\right) = Cont\left(j \mid 1\right) - P_j$$

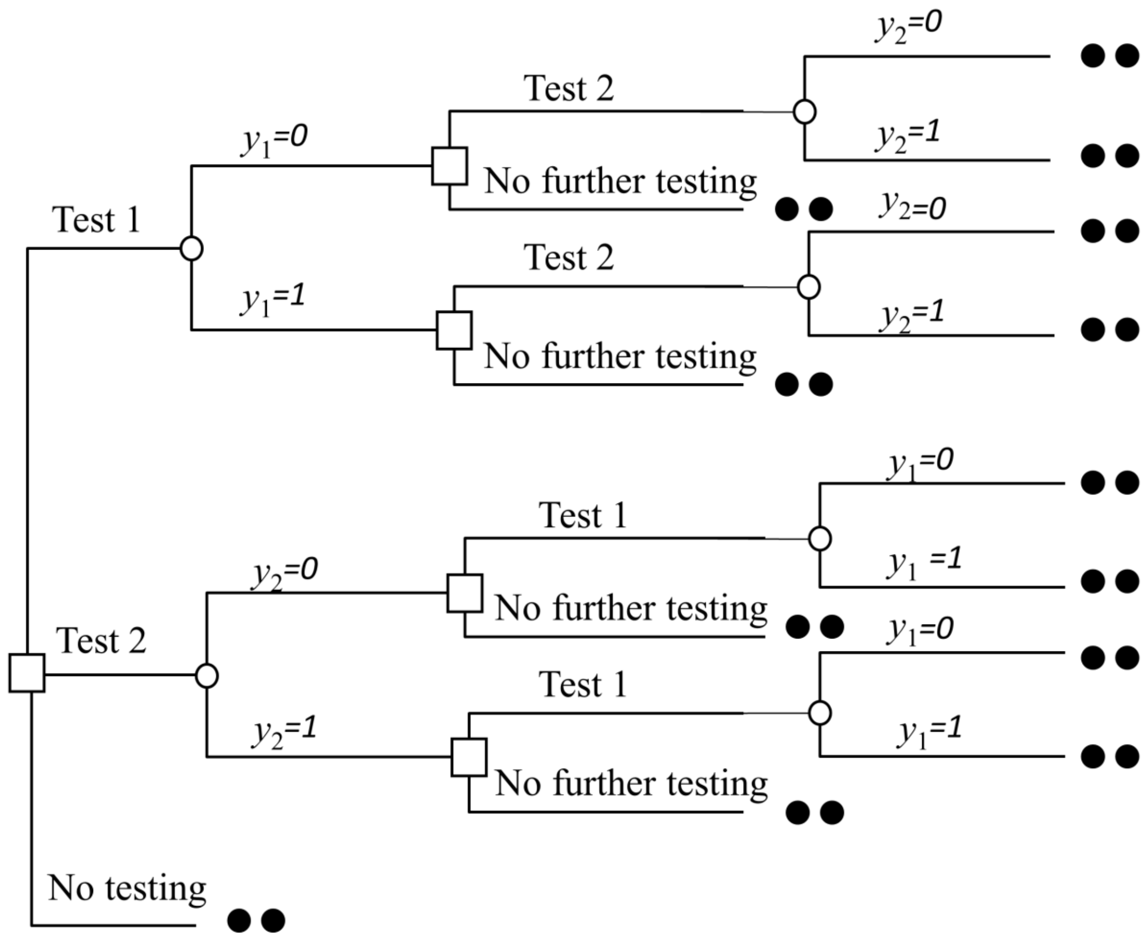
$$Cont\left(j \mid 1\right) = \int \max \left\{ \begin{array}{l} \max_{k \neq 1, j} \left\{ CV\left(k \mid j, 1\right) \right\}, \\ \sum_{i=1}^{n} \max_{a_i} \left\{ E\left(v\left(x_i, a_i\right) \mid \boldsymbol{y}_1, \boldsymbol{y}_j\right) \right\} \end{array} \right\} p(\boldsymbol{y}_j \mid \boldsymbol{y}_1) d\boldsymbol{y}_j$$

Continue testing.

Stop testing.

Solution is again dynamic programming.

Sequential testing– bivariate illustration



Sequential information (bivariate data)

Value with no more testing (after first test):

$$PoV\left(\boldsymbol{y}_1\right) = \int \sum_{i=1}^{n} \max_{a_i \in A_i} \left\{ E\left(v\left(x_i, a_i\right) \mid \boldsymbol{y}_1\right) \right\} p(\boldsymbol{y}_1) d\boldsymbol{y}_1$$

Criterion for continued testing:

$$\int_{\boldsymbol{y}_2} \sum_{i=1}^{n} \max_{a_i \in A_i} \left\{ E\left(v\left(x_i, a_i\right) \mid \boldsymbol{y}_1, \boldsymbol{y}_2\right) \right\} p\left(\boldsymbol{y}_2 \mid \boldsymbol{y}_1\right) d\boldsymbol{y}_2 - P_2 > \sum_{i=1}^{n} \max_{a_i \in A_i} \left\{ E\left(v\left(x_i, a_i\right) \mid \boldsymbol{y}_1\right) \right\}$$

$$PoV_{seqtest}\left(\boldsymbol{y}_1\right) = \int \max \left\{ \begin{array}{l} \int_{\boldsymbol{y}_2} \sum_{i=1}^{n} \max_{a_i} \left\{ E\left(v\left(x_i, a_i\right) \mid \boldsymbol{y}_1, \boldsymbol{y}_2\right) \right\} p\left(\boldsymbol{y}_2 \mid \boldsymbol{y}_1\right) d\boldsymbol{y}_2 - P_2, \\ \sum_{i=1}^{n} \max_{a_i} \left\{ E(v\left(x_i, a_i\right) \mid \boldsymbol{y}_1) \right\} \end{array} \right\} p(\boldsymbol{y}_1) d\boldsymbol{y}_1$$

Continue testing when the additional expected value of more testing exceeds the price.

Dynamic programming

The exact solution to sequential testing is only available even in small-size discrete models.

Various approximate strategies exist. (Approximate dynamic programming).

Myopic (near-sighted) is a common strategy for sequential problems.
It considers only one-stage at a time, not looking into the ‘future’:
(A Heuristic solution to the dynamic program.)

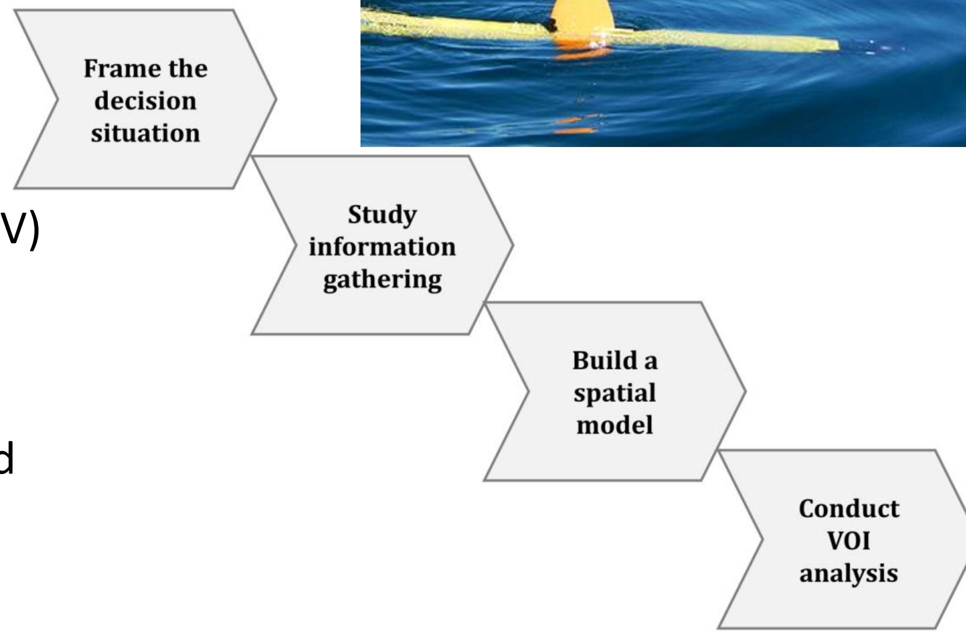
Myopic strategy for information

- Find best **first data design**, using one-stage, if any give positive VOI. 1 level
- **Collect first data** (by simulation) using best design.
- **Update** probability distributions, conditional on the data.
- Find **second best design**, using one-stage, in new model, if any give positive VOI. 2 level
- **Collect second data** (by simulation from new model) using best design.
- **Update** probability distributions, conditional on the data.
- Find **third best data**, using one-stage, in new model, if any give positive VOI. 3 level

.....

AUV data for ocean temperatures

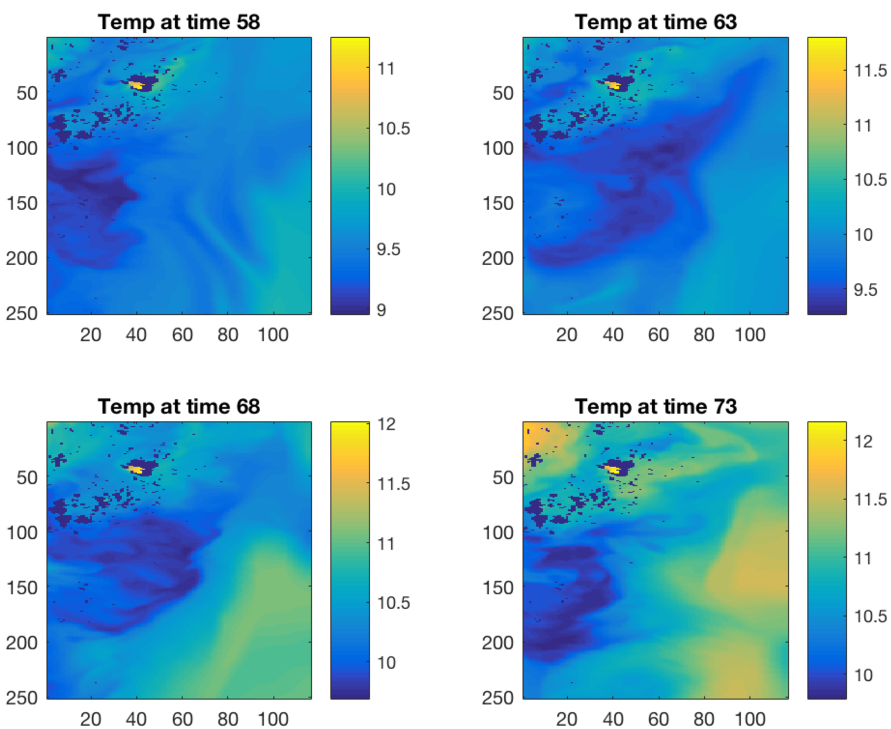
- Goal (value) is to detect large spatial gradients in ocean temperature.
- Autonomous underwater vehicle (AUV) information. Where? And in what sequence?
- Model for temperature is represented by Gaussian spatial process.
- VOI analysis uses analytical approach and myopic heuristics.



Fossum et al., 2018, Journal of field robotics.

Mapping ocean temperature variability

Satellite data and ocean models realizations are used to build Gaussian prior mean and covariance.

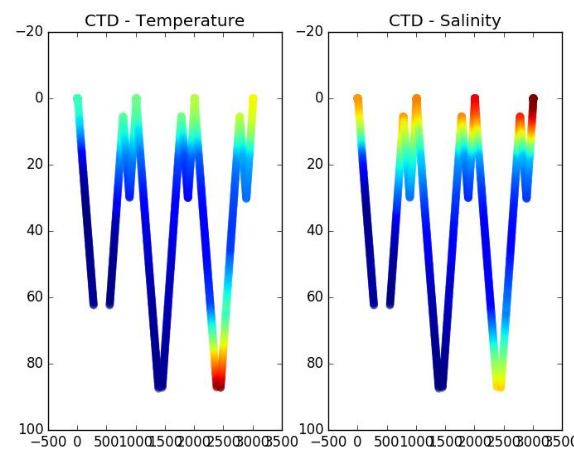


Area outside Trondheim fjord.

Possible questions:

- Environmental challenges
- Fish farming
- Algae bloom

Typical AUV data



Gaussian prior and likelihood

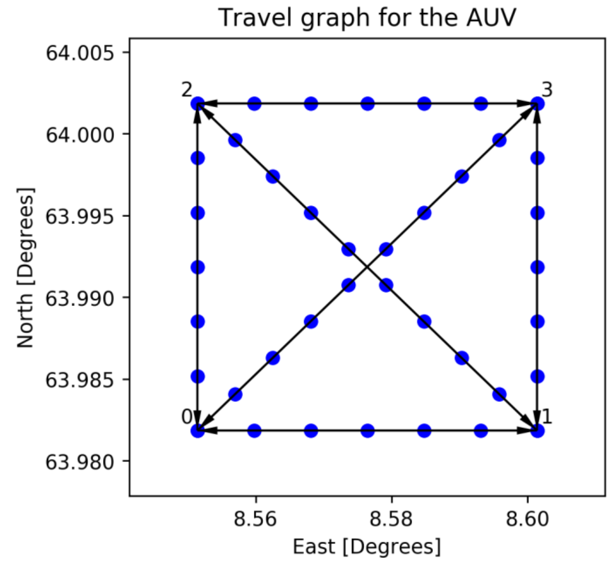
$$p(\boldsymbol{x}) = N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Gaussian spatial process prior for temperatures (learned from current knowledge).

$$\boldsymbol{y} = \boldsymbol{F}\boldsymbol{x} + N(\boldsymbol{0}, \tau^2 \boldsymbol{I})$$

$$p(\boldsymbol{y} \mid \boldsymbol{x}) = N(\boldsymbol{F}\boldsymbol{x}, \tau^2 \boldsymbol{I})$$

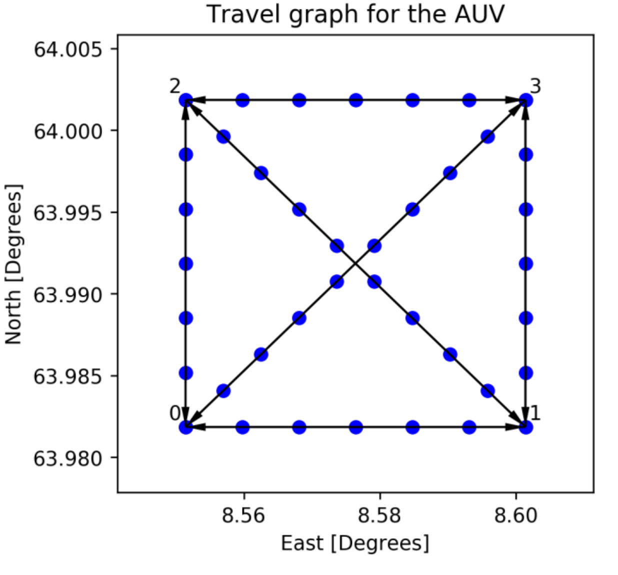
Likelihood, design matrix, picks data locations, for every time step.



Goal of surveying

The main task for the AUV is to detect large gradients in temperature which are linked to algea bloom.

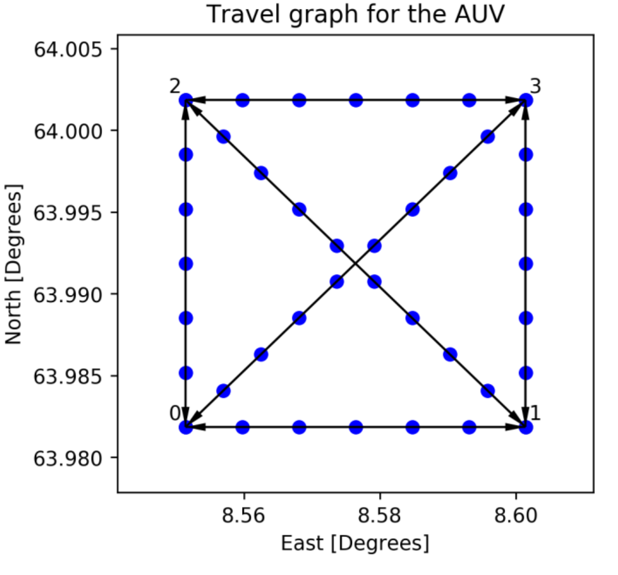
Waypoints in survey design for AUV.



Adaptive sequential algorithm

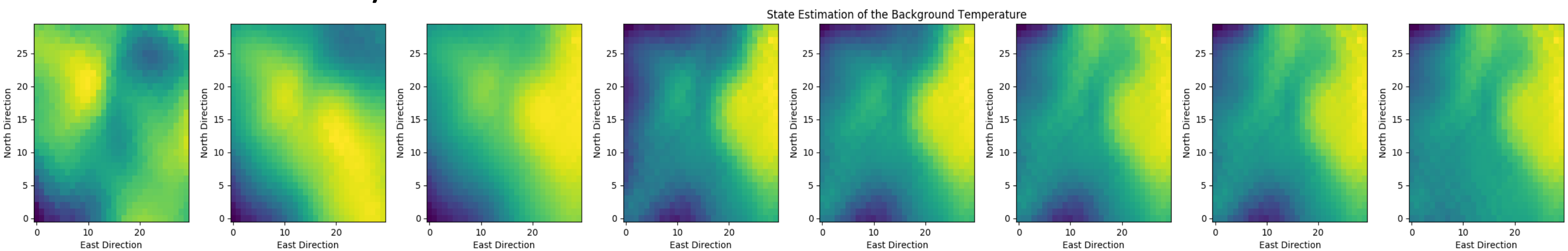
1. Find next best survey line (if any) from analytic VOI, of all possible survey lines.
2. Collect temperature data along currently best survey line.
3. Update temperature model in entire spatial domain given survey data.
4. Go to 1.

Myopic heuristic for dynamic program.

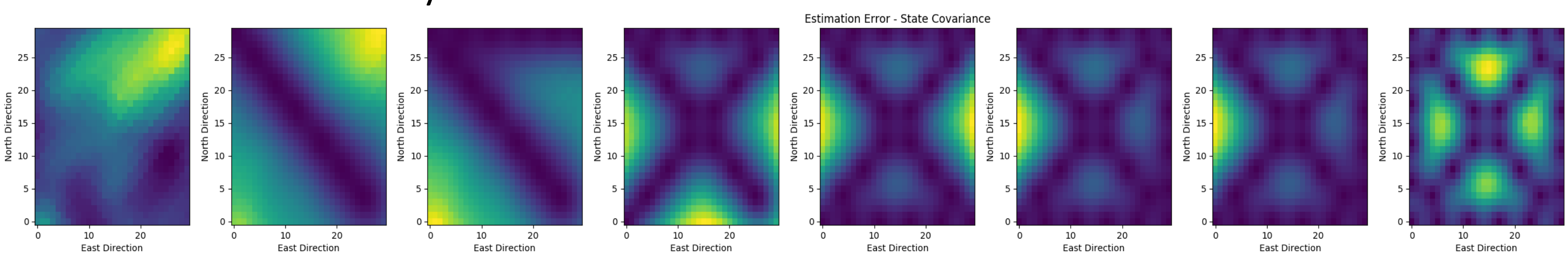


Results of adaptive algorithm

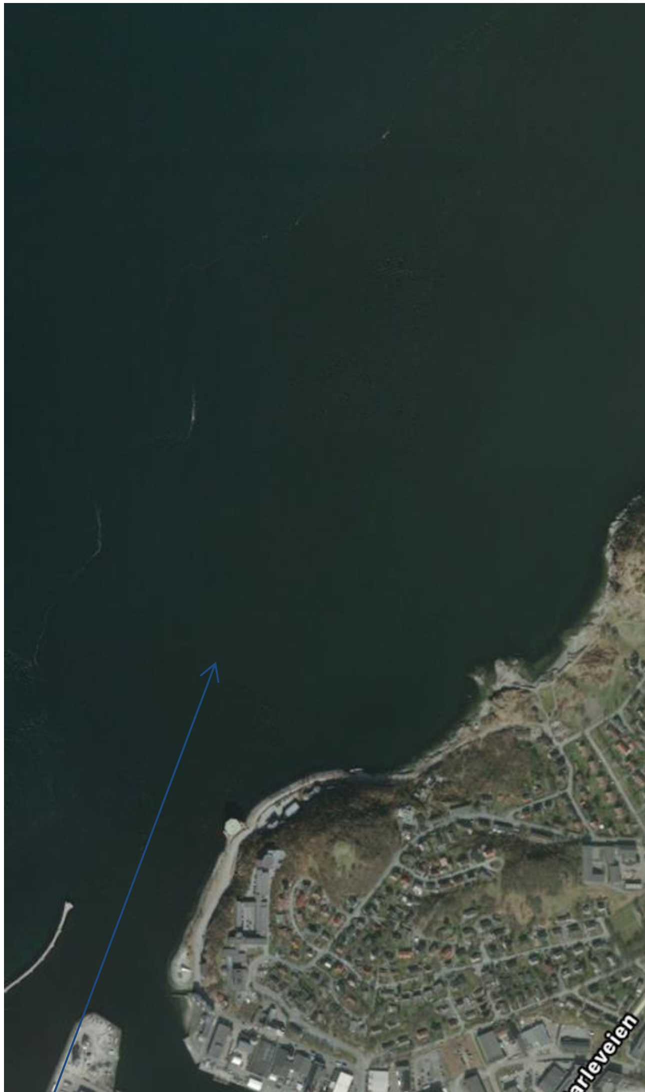
Mean of one survey



Variance of one survey



Fresh cold water & salt warm water



River mouth

River mouth

Excursion sets and excursion probabilities

$$ES_a = \{\mathbf{s} : x(\mathbf{s}) < a\}$$

Connections to active learning.

$$\boldsymbol{x} \sim GP(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$EP_a(\mathbf{s}) = P(x(\mathbf{s}) < a)$$

$$\boldsymbol{y}_d = \boldsymbol{A}_d \boldsymbol{x} + N(\boldsymbol{0}, \boldsymbol{T}_d)$$

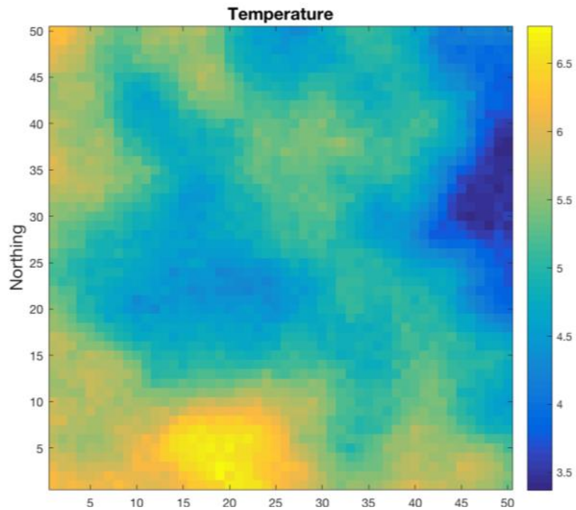
$$EP_a(\mathbf{s} \mid \boldsymbol{y}_d) = P(x(\mathbf{s}) < a \mid \boldsymbol{y}_d)$$

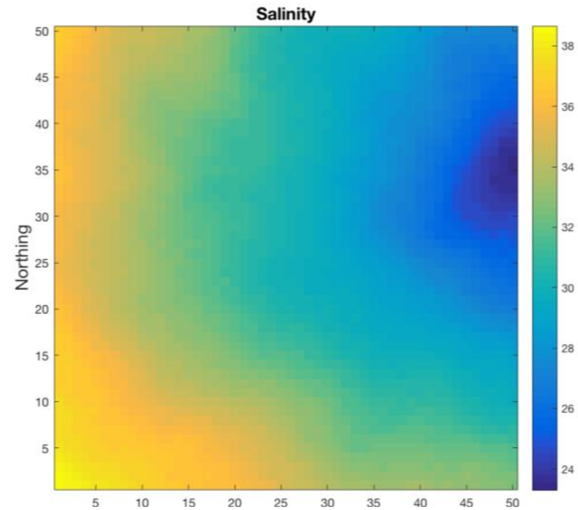
Criterion for path selection:

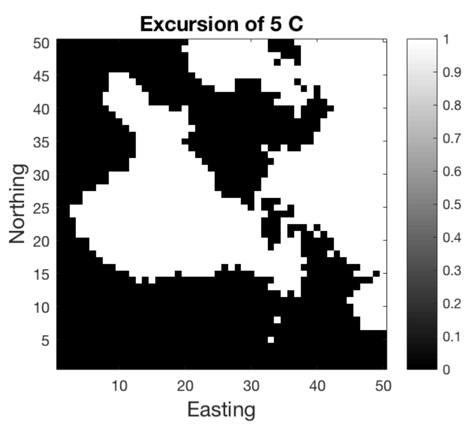
$$d^* = \arg\min_d \left\{ \int\int \underbrace{EP_a(\mathbf{s} \mid \boldsymbol{y}_d)(1 - EP_a(\mathbf{s} \mid \boldsymbol{y}_d}))\, p(\boldsymbol{y}_d)\, d\boldsymbol{y}_d\, d\mathbf{s} \right\}$$

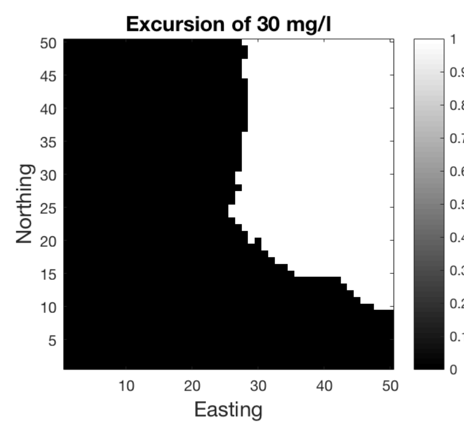
Closed form for Gaussian processes.

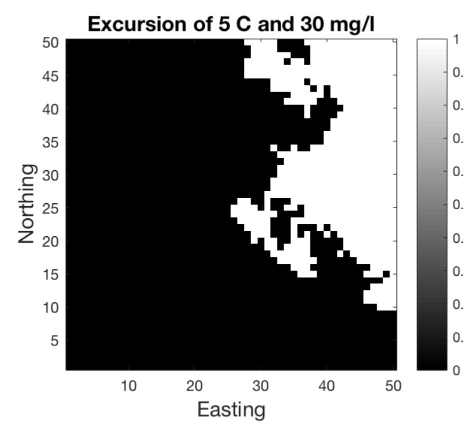
(Bivariate) excursion sets



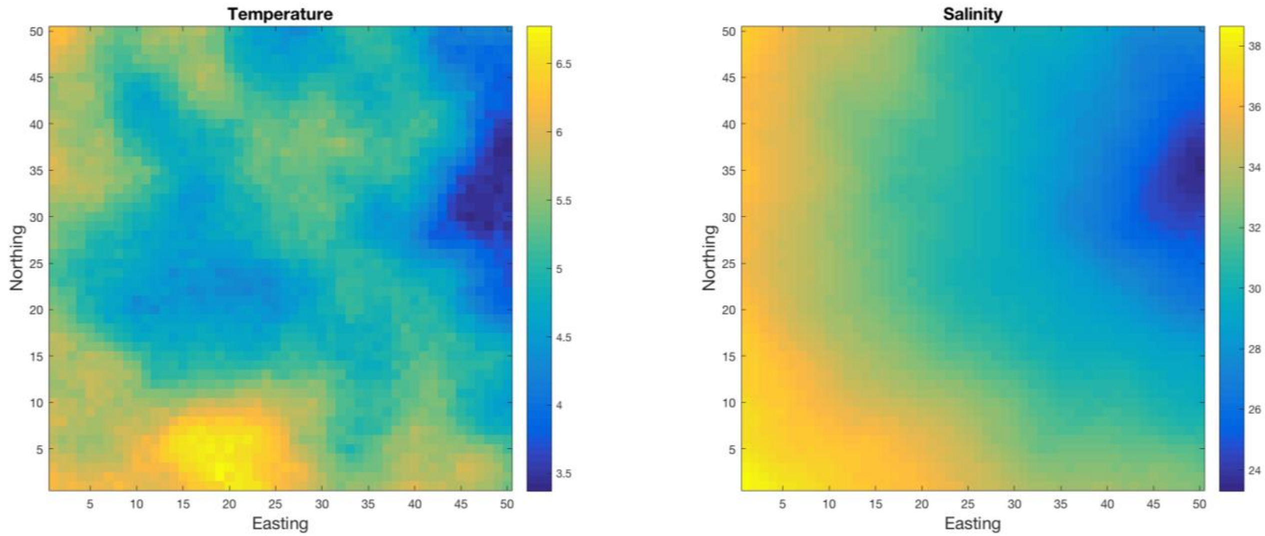








Bivariate excursion sets – closed form



$$\int EP_a(\mathbf{s} \mid \boldsymbol{y}_d)\left(1 - EP_a(\mathbf{s} \mid \boldsymbol{y}_d)\right) p(\boldsymbol{y}_d)\, d\boldsymbol{y}_d$$

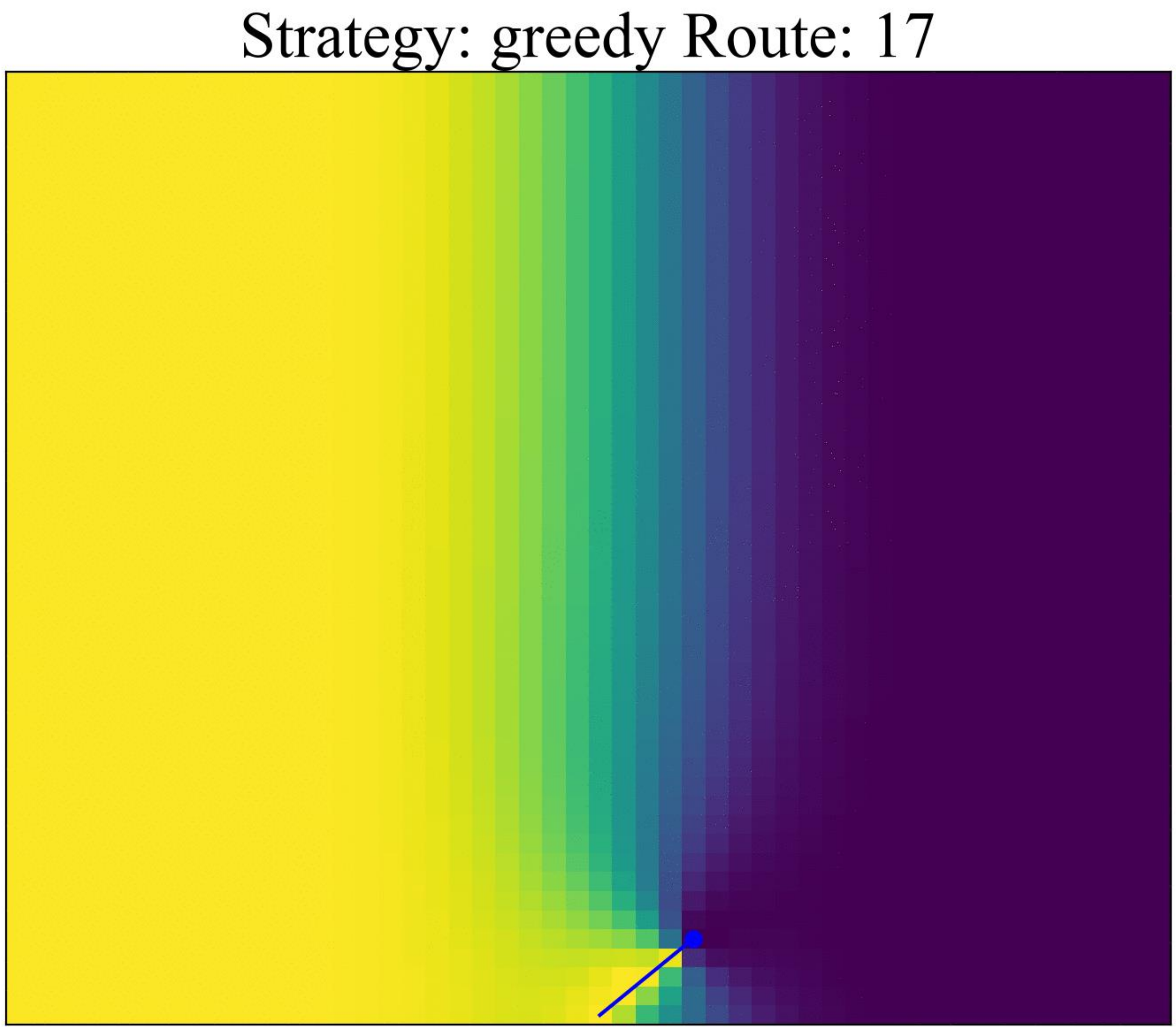
$$= \int P\left(x_{\text{temp}}(\mathbf{s}) < \alpha_{\text{temp}}, x_{\text{sal}}(\mathbf{s}) < \alpha_{\text{sal}} \mid \boldsymbol{y}_d\right) p(\boldsymbol{y}_d)\, d\boldsymbol{y}_d$$

$$= \Phi_2(\mathbf{0}_2, \boldsymbol{\beta}_2, \boldsymbol{R}_2) - \Phi_4(\mathbf{0}_4, \boldsymbol{\beta}_4, \boldsymbol{R}_4)$$

Multivariate Gaussian cumulative distribution function

Standard matrix –vector computations.

Myopic path selection for excursions



Real time excursion probability (blue = cold fresh water, yellow = salt warm water.

Wrap up:

- VOI (Active learning) is applied to sequential search for good data designs.
- The design will depend on the data, and the results can be averaged over the data, to approximate the value of different strategies.

- For larger-scales operation, the process is spatio-temporal – extensions required.

Illustration: Sequential VOI in Gaussian models

Consider again the 25x25 grid, with a Gaussian process prior for profits (like in the forestry example).

Assume the situation from with low decision flexibility, goal is to classify total (sum of) profits from all units.

Use the myopic strategy to find sequential data designs along the 25 North-South lines. The price of a test is P=0.1.

How many tests are done before we stop? (varies with data samples)
What tests are usually done? (varies with data samples)

By playing the game over many runs, we can study properties of the approach.

Myopic scheme

1. Find best single NS line, if any.

$$\boldsymbol{R}_j = \boldsymbol{\Sigma F}_j^t \left(\tau^2 \boldsymbol{I} + \boldsymbol{F}_j \boldsymbol{\Sigma F}_j^t\right)^{-1} \boldsymbol{F}_j \boldsymbol{\Sigma},$$

$$r_{w,j} = \sqrt{\sum \sum R_{ii',j}}, \qquad \mu_w = \sum \mu_i$$

$$PoV\left(\boldsymbol{y}_j\right) = \left(\mu_w \Phi\left(\mu_w / r_{w,j}\right) + r_{w,j} \phi\left(\mu_w / r_{w,j}\right)\right) - P_j$$

2. Collect data for this line. $\boldsymbol{y}_j$

3. Update the model

$$\boldsymbol{\mu} = \boldsymbol{\mu} + \boldsymbol{\Sigma F}_j^t \left(\tau^2 \boldsymbol{I} + \boldsymbol{F}_j \boldsymbol{\Sigma F}_j^t\right)^{-1} \left(\boldsymbol{y}_j - \boldsymbol{F}_j \boldsymbol{\mu}\right)$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma} - \boldsymbol{R}_j,$$

4. Stop testing or continue testing.

$$Stop = \max\{0, \mu_w\}, \qquad \mu_w = \sum_{i=1}^{n} \mu_i$$

Find largest among all k. →

$$Cont\left(\boldsymbol{y}_k\right) = \left(\mu_w \Phi\left(\mu_w / r_{w,k}\right) + r_{w,k} \phi\left(\mu_w / r_{w,k}\right)\right) - P_k$$

Use updated mean and covariances.

Etc....