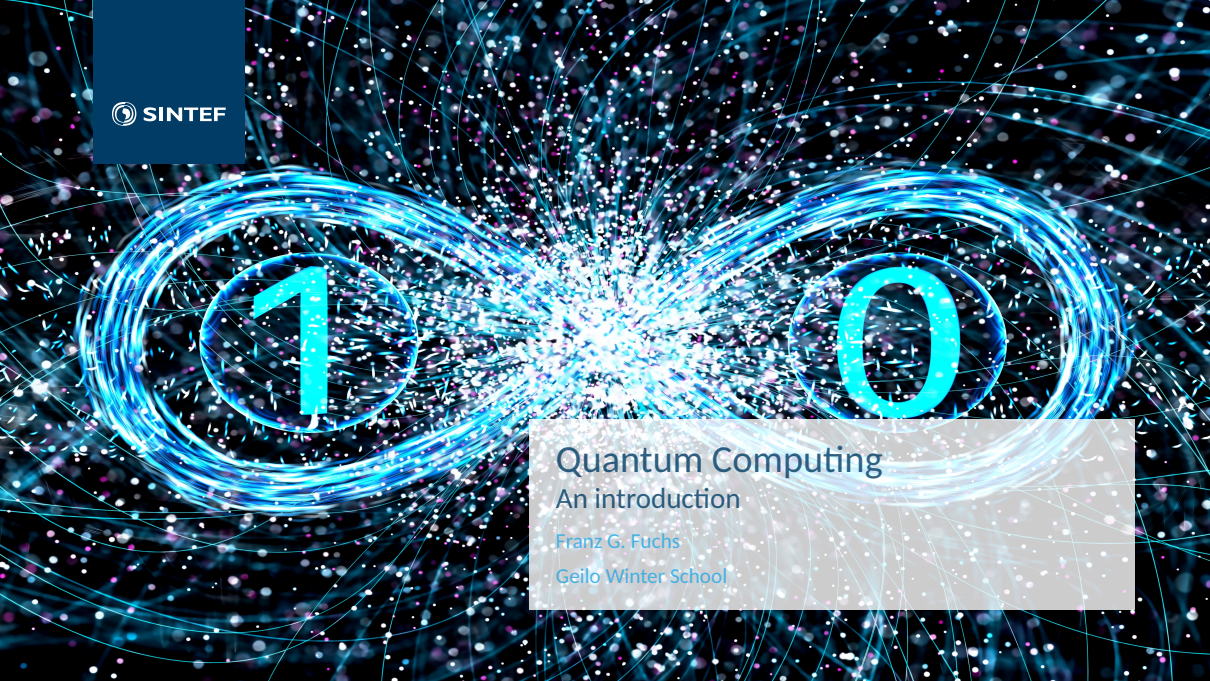

Quantum Computing

An introduction

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Dirac/”Bra-ket” notation

- common notation for quantum states i.e. vectors in a complex Hilbert spaces V
- $|\rangle$ denotes a vector in a vector space V
- $\langle|$ denotes a linear functional on V, i.e. is an element of V^*
- we can identify a vector with a linear functional, i.e. a ”ket” with a ”bra”, and vice versa
- $\langle|\rangle : V \times V \to \mathbb{C}$ denotes the inner product
- $|\rangle\langle| : V \times V \to V \otimes V$ denotes the outer product

A quantum bit

Postulate 1 [Nielsen and Chuang(2000), page 80]

Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the **state space** of the system. The system is completely described by its state vector, which is a **unit vector** in the system's state space.

A quantum bit

A quantum bit (qubit) is a quantum mechanical system with a two-dimensional state space. A state $|\Phi\rangle$ is a unit vector in $\mathbb{C}^2$. Given an orthonormal basis $|\varphi_0\rangle, |\varphi_1\rangle$, a qubit can be written as

$$|\Phi\rangle = a_0 |\varphi_0\rangle + a_1 |\varphi_1\rangle \text{, with } a_0, a_1 \in \mathbb{C} \text{ and } \langle\Phi|\Phi\rangle = |a_0|^2 + |a_1|^2 = 1. \quad (1)$$

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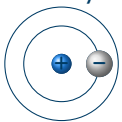
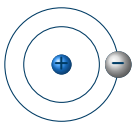
An example using states of hydrogen atoms

ground state



$$|\varphi_0\rangle = |0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

first excited state



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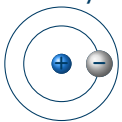

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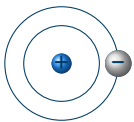

$$|\varphi_1\rangle = |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Another example is photon polarization.

A quantum bit

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In contrast to classical mechanics, a **superposition** of basis states is possible. An example is the state $|\Phi\rangle = -\frac{1}{\sqrt{2}} |0\rangle + i\frac{1}{\sqrt{2}} |1\rangle$.

Bloch sphere and superposition

The general state of a qubit can be written using polar representation

$$|\Phi\rangle = r_0 e^{i\theta_0} |0\rangle + r_1 e^{i\theta_1} |1\rangle . \tag{2}$$

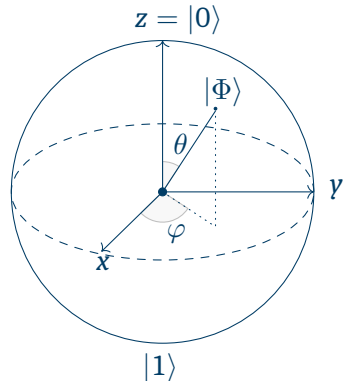
The global phase is irrelevant (for reasons explained later), we can multiply the state with $e^{-i\theta_0}$ and our (equivalent) state is

$$|\Phi\rangle = r_0 |0\rangle + r_1 e^{i\theta} |1\rangle , \quad \theta = \theta_1 - \theta_0. \tag{3}$$

Using that we have a unit vector, we can write

$$|\phi\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2) e^{i\phi} |1\rangle , \tag{4}$$

where $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$.



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Multiple qubits

Postulate 4 [Nielsen and Chuang(2000), page 94]

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\Phi_i\rangle$, then the joint state of the total system is $|\Phi_1\rangle \otimes |\Phi_2\rangle \otimes \cdots \otimes |\Phi_n\rangle$.

Reminder: Tensor product

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} \tag{5}$$

$$\begin{aligned} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \otimes \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} &= \begin{pmatrix} a_{1,1} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} & a_{1,2} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \\ a_{2,1} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} & a_{2,2} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{pmatrix}. \end{aligned} \tag{6}$$

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Multiple qubits

The general state $|\Phi\rangle$ of n qubits is a unit vector in $(\mathbb{C}^2)^{\otimes n} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{\text{n times}}$.

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Multiple qubits

The general state $|\Phi\rangle$ of n qubits is a unit vector in $(\mathbb{C}^2)^{\otimes n} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{\text{n times}}$.

Using the standard basis for $\mathbb{C}^2$, a basis for $(\mathbb{C}^2)^{\otimes n}$ is given by the following 2^n vectors

$$
\begin{aligned}
|0\rangle_n &:= |\underbrace{00\ldots00}_{\text{n digits}}\rangle = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |0\rangle = \begin{pmatrix}1,0 & \ldots & 0,0\end{pmatrix}^\mathsf{T} \\
|1\rangle_n &:= |\underbrace{00\ldots01}_{\text{n digits}}\rangle = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |1\rangle = \begin{pmatrix}0,1 & \ldots & 0,0\end{pmatrix}^\mathsf{T} \\
&\vdots \\
|2^n-1\rangle_n &:= |\underbrace{11\ldots11}_{\text{n digits}}\rangle = |1\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle \otimes |1\rangle = \begin{pmatrix}0,0 & \ldots & 0,1\end{pmatrix}^\mathsf{T}
\end{aligned}
\tag{7}
$$

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Multiple qubits

A general state can therefore be expressed as

$$|\Phi\rangle = \sum_{i=0}^{2^n-1} c_i |i\rangle = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2^n-2} \\ c_{2^n-1} \end{pmatrix}, \quad \sum_{i=0}^{2^n-1} |c_i|^2 = 1, \quad c_i \in \mathbb{C}. \tag{8}$$

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Remark.

- The space $(\mathbb{C}^2)^{\otimes n}$ is a 2^n-dimensional space. The dimension grows exponentially with the number of qubits.
- The state space of n classical bits, i.e., a binary string $\{0,1\}^n$ is an n-dimensional space. The dimension grows linearly with the number of bits.

Product states and entanglement

A quantum state $|\Phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a **product state** if it can be expressed as a tensor product of n single qubits $|\Phi_i\rangle$, i.e.,

$$|\Phi\rangle = \underbrace{\Phi_1 \otimes \cdots \otimes \Phi_n}_{\text{n times}} \tag{9}$$

Otherwise, it is **entangled**.

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Examples.

- Product state: $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- Entangled state: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Important states and conventions

- Two-qubit Bell states

  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
  $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
  $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

  (They form a maximally entangled basis, known as the Bell basis, of the four-dimensional Hilbert space for two qubits.)
- Superposition states

  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
  $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- Sometimes one writes $|\Phi_1\rangle\,|\Phi_2\rangle$, which is short hand for $|\Phi_1\rangle \otimes |\Phi_2\rangle$.

Quantum evolution

Postulate 2 [Nielsen and Chuang(2000), page 81]

The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\Phi\rangle$ of the system at time $t1$ is related to the state $|\Phi'\rangle$ of the system at time $t2$ by a **unitary operator** U which depends only on the times $t1$ and $t2$,

$$|\Phi'\rangle = U|\Phi\rangle \tag{10}$$

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Operations on qubits

An operation applied by a quantum computer, which is also called a **gate**, to n qubits is a **unitary matrix** $\mathbb{C}^{2^n \times 2^n}$.

- A matrix is U unitary, if $U^\dagger U = UU^\dagger = I$.
- Unitary matrices are norm-preserving, i.e., $\| U |\Phi\rangle \| = \| |\Phi\rangle \|$. This means that we get back a quantum state, which is a unit vector.
- Quantum operations are linear.
- Quantum operations are reversible.

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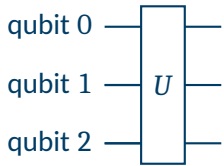
This seems restrictive at first, but:

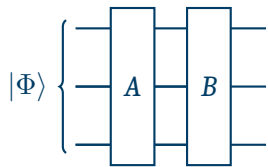
- A universal quantum computer is Turing-complete [Deutsch(1985)].
- All computations (including classical computations) can be madereversible [Bennett(1973)].

Notation for quantum circuits

Wires represent qubits and **gates** are operations:

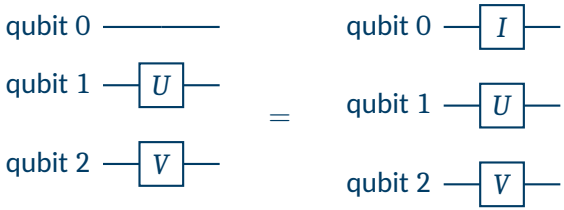


Serially wired gates. The state $BA\,|\Phi\rangle$ is represented as:



Notation for quantum circuits

Parallel gates:



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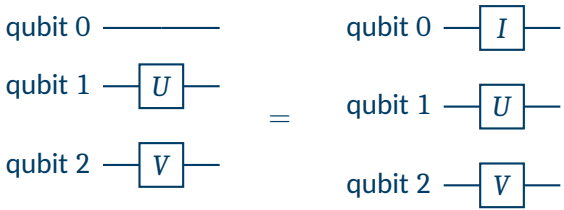

- If we have a product state $|\psi_0\rangle \otimes |\psi_1\rangle \otimes |\psi_2\rangle$ then we have

$$(I \otimes U \otimes V)\,|\psi_0\rangle \otimes |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_0\rangle \otimes U\,|\psi_1\rangle \otimes V\,|\psi_2\rangle \tag{11}$$

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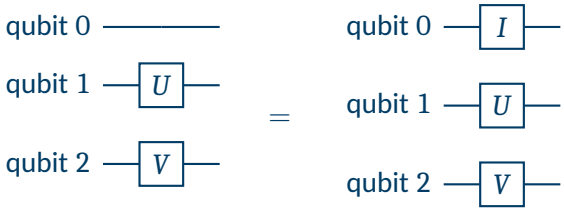

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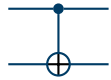
- But for a general (entangled) state $|\Psi\rangle$ the action of $I \otimes U \otimes V$ cannot be determined in such a simple way. We need to explicitly calculate the effect of the $2^n \times 2^n$ matrix on the state $|\Psi\rangle$. This is essentially the reason why we in general need exponential amounts of memory (or time) to keep track of the full state in 2^n-dimensional complex space.

Examples of 1 qubit gates

- Hadamard gate $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We have that
  $H^2 = I, H|0\rangle = |+\rangle, H|1\rangle = |-\rangle, H|+\rangle = |0\rangle, H|-\rangle = |1\rangle.$
- Pauli gates $X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have that
  $X^2 = I, X|0\rangle = |1\rangle, X|1\rangle = |0\rangle, X|+\rangle = |+\rangle, X|-\rangle = -|-\rangle.$
- Pauli gates $Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. We have that $Y^2 = I, Y|0\rangle = i|1\rangle, Y|1\rangle = -i|0\rangle.$
- Pauli gates $Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We have that $Z^2 = I, Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle.$
- Phase shift gates $R_\Phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Phi} \end{pmatrix}$.
- Square root of NOT gate $\sqrt{X} = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+1 \end{pmatrix}$. We have that $\sqrt{X}\sqrt{X} = X$.
- ...

Examples of 2 qubit gates

- controlled not gate $CNOT = CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$ 

  It has the effect
  $CNOT\,|00\rangle = |00\rangle\,, CNOT\,|01\rangle = |01\rangle\,, CNOT\,|10\rangle = |11\rangle\,, CNOT\,|11\rangle = |10\rangle.$

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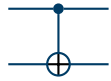
- controlled not gate $CNOT = CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$ 

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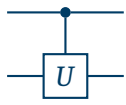
- controlled U gate $CU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{pmatrix} =$ 

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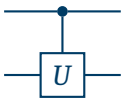
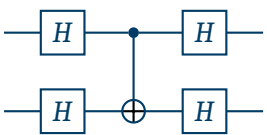
- controlled U gate $CU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{pmatrix} =$ 

- Note that  $=$ 

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

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Universal quantum gate sets

A set G of quantum gates universal if one can approximate any unitary transformation on any number of qubits with gates from G to any desired precision ε, i.e. there is a sequence of gates $g_1, \ldots g_k \in G$, such that

$$\|U - U_k \ldots U_2 U_1\| \leq \varepsilon. \tag{12}$$

- The operator norm is defined by $\|U - U'\| = \max_{|v\rangle, with \||v\rangle\|=1} \| (U - U') |v\rangle \|$.
- $U_i = I^l \otimes g_i \otimes I^m$ with appropriate l, m.

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Examples of universal gate sets:

- $G = \{CNOT, H, S = R_{\pi/2}, T = R_{\pi/4}\}$
- $G = \{CNOT, U(\theta, \phi, \lambda)\}$, where $U(\theta, \phi, \lambda) = \begin{pmatrix} e^{-i(\phi+\lambda)/2}\cos(\theta/2) & -e^{-i(\phi-\lambda)/2}\sin(\theta/2) \\ e^{i(\phi-\lambda)/2}\sin(\theta/2) & e^{i(\phi+\lambda)/2}\sin(\theta/2) \end{pmatrix}$

Solovay-Kitaev theorem

Let G be a universal gate set that is closed under inverses (i.e. if $g \in G$ then $g^{-1} \in G$) for $SU(n)$ and $\varepsilon > 0$ a desired accuracy. Then there is a constant c such that for any $U \in SU(n)$ there exits a finite sequence S of gates from G of length $\mathcal{O}(log^c(1/\varepsilon))$ such that $d(U, S) < \varepsilon$.

This SK algorithm provides a proof of the theorem and provides an algorithm to find the sequence S efficiently on a classical computer with running time $\mathcal{O}(log^{2.71}(1/\varepsilon))$.

Computational complexity

For an efficient algorithm we require that the circuit contains polynomially many gates in the number of qubits n and each gate has a compact representation in the universal gate set provided by the quantum computer.

Gottesman-Knill theorem

Beware! A quantum circuit using only the following elements can be simulated efficiently on a classical computer:

- Preparation of qubits in computational basis states,
- Quantum gates from the Clifford group (Hadamard gates, controlled NOT gates, Phase Gate), and
- Measurements in the computational basis.

How do we obtain information?

How do we obtain information?

Postulate 3 [Nielsen and Chuang(2000), page 84]

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. [..]If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \tag{13}$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{|M_m|\psi\rangle|} = \frac{M_m|\psi\rangle}{\sqrt{p(m)}}. \tag{14}$$

The measurement operators satisfy the completeness equation $\sum_m M_m^\dagger M_m = I$.

Measurement

- The completeness equation expresses the fact that probabilities sum to one:

$$\sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = \langle\psi|\sum_m M_m^\dagger M_m|\psi\rangle = \langle\psi|\psi\rangle = 1 \tag{15}$$

SINTEF

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- An important example is "measurement of a qubit in the **computational basis**": $M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|$. Notice $M_i^\dagger = M_i$, and $M_i M_i = M_i$ for $i \in \{0, 1\}$. Given a state $|\psi\rangle = a|0\rangle + b|1\rangle$, we have that
  - $p(0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle = \overline{a}\langle 0|0\rangle\langle 0|0\rangle a = |a|^2$, and the state after measurement is $M_0|\psi\rangle/|a| = a/|a||0\rangle = e^{i\theta_a}|0\rangle$.
  - $p(1) = |b|^2$ and the resulting state is $b/|b||1\rangle = e^{i\theta_b}|1\rangle$
- Measurement w.r.t. to the $|\pm\rangle$ basis.

$$\widetilde{M}_0 = 1/\sqrt{2}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \widetilde{M}_1 = 1/\sqrt{2}\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}$$

  - $p(0) = 1/2(\overline{a}+\overline{b})(a+b)$, and the state after measurement is $\frac{a+b}{\sqrt{2p(0)}}|0\rangle$.
  - $p(1) = 1/2(\overline{a}-\overline{b})(a-b)$ and the resulting state is $\frac{a-b}{\sqrt{2p(1)}}|1\rangle$

SINTEF

Measurement

- Let's say we want to measure a state $|\psi\rangle$ in the basis given by a set of orthonormal vectors u_i.
- However, we can only "physically" measure in the computational basis $P_i = |i\rangle \langle i|$.

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Idea: Apply basis change to computational basis before measurement.
The way to achieve this is to construct the unitary matrix U, where the columns consist of the vectors u_i and apply the inverse of U before measurement.

$$p_U(m) = \langle\psi|UP_m^\dagger P_m U^\dagger|\psi\rangle = \left\langle\psi'\middle|P_m^\dagger P_m\middle|\psi'\right\rangle, \quad \text{with } \left|\psi'\right\rangle = U^\dagger\left|\psi\right\rangle. \tag{16}$$

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This is how we ended up with the matrices $\widetilde{M_0}, \widetilde{M_1}$ on the previous slide.

Measurement

- A word of caution: It is wrong to think of a quantum state as a probability distribution.
- Coefficients are complex numbers unrestricted in sign, but probabilities are real, positive numbers.
- A quantum state **induces** a probability distribution through measurement.

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- Coefficients are complex numbers unrestricted in sign, but probabilities are real, positive numbers.
- A quantum state **induces** a probability distribution through measurement.
- Measurement is **irreversible**.
- Global phase: Consider $|\phi\rangle = e^{-i\theta}|\psi\rangle$. Then we have

$$p(m) = \langle\phi|M_m^\dagger M_m|\phi\rangle = e^{i\theta}\langle\psi|M_m^\dagger M_m e^{-i\theta}|\psi\rangle = \langle\psi|M_m^\dagger M_m|\psi\rangle \tag{17}$$

Expectation value of an observable

Given a state $|\phi\rangle$ and an observable A, the expectation value of A in the state ϕ is given by

$$\langle A \rangle_\phi := \langle \phi | A | \phi \rangle = \sum_i \lambda_i |\langle \phi | \psi_i \rangle|^2 . \tag{18}$$

Here, A is a self-adjoint operator on the Hilbert space $\mathbb{C}^{\otimes n}$, and $\{\lambda_i, |\psi_i\rangle\}$ is the set of eigenvalues and eigenvectors of A.

No-cloning principle

Let $|\phi\rangle$ be an arbitrary quantum state on n qubits.
$\nexists$ a unitary matrix that maps $|\phi\rangle \otimes |0\rangle$ to $|\phi\rangle \otimes |\phi\rangle$.

No-cloning principle

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Proof.
Suppose there exists such a U. Then we have

$$\begin{aligned} U|\phi_1\rangle \otimes |0\rangle &= |\phi_1\rangle \otimes |\phi_1\rangle \\ U|\phi_2\rangle \otimes |0\rangle &= |\phi_2\rangle \otimes |\phi_2\rangle \end{aligned} \tag{19}$$

It follows that

$$\begin{aligned} \langle\phi_1|\phi_2\rangle &= \langle\phi_1|\phi_2\rangle \langle 0|0\rangle = (\langle\phi_1| \otimes \langle 0|)(|\phi_2\rangle \otimes |0\rangle) \\ &= (\langle\phi_1| \otimes \langle 0|)U^\dagger U(|\phi_2\rangle \otimes |0\rangle) = (\langle\phi_1| \otimes \langle\phi_1|)(|\phi_2\rangle \otimes |\phi_2\rangle) = \langle\phi_1|\phi_2\rangle^2 \end{aligned} \tag{20}$$

This is only true if $\langle\phi_1|\phi_2\rangle$ is 0 or 1. So $|\phi_1\rangle$, $|\phi_2\rangle$ are not general states.

Table of Contents

Quantum evolution

Postulate 2 [Nielsen and Chuang(2000), page 81]

The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\Phi\rangle$ of the system at time t_1 is related to the state $|\Phi'\rangle$ of the system at time t_2 by a **unitary operator** U which depends only on the times t_1 and t_2,

$$|\Phi'\rangle = U|\Phi\rangle \tag{21}$$

Let us derive the postulate.

Schrödinger Equation

The evolution of an isolated pure quantum state $|\Phi\rangle$ is described by the Schrödinger equation ($\hbar = 1$)

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H|\Phi(t)\rangle\,, \tag{22}$$

where H is the Hamiltonian of the system.
The Hamiltonian H is a Hermitian matrix, i.e., $H = H^\dagger$.

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What does e to the power of a (Hermitian) matrix mean?

Exponentials of matrices

Exponential of a matrix defined through standard Taylor series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{24}$$

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Example: Easy, if A is a diagonal matrix

$$e^{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} \tag{25}$$

SINTEF

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What about general Hermitian matrices?

Exponentials of Hermitian matrices

Theorem

For a Hermitian matrix $H \in \mathbb{C}^{n\times n}$ there exist n orthonomal eigenvectors and all eigenvalues are real. The matrix H admits the eigendecomposition

$$H = V\Lambda V^{\dagger}, \tag{26}$$

where the columns of V consist of the n orthonormal eigenvectors of A and the diagonal entries of Λ are given by the corresponding eigenvalues.

SINTEF

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where the columns of V consist of the n orthonormal eigenvectors of A and the diagonal entries of Λ are given by the corresponding eigenvalues.

The exponential of a Hermitian matrix H can be calculated as

$$e^H = \sum_{k=0}^{\infty} \frac{(V\Lambda V^\dagger)^k}{k!} = \sum_{k=0}^{\infty} \frac{V(\Lambda)^k V^\dagger}{k!} = Ve^\Lambda V^\dagger \tag{27}$$

SINTEF

Quantum evolution is unitary.

Theorem

For a Hermitian matrix H and $t \in \mathbb{R}$, the matrix $U = e^{-iHt}$ is a unitary matrix, i.e., $UU^\dagger = U^\dagger U = I$.

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Proof: We have that

$$\begin{aligned} e^{-iHt}\left(e^{-iHt}\right)^\dagger &= Ve^{-i\Lambda t}V^\dagger \left(Ve^{-i\Lambda t}V^\dagger\right)^\dagger = Ve^{-i\Lambda t}V^\dagger Ve^{i\Lambda t}V^\dagger = I, \\ \left(e^{-iHt}\right)^\dagger e^{-iHt} &= \cdots = I, \end{aligned} \tag{28}$$

Can we find a Hamiltonian for a given unitary matrix?

Theorem

Given a unitary matrix U we can always find a Hermitian matrix H such that $U = e^{-iHt}$.

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- Diagonalize $U = VDV^\dagger$. For all j find λ_j such that $D_{jj} = e^{-i\lambda_j t}$.
- Not unique, since we can multiply with $e^{i2\pi k}$, $k \in \mathbb{Z}$.

SINTEF

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- Not unique, since we can multiply with $e^{i2\pi k}$, $k \in \mathbb{Z}$.
- Note that, if $\lambda_i, |\Phi_i\rangle$ is an eigenpair of H, then $e^{-i\lambda_i t}, |\Phi_i\rangle$ is an eigenpair of $U = e^{-iHt}$.

Physics jargon

- Physicist call eigenvalues of a Hamiltonian for **energies**.
  - These values are amounts of energy the system can have.
  - They are all real and can be order from smallest to largest, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
- To each energy λ_j corresponds to an **energy eigenstate**.
  - The energy eigenstate $|v_1\rangle$ corresponding to the lowest energy is called **ground state**.
  - The energy eigenstate $|v_2\rangle, |v_3\rangle, \ldots$ are called **first excited state**, **second excited state**, ...

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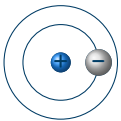
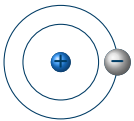
Example:

Electron sitting in the lowest shell is in the ground state



First excited state has the electron in the next shell up



The adiabatic theorem

”A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian’s spectrum.”

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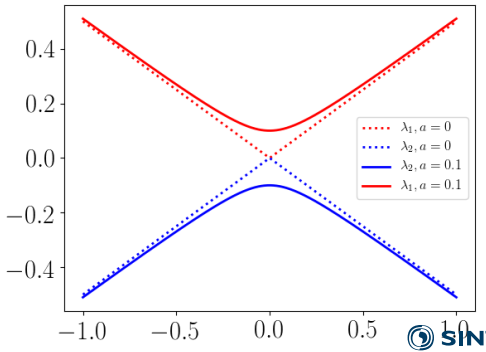
Consider a time dependent Hamiltonian

$$H(t) = \begin{pmatrix} \alpha t & a \\ a & -\alpha t \end{pmatrix} \tag{29}$$

$$\lambda_{1,2} = \pm\sqrt{a^2 + (\alpha t)^2} \tag{30}$$

The probability of a diabatic transition is given by (Landau-Zener)

$$P_D = e^{2\pi a^2/|\alpha|} \tag{31}$$



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Qauntum annealing

A predecessor of QAOA, quantum annealing (QA) has been widely studied for the purpose of solving combinatorial optimization problems. To find the MaxCut configuration that maximizes $\langle H_C \rangle$, we consider the following simple QA protocol:

$$H_{QA}(s) = -(sH_C + (1-s)H_B), \quad s = t/T \tag{32}$$

- Ground state for $s = 0$ is $|+\rangle^{\otimes n}$.
- Ground state for $s = 1$ corresponds to solution encoded in H_C.

Methods to solve combinatorial optimization problems

- In **adiabatic QA**, the algorithm relies on the adiabatic theorem to remain in the instantaneous ground state along the annealing path, and solves the computational problem by finding the ground state at the end. To guarantee success, the necessary run time of the algorithm typically scales as $T = \mathcal{O}(1/\Delta_{\text{min}}^2)$, where $\Delta_{\text{min}} = \min_{s\in[0,1]}(\lambda_2(t) - \lambda_1(t))$ is the minimum spectral gap. It turns out that for hard instances, Δ_{min} is exponentially small with respect to the problem size.

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- Classical **simulated annealing** mimics adiabatic QA. But also takes exponential amount of time in the worst case.
- The **adiabatic algorithm (QAOA)** can (at best) achieve Grover speedup.

Solving combinatorial optimization problems

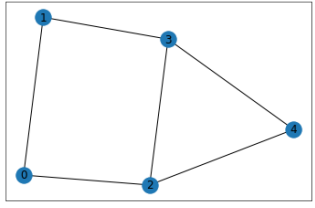
Example: weighted MAXCUT

- Given a graph $G = (V, E)$ consisting of vertices V and edges E width weights $w_{i,j} > 0$, for $(i,j) \in E$.
- A cut is defined as a partition of the vertices V into two disjoint subsets $S, \overline{S}$.
- The cost function to be maximized is the sum of weights of edges with vertices in the two different subsets.

Assign $x_i = \begin{cases} -1, & \text{if edge } i \text{ is in set } S \\ +1, & \text{otherwise} \end{cases}$, then the cost function is given by

$$C(x) = \sum_{(i,j)\in E} w_{i,j} \frac{1}{2}(1 - x_i x_j) \tag{33}$$

$$V = \{0, 1, 2, 3, 4\}$$
$$E = \{(0, 1, 1.0), (0, 2, 2.0), (2, 3, 1.0), (3, 1, 2.0), (3, 4, 1.0), (4, 2, 1.0)\}$$



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Types of approaches

Solving NP hard optimization problems.

- **Heuristic algorithms**. No polynomial run time guarantee; appear to perform well on some instances.
- **Approximate algorithms**. Efficient and provide provable guarantees.

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- **Heuristic algorithms**. No polynomial run time guarantee; appear to perform well on some instances.
- **Approximate algorithms**. Efficient and provide provable guarantees.
  With high probability we get a solution x^* such that

$$\frac{C(x^*)}{\max_x C(x)} \geq \alpha, \tag{34}$$

  where $0 < \alpha \leq 1$ is the approximation ratio.

Classical solution

- Calculating the cost of all partitions takes exponential time.

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- Polynomial time algorithm is randomized partitioning: for each edge $(i,j) \in E$ choose randomly S or $\overline{S}$ with 50%. Therefore, the expectation value of a cut produced by random assignment can be written as follows:

$$\sum_{(i,j)\in E} w_{i,j} * Pr((i,j) \in \text{cut}) = \frac{1}{2} \sum_{e\in E} w_e \tag{35}$$

This produces a cut with expectation value of at least 0.5 times the maximum cut, since $\sum_{e\in E} w_e$ is an upper bound.

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  This produces a cut with expectation value of at least 0.5 times the maximum cut, since $\sum_{e\in E} w_e$ is an upper bound.
- Other polynomial approaches exist that involve semi-definite programming which give cuts of expected value at least 0.87856 times the maximum cut.

Express problem as ground state of Hamiltonian

- For each vertex we define $|x_i\rangle = \begin{cases} |0\rangle, & \text{if vertex } i \in S \\ |1\rangle, & \text{if vertex } i \in \overline{S} \end{cases}$
- Observe that

$$\begin{aligned} \sigma_z |0\rangle &= |0\rangle \\ \sigma_z |1\rangle &= -|1\rangle \end{aligned} \tag{36}$$

- The Hamiltonian encoding our problem is therefore

$$H_C = \sum_{(i,j)\in E} w_{i,j} \frac{1}{2}(I^n - I^a \otimes \sigma_z^i \otimes I^b \otimes \sigma_z^j \otimes I^c), \tag{37}$$

where I^m denotes the identity matrix in $(C^2)^{\otimes m}$

Barbell example

Barbell example

Observe that, e.g.,

- $H_c \left|00\right\rangle = 1/2(I \otimes I - \sigma_z \otimes \sigma_z) \left|00\right\rangle = 1/2(\left|00\right\rangle - \sigma_z \left|0\right\rangle \otimes \sigma_z \left|0\right\rangle) = 1/2(\left|00\right\rangle - \left|0\right\rangle \otimes \left|0\right\rangle) = 0 \left|00\right\rangle$
- $H_c \left|01\right\rangle = 1/2(I \otimes I - \sigma_z \otimes \sigma_z) \left|01\right\rangle = 1/2(\left|01\right\rangle - \sigma_z \left|0\right\rangle \otimes \sigma_z \left|1\right\rangle) = 1/2(\left|01\right\rangle - \left|0\right\rangle \otimes (-\left|1\right\rangle)) = 1 \left|01\right\rangle$

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This means that

- $\left|00\right\rangle$, and $\left|11\right\rangle$ are eigenkets of $-H_C$ with eigenvalue 0.
- $\left|01\right\rangle$, and $\left|10\right\rangle$ are eigenkets of $-H_C$ with eigenvalue -1.

QAOA

(weighted) Max-Cut

$$H_C = \sum_{(j,k)\in E} \frac{1}{2} w_{i,j} \left(I - \sigma_z^i \sigma_z^j \right) \tag{38}$$

- H_C is sum of $|E|$ local terms
- H_C is a diagonal matrix

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- H_C is sum of $|E|$ local terms
- H_C is a diagonal matrix

$$H_B = \sum_{i \in nodes} \sigma_x^i \tag{39}$$

- H_B has only off-diagonal non-zero entries
- H_B induces a swap operation between neighboring qubits, and thus can move the excitation around for the purpose of state transfer

How to find quantum gates for QA?

We need to find gates for

$$e^{-iH_{QA}(s)}, \tag{40}$$

where

$$H_{QA}(s) = -(sH_C + (1-s)H_B), \quad s = t/T \tag{41}$$

Adding of Hamiltonians

If H_1, H_2 are matrices (Hamiltonians), then

$$e^{H_1+H_2} \neq e^{H_1}e^{H_2}, \tag{42}$$

except when H_1 and H_2 commute, i.e., $H_1H_2 = H_2H_1$.

SINTEF

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except when H_1 and H_2 commute, i.e., $H_1H_2 = H_2H_1$.
Trotterization, (Lie-Trotter-Suzuki product formula[Trotter(1959), Suzuki(1976)])

$$e^{-i(H_1+H_2)t} = \left(e^{-iH_1\frac{t}{n}}e^{-iH_2\frac{t}{n}}\right)^n + \mathcal{O}\left(\frac{t^2}{n}\right) \tag{43}$$

First and second order versions

$$\begin{aligned} e^{-i(H_1+H_2)t} &= e^{-iH_1t}e^{-iH_2t} + \mathcal{O}\left(t^2\right) \\ e^{-i(H_1+H_2)t} &= e^{-iH_1t/2}e^{-iH_2t}e^{-iH_1t/2} + \mathcal{O}\left(t^3\right) \end{aligned} \tag{44}$$

Overall QAOA

1. Using $2p$ parameters $\gamma = \gamma_1, \ldots, \gamma_p$, $\beta = \beta_1, \ldots, \beta_p$, prepare state

$$|\Psi(\gamma, \beta)\rangle = U_{B_p} U_{C_p} \ldots U_{B_1} U_{C_1} |+\rangle^{\otimes n}, \tag{45}$$

   where the operators have the explicit form

$$\begin{aligned} U_{B_l} &= e^{-i\beta_l H_B} = \prod_{j=1}^{n} e^{-i\beta_l \sigma_x^j}, \\ U_{C_l} &= e^{-i\beta_l H_C} = \prod_{(j,k)\in E} e^{-i\gamma_l w_{j,k}/2(I-\sigma_z^j \sigma_z^k)}, \end{aligned} \tag{46}$$

2. Obtain $\langle \Psi(\gamma, \beta) | H_C | \Psi(\gamma, \beta) \rangle$.
3. Run an outer, classical, optimization loop to find γ, β that minimizes the expectation value $\langle \Psi(\gamma, \beta) | H_C | \Psi(\gamma, \beta) \rangle$.

How to obtain the expectation value

H_C is a diagonal Hamiltonian, and we have that

$$H_C = \sum_{x \in \{0,1\}^n} C(x)|x\rangle\langle x| \tag{47}$$

Therefore,

$$\begin{aligned}\langle \Psi_p(\vec{\gamma},\vec{\beta})|H|\Psi_p(\vec{\alpha},\vec{\beta})\rangle &= \langle \Psi_p(\vec{\gamma},\vec{\beta})| \sum_{x \in \{0,1\}^n} C(x)|x\rangle\langle x||\Psi_p(\vec{\alpha},\vec{\beta})\rangle \\ &= \sum_{x \in \{0,1\}^n} C(x)\langle \Psi_p(\vec{\gamma},\vec{\beta})|x\rangle\langle x|\Psi_p(\vec{\alpha},\vec{\beta})\rangle = \sum_{x \in \{0,1\}^n} C(x)p(x)\end{aligned} \tag{48}$$

SINTEF

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Remember that, given a random outcome x', we only need to calculate the cost function once.

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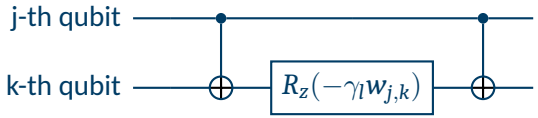
How to implement with gates efficiently?

$e^{-i\gamma_l w_{j,k}/2(I-\sigma_z^j\sigma_z^k)}$ can be implemented as



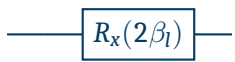
- Observe that $e^{-i\gamma_l w_{j,k}/2I}$ is a global phase and can be ignored
-

$$(CX)(I\otimes Rz(\theta))(CX) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}e^{-i\theta/2}&0\\0&e^{i\theta/2}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$= \begin{pmatrix}e^{-i\theta/2}&0&0&0\\0&e^{i\theta/2}&0&0\\0&0&e^{i\theta/2}&0\\0&0&0&e^{-i\theta/2}\end{pmatrix} = e^{-i\theta/2\sigma_z\sigma_z}$$

How to implement with gates efficiently?

$e^{-i\beta_l \sigma_x^j}$ can be implemented as



$$Rx(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{50}$$

Table of Contents

Errors and what to do about it

- Inherent noise in quantum devices leads to wrong estimates of the expectation values of observables (as we have seen during the coding sessions).
- Getting rid of (most of) the noise inherent in quantum computing is a critical step toward making it useful for practical applications.
- *Quantum error correction (QEC)* can only be achieved by increasing quantum resources (ancillary qubits). The first scheme was proposed by [Shor(1995)] and many other schemes were proposed since then, e.g., the class called stabilizer codes, see [Gottesman(1997)].
- However, the number of ancillary qubits needed to achieve QEC depends intrinsically on the error rates and is out of reach for NISQ devices.
- *Quantum error mitigation (QEM)*, on the other hand can be achieved with additional classical resources only and is therefore applicable to NISQ devices.

Density matrices

In finite dimensional space, the density operator is of the form

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \tag{51}$$

where the coefficients p_j are non-negative and add up to one. The expectation value of an operator A can be calculated through

$$\begin{aligned}\langle A\rangle &= \sum_j p_j \langle\psi_j|A|\psi_j\rangle = \sum_j p_j \operatorname{tr}\left(|\psi_j\rangle\langle\psi_j|A\right) = \sum_j \operatorname{tr}\left(p_j|\psi_j\rangle\langle\psi_j|A\right) \\ &= \operatorname{tr}\left(\sum_j p_j|\psi_j\rangle\langle\psi_j|A\right) = \operatorname{tr}(\rho A),\end{aligned} \tag{52}$$

Error mitigation

The ideal action of a gate is given by a unitary operator U transforming a state $|\phi\rangle$ into $U|\phi\rangle$.

- *Coherent noise* means that a small perturbation $\widetilde{U}$ of U is executed, where $\widetilde{U}$ is still unitary and preserves the purity of the input state $|\phi\rangle$.

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  An example is a slight over-rotation.
- *Incoherent noise* does not preserve the purity of the state. This type of noise comes from the (unwanted) interaction with the environment. In this case the evolution must be described through density matrices and Kraus operators.

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  An example of incoherent noise is amplitude damping modeling relaxation from an excited state to the ground state. For a single qubit with decay probability p, the density matrix $\rho = |\phi\rangle\langle\phi|$ is mapped to $K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger$ with
  $K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$

Error mitigation

Different types of techniques have been presented in the literature that can be used to mitigate the influence of noise on the ideal circuit.

- *Probabilistic error cancellation.* The main idea is to represent the ideal circuit as a quasi-probabilistic mixture of noisy ones. The circuit depth and width remain unchanged with this method.
- *Extrapolation techniques.* The main idea is to amplify the noise deliberately in a controlled way. The information of the dependence of the expectation value on the noise level is used to extrapolate back to the zero noise level. The circuit width remains unchanged, but the circuit depth is longer (or gate times are prolonged in case of phase control).

Probabilistic error cancellation

- [Temme et al.(2017)Temme, Bravyi, and Gambetta] present the method together with numerical evidence.
- [Song et al.(2019)Song, Cui, Wang, Hao, Feng, and Li] demonstrate an error mitigation protocol based on gate set tomography and quasi probability decomposition. One- and two-qubit circuits are tested on a superconducting device, and computation errors are successfully suppressed.
- Process tomography is not feasible for more than a few qubits since it scales exponentially with the number of qubits.
- In addition, process tomography is sensitive to state preparation and measurement (SPAM) errors. Gate set tomography can take these errors into account, but the scaling becomes even worse.

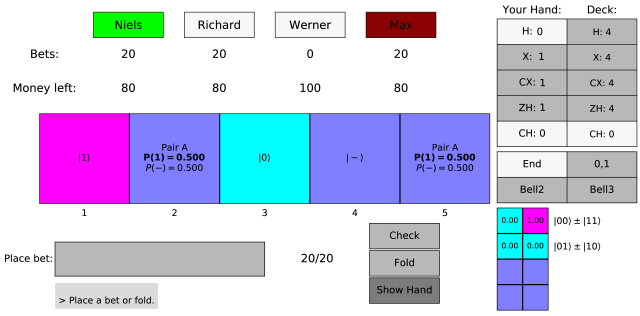
Extrapolation techniques

- [Temme et al.(2017)Temme, Bravyi, and Gambetta] and [Li and Benjamin(2017)] introduced the technique and provide numerical evidence.
- [Endo et al.(2018)Endo, Benjamin, and Li] extend the work in order to design efficient QEM circuits.
- [Kandala et al.(2019)Kandala, Temme, Córcoles, Mezzacapo, Chow, and Gambetta] demonstrate tremendous improvements in the accuracy of VQE on real quantum hardware. They use pulse control.

Quantum Poker

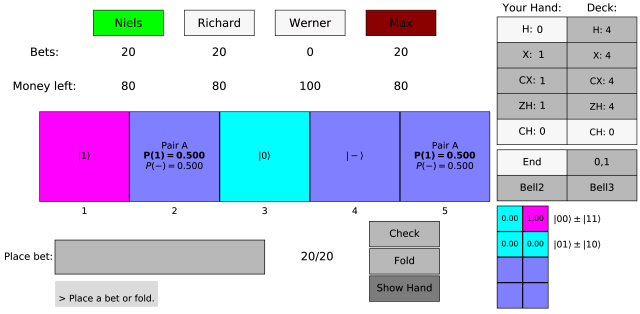
- Shortage of talent predicted. $\Rightarrow$ Design fun game to increase interest.
- Available at https://github.com/sintefmath/QuantumPoker and [Fuchs et al.(2019)Fuchs, Falch, and Johnsen]

Quantum Poker



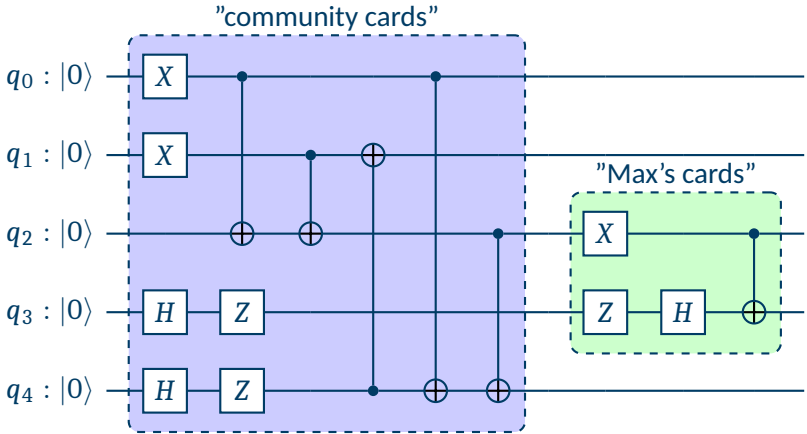
- Resembles Texas Hold 'em Poker using 5 qubits as community cards and quantum gates as player cards
- At the end, your qubits are measured, and your score is the number of 1's measured

Quantum Poker



- Resembles Texas Hold 'em Poker using 5 qubits as community cards and quantum gates as player cards
- At the end, your qubits are measured, and your score is the number of 1's measured
  $\Longrightarrow$ apply gates s.t. your qubits are likely to collapse to $|1\rangle$.

An example of a ”hand”



SINTEF

Ideal Simulator

- The state that Max creates is given by $|\phi_{\text{Max}}\rangle = \frac{1}{\sqrt{2}}(|01101\rangle + |11111\rangle)$.
- A state $|\phi\rangle = \sum_i \alpha_i |i\rangle$ induces a probability distribution $P_{|\phi\rangle}(i) = |\alpha_i|^2$.
- For Max's circuit this distribution is thus given by a 50% chance of being in either state $|01101\rangle$ and $|11111\rangle$.
- The expectation value for Max's circuit is thus $\langle A \rangle_{|\phi_{\text{Max}}\rangle} = 4$.

Observable

To match the objective of our game, we need to define an observable A such that $\langle A \rangle_\phi$ is equal to the expected number of ones in the computational basis. This can be done by choosing

$$A = \sum_{i=1}^{2^5} b(i) P_i, \tag{53}$$

where $b(i)$ is a function returning the number of ones of the binary representation of i, and $P_i = |i\rangle \langle i|$ is the measurement operator in the computational basis. A is a diagonal matrix with eigensystem $\{b(i), |i\rangle\}$.

Observable

The matrix A can also be constructed via the number operator in the second quantization (a formalism used to describe and analyze quantum many-body systems), which is given by

$$A = \sum_i N_i, \quad \text{where } N_i = a_i^\dagger a_i. \tag{54}$$

The creation and annihilation operators are given by

$$\begin{aligned} a_i^\dagger &= I^{\otimes n-i-1} \otimes Q^+ \otimes \sigma_z^{\otimes i}, \\ a_i &= I^{\otimes n-i-1} \otimes Q^- \otimes \sigma_z^{\otimes i}, \end{aligned} \tag{55}$$

and the raising and lowering operator is given by

$$Q^\pm = \frac{1}{2}\left(\sigma_x \mp i\sigma_y\right), \tag{56}$$

i.e, $Q^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Q^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. As an example, for two qubits A is a diagonal matrix with entries $(0, 1, 1, 2)$, from upper left to lower right.

How to get the expectation value

Since A is diagonal, it is straight forward to calculate the expectation value as

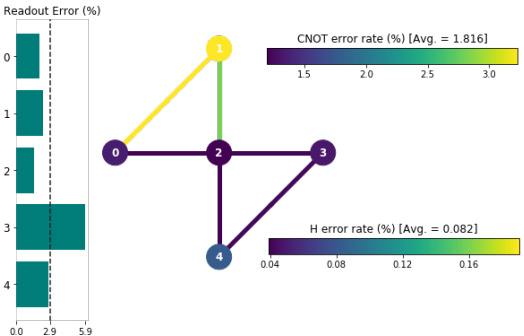
$$\langle\Phi|A|\Phi\rangle = \langle\Phi| \sum_{i=1}^{2^5} b(i)\,|i\rangle\,\langle i|\Phi\rangle = \sum_{i=1}^{2^5} b(i)\,\langle\Phi|i\rangle\,\langle i|\Phi\rangle = \sum_{i=1}^{2^5} b(i)p(i). \tag{57}$$

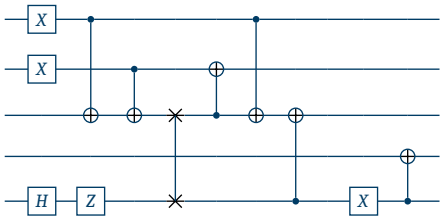
This means we can measure the state $|\Phi\rangle$ in the computational basis and multiply the resulting bit strings with $b(i)$ to get the expectation value.

Circuit Mapping

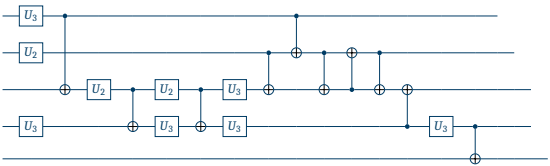
- only a subset of qubits are physically connected
- on IBM's QX devices CNOT gates can only be applied to qubits that are connected by a bus resonator
- additional gates, such as SWAP or BRIDGE gates, need be used to transform the circuit into an equivalent one that obeys the connectivity graph.
- Inserting one SWAP or BRIDGE gate increases the number of CNOT gates by three.
- the noise level of two-qubit gate (CNOT) times and error rates are one order of magnitude higher than for single qubit gates
- One therefore wishes to find a mapping with the lowest number of CNOT gates.
- In general, the problem of finding an optimal mapping is $\mathcal{NP}$-complete problem[Wille et al.(2019)Wille, Burgholzer, and Zulehner].
- For Max's circuit it is easy to find an optimal mapping manually, using only one extra SWAP gate.

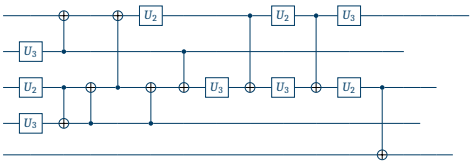
Equivalent circuit matching IBM's QX2





Transpiled circuits





SINTEF

The effect of noise on quantum computation

- Noise is inherent to quantum computers.
- Qiskit provides methods for automatic generation of approximate noise models matching a given hardware device.
- This enables us to simulate the effects of realistic noise on our computation before we run our circuits on a real quantum computer.
- Due to the influence of noise, the resulting expectation values converge to a value around 3.85 for the simulated noise model and 3.54 on the IBM QX2 device.

Expectation values

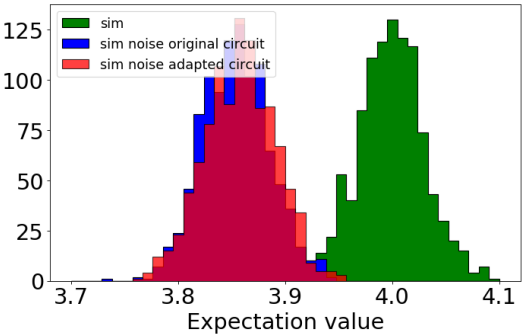


Figure: Distribution of sequence averages for 1024 repetitions with 1024 shots each.

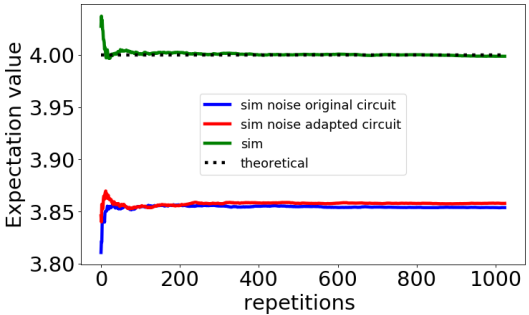


Figure: Convergence of sequence averages to the expectation value with respect to number of repetitions. Each repetition uses 1024 shots.

SINTEF

Error mitigation I

Basic assumption: the expectation value of an observable depends smoothly on a small noise parameter $\lambda \ll 1$ and admits the following power series,

$$\langle A \rangle_{|\phi\rangle}(\lambda) = \langle A \rangle^*_{|\phi\rangle} + \sum_{i=1}^{n} a_i \lambda^i + \mathcal{O}(\lambda^{i+1}), \tag{58}$$

where $\langle A \rangle^*_{\phi}$ is the zero noise value we are trying to recover.

Error mitigation II

- A better estimate of $\langle A \rangle_{\phi}^{*}$ is then constructed by combining these values in such a way that the lowest order terms in the power series cancel.
- Clearly, using $r_1 = 1$ generates the expectation value with the least noise.
- Amplification of noise with the factors $r_i > 1$ can either be achieved directly through pulse control or through modifying the circuit by adding certain extra gates.
- For IBM's QX devices pulse control is only accessible for their customers, which leaves us with the second possibility.

Pauli twirling

- Convert non-stochastic errors of CNOT gates into stochastic errors, see e.g. [Li and Benjamin(2017), section VII].
- One way to achieve this is to apply Pauli-twirling.
- In our case gates $\sigma^a, \sigma^b, \sigma^c, \sigma^d$ are inserted before and after each CNOT gate Λ, where σ^i is chosen from the twirling set consisting of the Pauli gates $\{\mathbb{1}, \sigma^x, \sigma^y, \sigma^z\}$.
- After randomly choosing σ^a, σ^b the gates σ^c, σ^d are then chosen to satisfy

$$\sigma^c \otimes \sigma^d = \mathrm{e}^{i\theta} \Lambda (\sigma^a \otimes \sigma^b) \Lambda^\dagger. \tag{59}$$

- The method is applicable, if the qualities of single-qubit gates are an order of magnitude smaller than two-qubit gates.

Pauli twirling

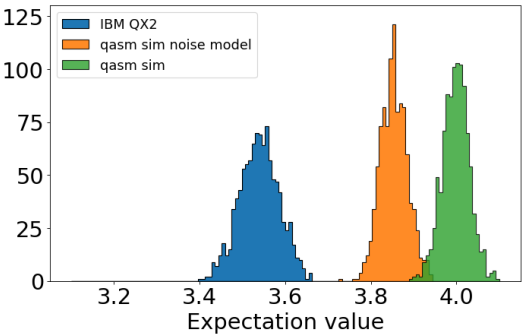


Figure: Transpiled circuit without Pauli twirling.

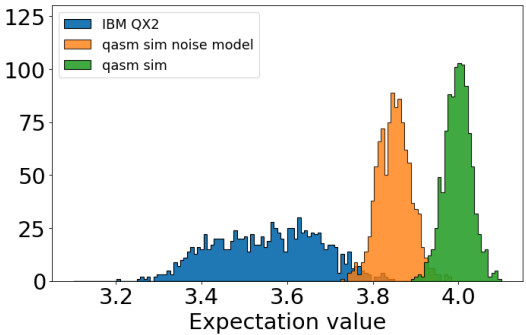


Figure: Transpiled circuit with Pauli twirling.

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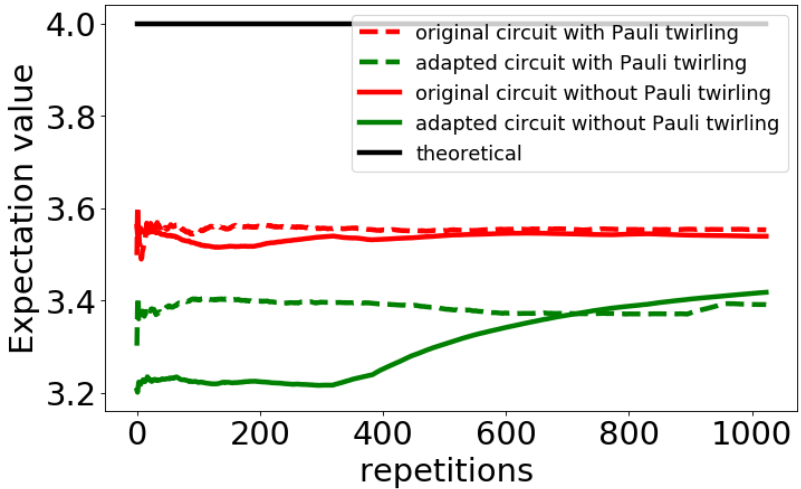
Expectation values



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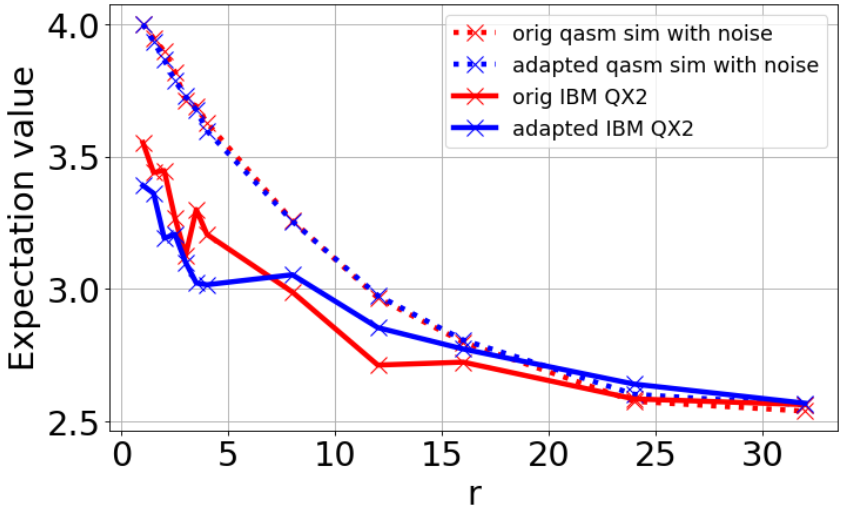
Noise amplification

- In order to amplify the strength of the noise, we will apply random Pauli gates with a probability proportional to the error rate of the CNOT gate between a given pair of qubits.
- More precisely this is means applying gates σ^e, σ^f randomly chosen form the set of Pauli gates $\{\mathbb{1}, \sigma^x, \sigma^y, \sigma^z\}$ after the twirled CNOT gates with probability $(r-1)\epsilon_{i,j}$.
- Here, $\epsilon_{i,j}$ is the two-qubit gate error rate between qubits q_i and q_j.
- On average this increases the error rate to the desired value $\epsilon_{\text{new}} = \epsilon_{i,j} + (r-1)\epsilon_{i,j} = r\epsilon_{i,j}$.

Dependence on the noise amplification factor.



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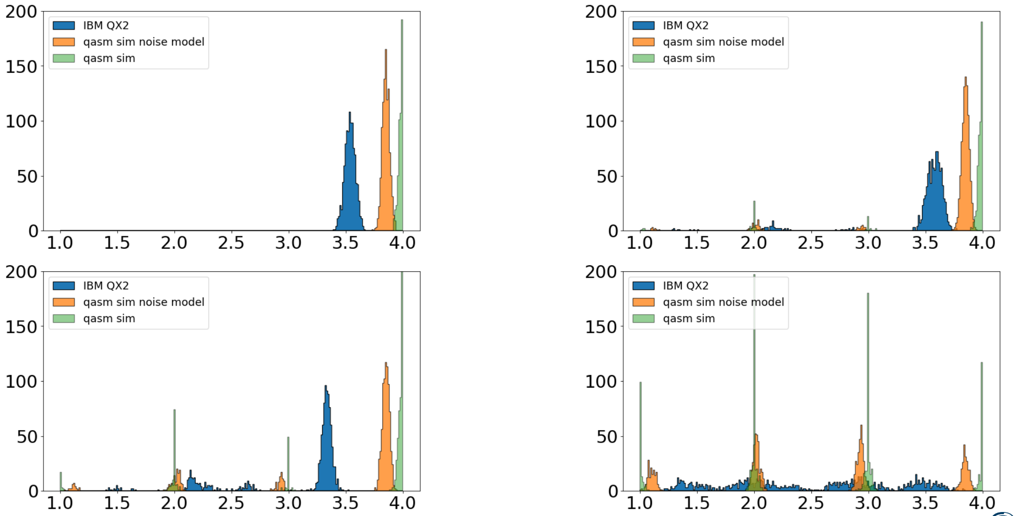
Effect of noise amplification factor



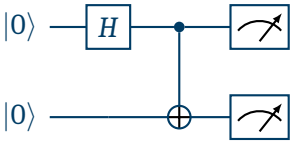
Error mitigation of measurement noise I

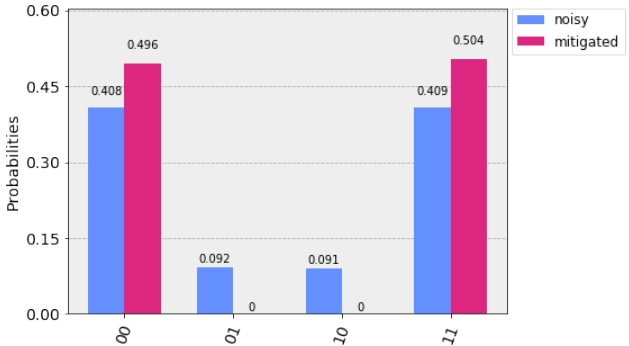
- Measurement or read-out error is another major source of error.
- Here we use the model that assumes spatially uncorrelated errors of a bit flip.
- We compute the probability that the state $|j\rangle$ is observed if the state $|i\rangle$ is prepared, i.e. the conditional probability $P(|i\rangle \mid |j\rangle)$.
- In the absence of errors $P(|i\rangle \mid |j\rangle) = \delta_{i,j}$, but we can see that there are off-diagonal nonzero entries.
- In order for the method to work, measurement errors must be at least one order of magnitude larger than state preparation and the execution of the X gate.
- requires an exponential amount (in the number of qubits) of states to be prepared and measured.

Error mitigation of measurement noise II

- Given $P(|i\rangle \mid |j\rangle)$, one can construct a filter to counteract the effect of measurement noise.
- Qiskit provides an implementation.

Example of error mitigation of measurement noise





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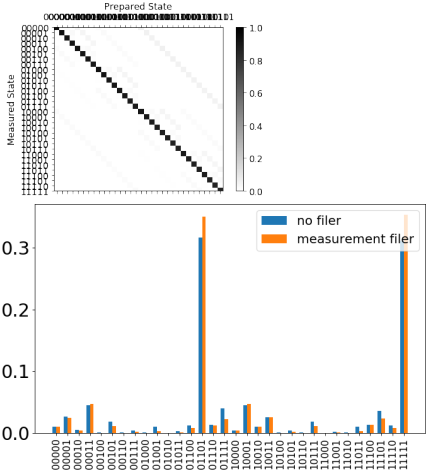
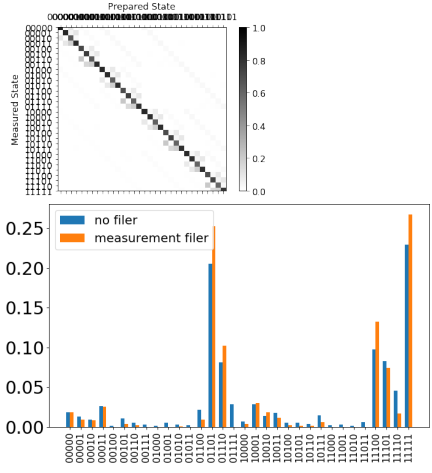
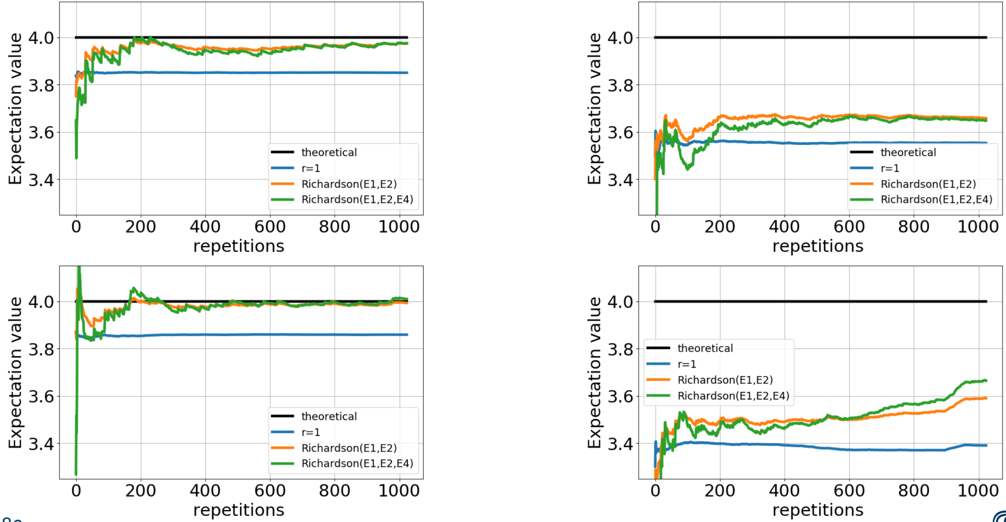
Measurement mitigation for Max’s circuit





Richardson extrapolation for Max's circuit



Relative error [%]

	no Pauli twirling				with Pauli twirling			
	E1	R(E2,E4)	R(E1,E2)	R(E1,E2,E4)	E1	R(E2,E4)	R(E1,E2)	R(E1,E2,E4)
orig qasm	3.8	1.5	0.2	0.2	3.8	1.6	0.8	0.5
adap qasm	3.6	0.6	0.8	0.9	3.5	1.2	0.0	0.4
orig QX2	11.5	3.1	10.0	12.4	11.2	7.8	8.5	8.8
adap QX2	14.6	12.6	10.6	9.9	15.2	15.8	10.2	8.4
orig qasm filter	2.0	0.4	1.7	2.2	1.9	0.4	1.3	1.6
adap qasm filter	1.6	1.5	1.3	1.3	1.5	0.9	2.2	2.6
orig QX2 filter	10.0	1.2	8.5	10.9	9.6	6.1	6.9	7.2
adap QX2 filter	13.1	11.1	8.9	8.2	13.8	14.4	8.6	6.6

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Charles H Bennett.
Logical reversibility of computation.
IBM journal of Research and Development, 17(6):525–532, 1973.

David Deutsch.
Quantum theory, the church–turing principle and the universal quantum computer.
Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 400(1818):97–117, 1985.

Suguru Endo, Simon C. Benjamin, and Ying Li.
Practical quantum error mitigation for near-future applications.
Physical Review X, 8(3), July 2018.
doi: 10.1103/physrevx.8.031027.
URL https://doi.org/10.1103/physrevx.8.031027.

Franz G. Fuchs, Vemund Falch, and Christian Johnsen.
Quantum poker – a pedagogical tool to introduce the basic concepts of quantum computing and error mitigation techniques for nisq devices, 2019.
URL https://arxiv.org/abs/1908.00044.

Daniel Gottesman.
Stabilizer codes and quantum error correction.
arXiv preprint quant-ph/9705052, 1997.

Abhinav Kandala, Kristan Temme, Antonio D. Córcoles, Antonio Mezzacapo, Jerry M. Chow, and Jay M. Gambetta.
Error mitigation extends the computational reach of a noisy quantum processor.
Nature, 567(7749):491–495, March 2019.
doi: 10.1038/s41586-019-1040-7.
URL https://doi.org/10.1038/s41586-019-1040-7.

Ying Li and Simon C. Benjamin.
Efficient variational quantum simulator incorporating active error minimization.
Physical Review X, 7(2), June 2017.
doi: 10.1103/physrevx.7.021050.
URL https://doi.org/10.1103/physrevx.7.021050.

M.A. Nielsen and I.L. Chuang.
Quantum Computation and Quantum Information.
Cambridge Series on Information and the Natural Sciences. Cambridge University Press, 2000.

L. F. Richardson and J. A. Gaunt.
The deferred approach to the limit. part i. single lattice. part II. interpenetrating lattices.
Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 226(636-646):299–361, January 1927.

doi: 10.1098/rsta.1927.0008.
URL https://doi.org/10.1098/rsta.1927.0008.

Peter W. Shor.
Scheme for reducing decoherence in quantum computer memory.
Physical Review A, 52(4):R2493–R2496, October 1995.
doi: 10.1103/physreva.52.r2493.
URL https://doi.org/10.1103/physreva.52.r2493.

Chao Song, Jing Cui, H. Wang, J. Hao, H. Feng, and Ying Li.
Quantum computation with universal error mitigation on a superconducting quantum processor.
Science Advances, 5(9):eaaw5686, September 2019.
doi: 10.1126/sciadv.aaw5686.
URL https://doi.org/10.1126/sciadv.aaw5686.

Masuo Suzuki.
Generalized trotter's formula and systematic approximants of exponential operators and inner derivations with applications to many-body problems.
Communications in Mathematical Physics, 51(2):183–190, 1976.

Kristan Temme, Sergey Bravyi, and Jay M. Gambetta.
Error mitigation for short-depth quantum circuits.
Physical Review Letters, 119(18), November 2017.
doi: 10.1103/physrevlett.119.180509.
URL https://doi.org/10.1103/physrevlett.119.180509.

Hale F Trotter.
On the product of semi-groups of operators.
Proceedings of the American Mathematical Society, 10(4):545–551, 1959.


Robert Wille, Lukas Burgholzer, and Alwin Zulehner.
Mapping quantum circuits to IBM QX architectures using the minimal number of SWAP and h operations.
In *Proceedings of the 56th Annual Design Automation Conference 2019 on - DAC 19*. ACM Press, 2019.
doi: 10.1145/3316781.3317859.
URL `https://doi.org/10.1145/3316781.3317859`.