UQ for high dimensional problems

with a view towards hyperbolic conservation laws

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Recap from last time

Convergence of Monte Carlo

$$\mathbb{E}((\mathbb{E}(X) - \frac{1}{M}\sum_{k=1}^{M} X_k)^2)^{1/2} = \frac{Var(X)^{1/2}}{M^{1/2}}.$$

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New notation for today:

$$\|f\|_{L^2(\Omega)} := \mathbb{E}(f^2)^{1/2} \qquad \text{for } f \in L^2(\Omega).$$

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Approximate:

$$\mathbb{E}(u(\cdot; T)) \approx \frac{1}{M} \sum_{k=1}^{M} u_k^{\Delta t, T}(\omega)$$

Error estimation

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Triangle inequality

$$\left\| \mathbb{E}(u(\cdot;T)) - \frac{1}{M}\sum_{k=1}^{M} u_k^{\Delta t,T} \right\|_{L^2(\Omega)} \leqslant \overbrace{\left\| \mathbb{E}(u(\cdot;T)) - \mathbb{E}\left(u_k^{\Delta t,T}\right) \right\|_{L^2(\Omega)}}^{=:\varepsilon_1} + \underbrace{\left\| \mathbb{E}(u_k^{\Delta t,T}) - \frac{1}{M}\sum_{k=1}^{M} u_k^{\Delta t,T} \right\|_{L^2(\Omega)}}_{=:\varepsilon_2}$$

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Assume* $|u(\omega;T) - u_k^{\Delta t,T}(\omega)| \leqslant C\Delta t^s$ for all ω, **then**:

*Can be replaced by L^2 assumption, can be shown for a large class of schemes/ODEs/RFs

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Hence,

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Equilibrating the error

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Want

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So choose

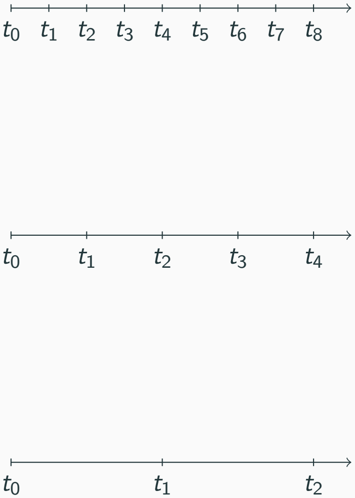
$$M = \mathcal{O}(\Delta t^{-2s}).$$

Multilevel Monte Carlo

Multilevel telescoping sum

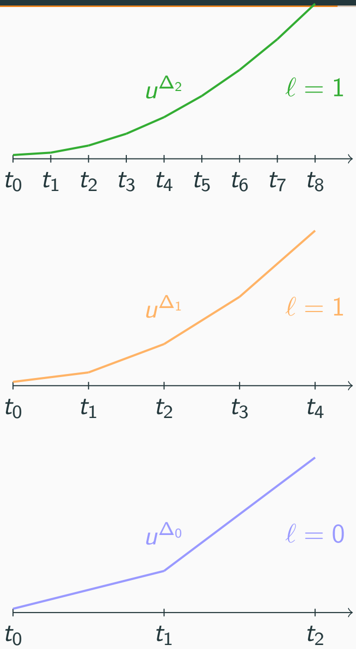
Numerical approximation u^{Δ}

Multilevel telescoping sum



Numerical approximation u^Δ

Multilevel telescoping sum



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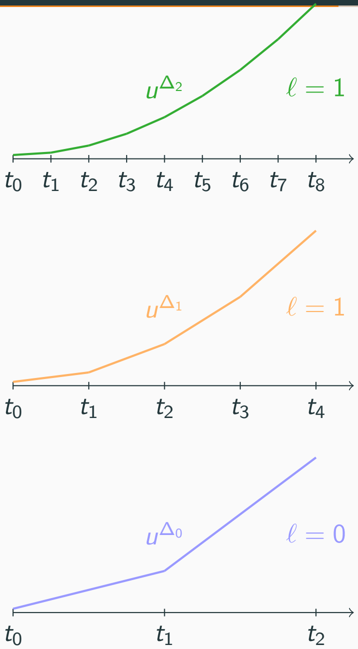
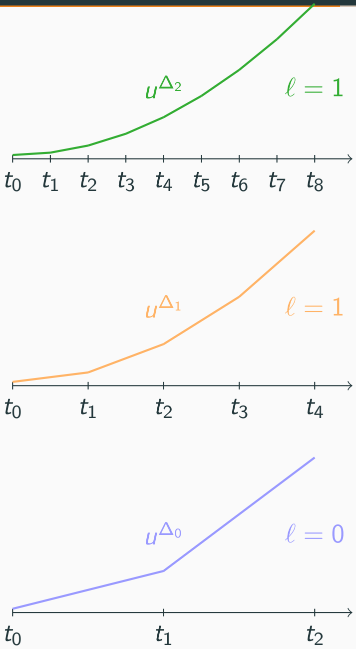


Numerical approximation u^{Δ}

$$u^{\Delta_1} = \left(u^{\Delta_1} - u^{\Delta_0}\right) + u^{\Delta_0}$$

Multilevel telescoping sum



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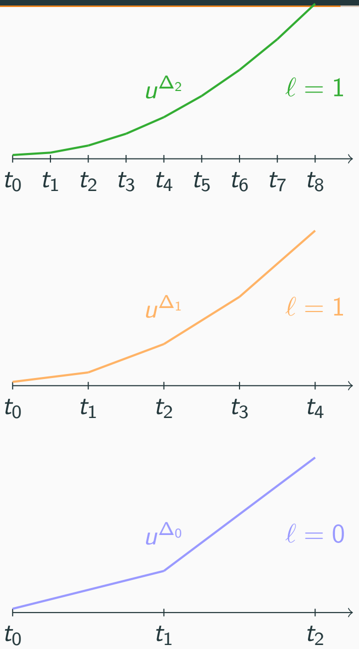

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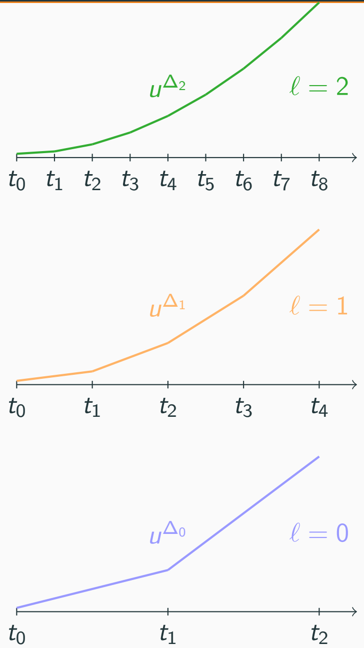
In general,

$$u^{\Delta_L} = \sum_{l=1}^{L} (u^{\Delta_\ell} - u^{\Delta_{\ell-1}}) + u^{\Delta_0}$$

Multilevel Monte Carlo [Giles, 2008; Heinrich 2001]



Random variable $u^{\Delta}(\omega)$

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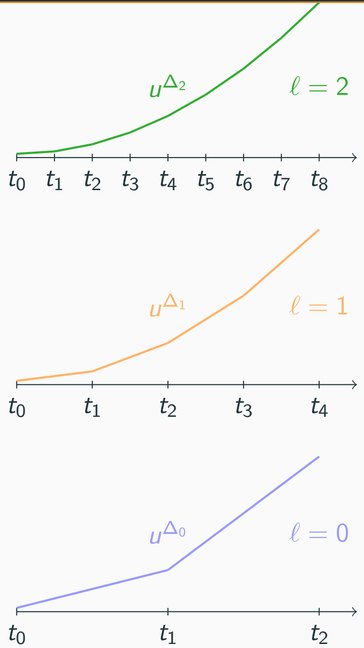

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$$\mathbb{E}(u^{\Delta_1}) = \mathbb{E}(u^{\Delta_1} - u^{\Delta_0}) + \mathbb{E}(u^{\Delta_0})$$

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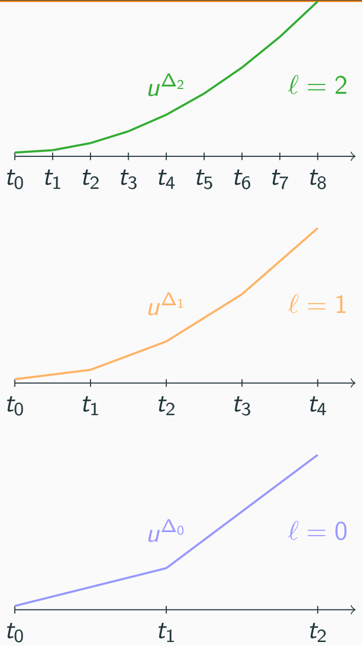

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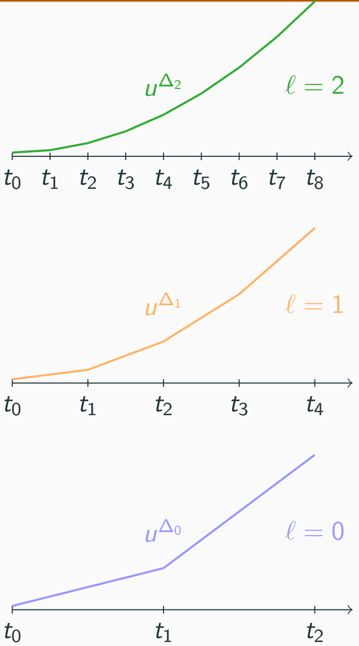

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Variance reduction

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Variance reduction

$$\begin{aligned}\mathbb{E}(u^{\Delta_1}) &= \mathbb{E}(u^{\Delta_1} - u^{\Delta_0}) + \mathbb{E}(u^{\Delta_0}) \\ &\approx \underbrace{\frac{1}{M_1}\sum_{k=1}^{M_1}\left(u_k^{\Delta_1} - u_k^{\Delta_0}\right)}_{\approx\mathbb{E}(u^{\Delta_1}-u^{\Delta_0})} + \underbrace{\frac{1}{M_0}\sum_{k=1}^{M_0} u_k^{\Delta_0}}_{\approx\mathbb{E}(u^{\Delta_0})}\end{aligned}$$

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Error

$$\|E_2(u) - \mathbb{E}(u)\|_{L^2(\Omega)} \leqslant \|\mathbb{E}(u) - \mathbb{E}(u^{\Delta_1})\|_{L^2(\Omega)} + \left\|E_2(u) - \mathbb{E}(u^{\Delta_1})\right\|_{L^2(\Omega)}.$$

Error analysis continued

So

$$\left\| E_2(u) - \mathbb{E}(u^{\Delta_1}) \right\|_{L^2(\Omega)} \leqslant \left\| \frac{1}{M_1} \sum_{k=1}^{M_1} \left(u_k^{\Delta_1} - u_k^{\Delta_0} \right) - \mathbb{E}(u^{\Delta_1} - u^{\Delta_0}) \right\|_{L^2(\Omega)} + \left\| \mathbb{E}(u^{\Delta_0}) - \frac{1}{M_0} \sum_{k=1}^{M_0} u_k^{\Delta_0} \right\|_{L^2(\Omega)}$$

Error analysis continued

So

$$\left\|E_2(u) - \mathbb{E}(u^{\Delta_1})\right\|_{L^2(\Omega)} \leqslant \left\|\frac{1}{M_1}\sum_{k=1}^{M_1}\left(u_k^{\Delta_1} - u_k^{\Delta_0}\right) - \mathbb{E}(u^{\Delta_1} - u^{\Delta_0})\right\|_{L^2(\Omega)}$$

$$+ \left\|\mathbb{E}(u^{\Delta_0}) - \frac{1}{M_0}\sum_{k=1}^{M_0} u_k^{\Delta_0}\right\|_{L^2(\Omega)}$$

$$\leqslant \frac{\operatorname{Var}(u^{\Delta_1} - u^{\Delta_0})^{1/2}}{M_1^{1/2}} + \frac{\operatorname{Var}(u^{\Delta_0})^{1/2}}{M_0^{1/2}}$$

Choosing the number of samples

Assume:

- Convergence

$$|u^{\Delta_1}(\omega) - u(\omega)| \leqslant C\Delta^s \qquad \text{for all } \omega \in \Omega$$

- Mesh refinement

$$\Delta_\ell = 2^\ell \Delta_0 \qquad \text{for all } \ell > 0.$$

- Computational work per sample:

$$\text{Work}(\Delta) = \mathcal{O}(\Delta^{-1}).$$

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hence

$$\left\| E_2(u) - \mathbb{E}(u^{\Delta_1}) \right\|_{L^2(\Omega)} \leqslant \frac{C\Delta_0^s}{M_1^{1/2}} + \frac{\mathrm{Var}(u^{\Delta_0})^{1/2}}{M_0^{1/2}}$$

Choosing the number of samples: Part II

$$\left\| E_2(u) - \mathbb{E}(u^{\Delta_1}) \right\|_{L^2(\Omega)} \leqslant \frac{C\Delta_0^s}{M_1^{1/2}} + \frac{\mathrm{Var}(u^{\Delta_0})^{1/2}}{M_0^{1/2}}$$

equilibrate the error

$$\frac{C\Delta_0^s}{M_1^{1/2}} \approx \frac{\mathrm{Var}(u^{\Delta_0})^{1/2}}{M_0^{1/2}} \approx C^{2s}\Delta_0^{2s}$$

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$$\frac{C\Delta_0^s}{M_1^{1/2}} \approx \frac{\operatorname{Var}(u^{\Delta_0})^{1/2}}{M_0^{1/2}} \approx C^{2s}\Delta_0^{2s}$$

That is

$$M_1 = 4$$

and

$$M_0 = \frac{4}{\Delta_1^{2s}}.$$

Work estimate for the new samples

$$\begin{aligned}
\text{Work}(M_0, M_1) &= M_0 \text{Work}(\Delta_0) + M_1 \text{Work}(\Delta_1) \\
&= M_0 \Delta_0^{-1} + M_1 \Delta_1^{-1} \\
&= \frac{4}{\Delta_0^{2s}} \Delta_0^{-1} + 4\Delta_1^{-1} \\
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Meanwhile, with Monte Carlo:

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$$\text{Work}(M) = \overbrace{\Delta_1^{-2s}}^{=M}\Delta_1^{-1} = 2^{2s+1}\Delta_0^{-2s-1}$$

for small Δ_0:

$$\text{Work}(M_0, M_1) < \text{Work}(M).$$

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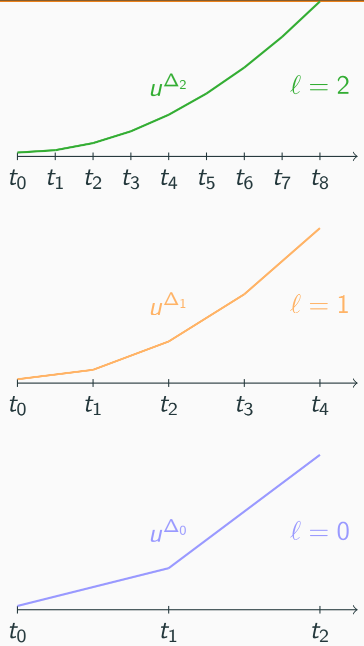

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