1 Practical Aspects of the Dual Weighted Residual (DWR) Method

Abstract. We discuss some practical aspects of the approximate evaluation of weighted a posteriori error estimators and the resulting mesh adaptation strategies. This includes mesh optimization, anisotropic mesh adaptation and h/p adaptation.

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Goal of simulation: Computation of certain quantities $J(u)$ from the solution of a continuous model

$$\mathcal{A}(u) = 0$$

with accuracy TOL, by using a discrete model

$$\mathcal{A}_h(u_h) = 0$$

of dimension N.

Goal of adaptivity: "Optimal" use of computing resources

- Minimal work for prescribed accuracy

$$N \to \text{min}, \quad TOL \ \text{given}$$

- Maximal accuracy for prescribed work

$$TOL \to \text{min}, \quad N \ \text{given}$$

Approaches:

- Error control by a posteriori *error estimates*
- Mesh adaptation by local *error indicators*

1.1 A model case

Dirichlet problem of the Laplacian on $\Omega \subset \mathbb{R}^2$:

$$-\Delta u = f \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0$$

Approximation by finite element Galerkin method

$$\begin{aligned} u \in V := H_0^1(\Omega): \qquad & a(u,\varphi) := (\nabla u, \nabla\varphi) = (f,\varphi) \quad \forall \varphi \in V \\ u_h \in V_h \subset V: \qquad & a(u_h,\varphi_h) = (f,\varphi_h) \quad \forall \varphi_h \in V_h \end{aligned}$$

'Galerkin orthogonality' for error $e := u - u_h$:

$$a(e,\varphi_h) = 0, \quad \varphi_h \in V_h$$

Error control with respect to some (linear) 'output functional' $J(\cdot)$:

$z \in V$ solution of the 'dual' ('adjoint') problem:

$$a(\varphi, z) = J(\varphi) \quad \forall \varphi \in V$$

$z_h \in V_h$ finite element approximation:

$$a(\varphi_h, z_h) = J(\varphi_h) \quad \forall \varphi_h \in V_h$$

Error representation (via 'Galerkin orthogonality')

$$\begin{aligned} J(e) &= a(e,z) = a(e, z - \psi_h), \quad \psi_h \in V_h \\ &= (f, z-\psi_h) - a(u_h, z-\psi_h) =: \rho(u_h)(z-\psi_h) \end{aligned}$$

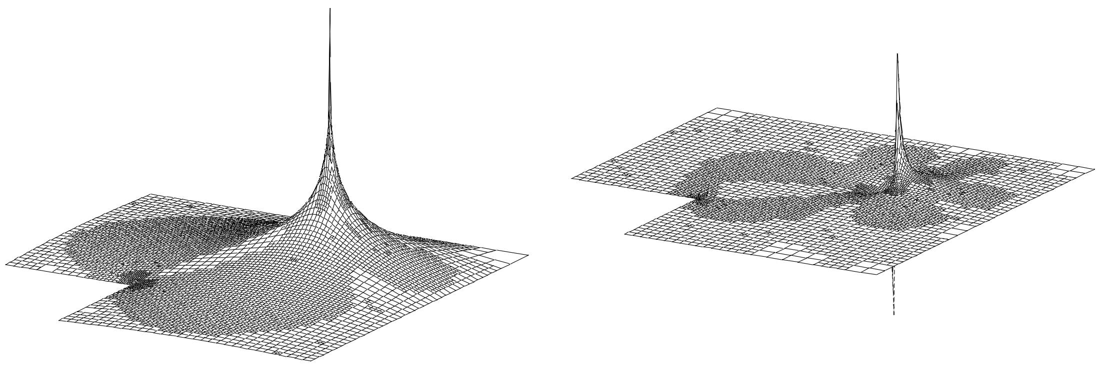


Figure. Examples of computed dual solutions for evaluating $u(a)$ and $\partial_1 u(a)$ (scaled differently)

Cell-wise integration by parts (Ω polygonal domain):

$$\begin{aligned}\rho(u_h)(z-\varphi_h) &= \sum_{K\in\mathbb{T}_h} \big\{(f+\Delta u_h, z-\psi_h)_K - (\partial_n u_h, z-\psi_h)_{\partial K}\big\} \\ &= \sum_{K\in\mathbb{T}_h} \big\{(\underbrace{f+\Delta u_h}_{=:\,R(u_h)}, z-\psi_h)_K + (\underbrace{-\tfrac{1}{2}[\partial_n u_h]}_{=:\,r(u_h)}, z-\psi_h)_{\partial K\setminus\partial\Omega}\big\}\end{aligned}$$

$[\nabla u_h]$ the jump of ∇u_h across the inter-element edges Γ,
cell and edge residuals $R(u_h)$ and $r(u_h)$:

$$\begin{aligned} R(u_h)_{|K} &:= f+\Delta u_h \\ r(u_h)_{|\Gamma} &:= \left\{ \begin{array}{rl} -\frac{1}{2}n\cdot[\nabla u_h], & \text{if } \Gamma\subset\partial K\setminus\partial\Omega \\ 0, & \text{if } \Gamma\subset\partial\Omega \end{array} \right\} \end{aligned}$$

L^2-norm error bound:

$$J(\varphi) = \|e\|^{-1}(e,\varphi), \qquad J(e) = \|e\|$$

The dual solution $z \in V \cap H^2(\Omega)$ satisfies

$$c_S := \|\nabla^2 z\| \le 1.$$

Using Hölder inequality in the error reporesentation yields

$$\|e\| \le \sum_{K\in\mathbb{T}_h} \rho_K\,\omega_K \le \Big(\sum_{K\in\mathbb{T}_h} h_K^4\rho_K^2\Big)^{1/2}\Big(\sum_{K\in\mathbb{T}_h} h_K^{-4}\omega_K^2\Big)^{1/2}$$

with residuals

$$\rho_K := \|R(u_h)\|_K + h_K^{-1/2}\|r(u_h)\|_{\partial K}$$

Strong interpolation estimate (à la Bramble/Hilbert):

$$\Big(\sum_{K\in\mathbb{T}_h} h_K^{-4}\big\{\|z-I_hz\|_K^2 + h_K\|z-I_hz\|_{\partial K}^2\big\}\Big)^{1/2} \le c_I\,\|\nabla^2 z\|$$

$$\|e\| \le c_I\Big(\sum_{K\in\mathbb{T}_h} h_K^4\rho_K^2\Big)^{1/2}\|\nabla^2 z\| \le c_Ic_S\Big(\sum_{K\in\mathbb{T}_h} h_K^4\rho_K^2\Big)^{1/2} =:\ \eta_{L^2}(u_h)$$

Example: L^2-Norm error estimates by duality arguments

$$-\nabla\cdot\{a\nabla u\} = f \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0$$

Dual problem

$$-\nabla\cdot\{a\nabla z\} = \|e\|^{-1}e \quad \text{in } \Omega, \quad z_{|\partial\Omega} = 0$$

A posteriori error estimates

- weighted:

$$\|e\| \le \eta^{\omega}_{L^2}(u_h) := c_I \sum_{K\in\mathbb{T}_h} h_K^2 \rho_K \omega_K$$

$$\omega_K := \|\nabla^2 z\|_K \approx \tilde{\omega}_K := \|\nabla_h^2 z_h\|_K$$

- global:

$$\|e\| \le \eta_{L^2}(u_h) := c_I c_S \Big(\sum_{K\in\mathbb{T}_h} h_K^4\, \rho_K^2 \Big)^{1/2}$$

$$c_S := \|\nabla^2 z\| \approx \tilde{c}_s := \Big(\sum_{K\in\mathbb{T}_h} \|\nabla_h^2 z_h\|_K^2 \Big)^{1/2}$$

The interpolation constant is $c_I \approx 0.2$. The functional $J(\cdot)$ is evaluated by replacing the unknown solution u in the righthand side of the dual problem by a patch-wise higher-order interpolation $I_h^{(2)} u_h$ of u_h,

$$e \approx I_h^{(2)} u_h - u_h$$

Effectivity index measures quality of error estimation

$$I_{\text{eff}} := \frac{\eta(u_h)}{|J(e)|}$$

Remark. '$I_{\text{eff}} \sim 1$' does not necessarily mean a good efficiency in computing the target quantity $J(u)$.

Numerical test (R. Becker 1996):

$$\Omega = (-1,1)^2, \quad a(x) = 0.1 + e^{3(x_1+x_2)}, \quad f \equiv 0.1$$

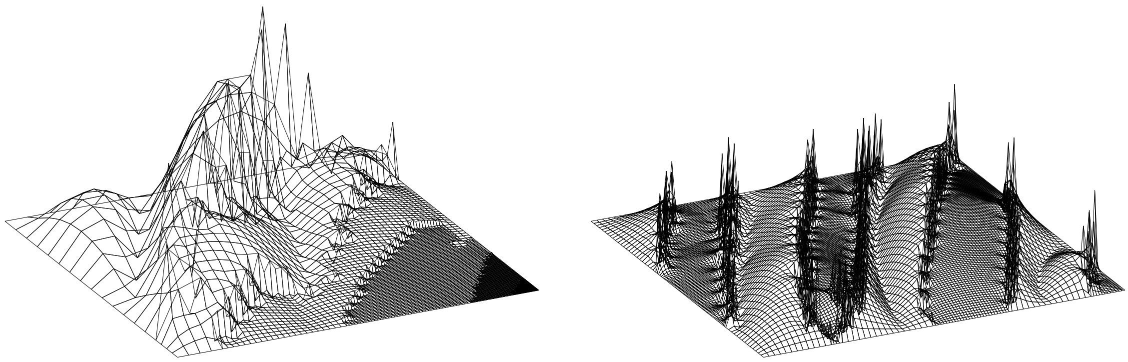


Figure. Pointwise errors obtained by η_{L^2} (left, scaled by 1/30) and $\eta_{L^2}^{\omega}$ (right, scaled by 1/10) on meshes with $N \sim 10000$ cells.

TOL	N	L	$\|e\|$	η_{L^2}	I_{eff}	$\tilde{c}_s$
4^{-1}	1132	9	9.03e-2	4.89e-1	5.41	2.52
4^{-2}	2836	9	6.40e-2	2.32e-1	3.62	3.02
4^{-3}	5884	10	2.13e-2	1.21e-1	5.68	3.26
4^{-4}	15736	11	7.36e-3	4.76e-2	6.46	3.55
4^{-5}	23380	11	5.59e-3	3.12e-2	5.58	3.39

TOL	N	L	$\|e\|$	$\eta_{L^2}^{\omega}$	I_{eff}	$\tilde{c}_s$
4^{-1}	52	4	3.18e-1	2.25e-1	0.70	....
4^{-2}	64	4	1.47e-1	1.52e-1	1.03	....
4^{-3}	148	5	1.08e-1	9.80e-2	0.90	....
4^{-4}	220	5	6.77e-2	5.24e-2	0.77	....
4^{-5}	592	6	2.21e-2	2.59e-2	1.17	....
4^{-6}	892	6	1.19e-2	1.54e-2	1.29	....
4^{-7}	2368	7	5.11e-3	7.17e-3	1.40	....

Table. Results obtained by η_{L_2} (top) compared to $\eta_{L_2}^{\omega}$ (bottom).

1.2 Evaluation of error estmators

Model problem on a polygonal domain $\Omega \subset \mathbb{R}^d$:

$$-\Delta u = f \;\text{ in } \Omega, \quad u = 0 \;\text{ on } \partial\Omega$$

A posteriori error representation for finite element approximation:

$$J(e) = E(u_h) := \sum_{K \in \mathbb{T}_h} \big\{ (R(u_h), z - I_h z)_K + (r(u_h), z - I_h z)_{\partial K} \big\}$$

A posteriori error estimate

$$|J(e)| \le |E(u_h)| \le \eta(u_h := \sum_{K \in \mathbb{T}_h} \eta_K$$

with the local 'error indicators'

$$\eta_K = \big| (R(u_h), z - I_h z)_K + (r(u_h), z - I_h z)_{\partial K} \big|$$

Aspects of relevance:

- sharpness of the global error bound $\eta(u_h)$
- effectivity of local error indicators η_K for mesh refinement

Remark. Already by this localization the asymptotic sharpness of the error estimate may get lost. One should use the approximate error representation $\tilde{E}(u_h)$, avoiding any localization.

Technical details of mesh adaptation (using 'hanging nodes')

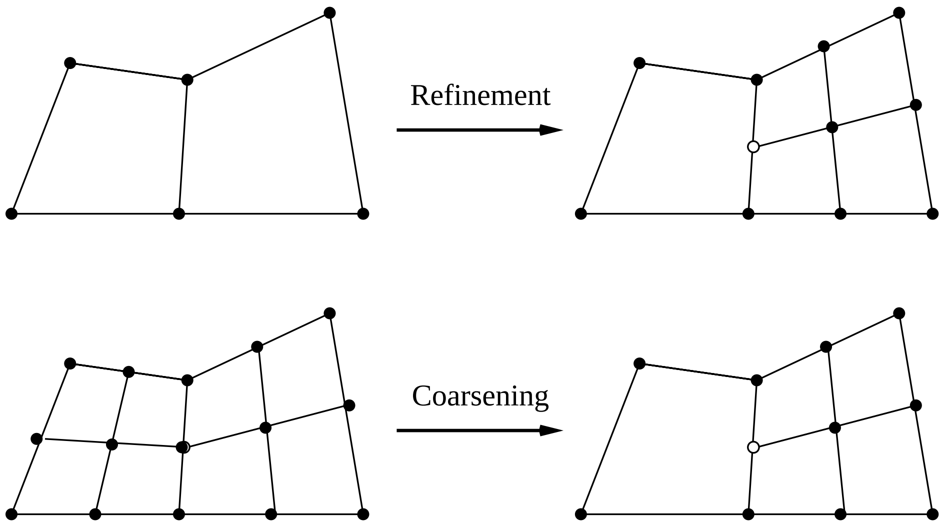


Figure. Refinement and coarsening in quadrilateral meshes

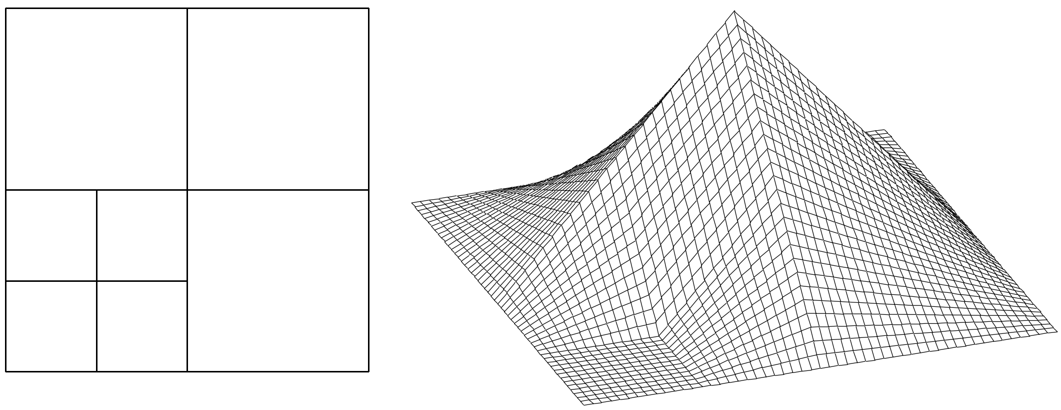


Figure. Q_1 nodal basis function on a patch of cells with hanging nodes

1. *Approximation by higher order methods:*
   The dual problem is solved by using *biquadratic* finite elements on the current mesh yielding an approximation $z_h^{(2)}$ to z:

$$\eta_K^{(1)} = \left|(R(u_h), z_h^{(2)} - I_h z_h^{(2)})_K + (r(u_h), z_h^{(2)} - I_h z_h^{(2)})_{\partial K}\right|$$

$$\Rightarrow \qquad \lim_{TOL\to 0} I_{\text{eff}} = 1$$

2. *Approximation by higher order interpolation:*
   Patchwise *biquadratic* interpolation of the bilinear approximation $z_h^{(1)}$ on the current mesh yields an approximation $I_{2h}^{(2)} z_h^{(1)}$ to z:

$$\eta_K^{(2)} = \left|(R(u_h), I_{2h}^{(2)} z_h^{(1)} - z_h^{(1)})_K + (r(u_h), I_{2h}^{(2)} z_h^{(1)} - z_h^{(1)})_{\partial K}\right|$$

$$\Rightarrow \qquad \lim\inf_{TOL\to 0} I_{\text{eff}} \sim 1 - 2$$

3. *Approximation by difference quotients:*
   Interpolation estimate à la Bramble/Hilbert yields

$$\|z - I_h z\|_K + h_K^{1/2} \|z - I_h z\|_{\partial K} \le c_I h_K^2 \|\nabla^2 z\|_K$$

   The second derivative $\nabla^2 z$ is replaced by a suitable second-order difference quotient $\nabla_h^2 z_h^{(1)}$ of the approximate dual solution:

$$\eta_K^{(3)} = c_I h_K^{-1/2} \rho_K \|[\partial_n z_h^{(1)}]\|_{\partial K}$$

$$\Rightarrow \qquad \lim\inf_{TOL\to 0} I_{\text{eff}} \sim 5 - 10$$

L	N	$I_{\text{eff}}^{(1)}$	$I_{\text{eff}}^{(2)}$	$I_{\text{eff}}^{(3)}$
1	4^3	6.667	12.82	2.066
2	4^4	1.253	13.51	45.45
3	4^5	1.052	3.105	9.345
4	4^6	1.007	2.053	7.042
5	4^7	1.003	1.886	6.667

Table. Efficiency of *weighted* error indicators for controlling the point-error $J(e) = |e(0)|$

1.3 Mesh adaptation strategies

A posteriori error indicators

$$|J(e)| \approx \eta := \sum_{K\in\mathbb{T}_h} \eta_K, \quad N := \#\{K \in \mathbb{T}_h\}$$

- 'Error balancing' strategy: Equilibrate by iteration

  $$\eta_K \approx \frac{TOL}{N} \quad \Rightarrow \quad \eta \approx TOL$$

  'optimal' mesh characterized by equilibrated error indicators.

- 'Fixed error reduction' strategy: Order cells according to

  $$\eta_{K,1} \geq \cdots \geq \eta_{K,i} \geq \cdots \geq \eta_{K,N}$$

  For prescribed rates $X\%$ and $Y\%$ form

  $$\sum_{i=1}^{N_*} \eta_{K,i} \approx X\,\eta, \qquad \sum_{i=N^*+1}^{N} \eta_{K,i} \approx Y\,\eta$$

  Cells $K_1,\dots,K_{N_*}$ are refined and cells $K_{N^*},\dots,K_N$ coarsened.

  Alternatively, refine $X\%$ and coarsen $Y\%$ of cells with largest and smalles error indicator, respectively. This allows to keep number of cells almost constant in the course of the mesh adaptation process.

- 'Mesh optimization' strategy:

  $$\eta = \sum_{K\in\mathbb{T}_h} \eta_K \approx \int_\Omega h(x)^2\Phi(x)\,dx \;\to\; min, \qquad N_{max} \text{ given}$$

  with $\Phi(u(x), z(x))$ a mesh-independent weighting function. Calculus of variations yields 'optimal' mesh-distribution function

  $$h_{\text{opt}}(x) = \Big(\frac{W}{N_{\text{max}}}\Big)^{1/2} \Phi(x)^{-1/4}, \qquad W := \int_\Omega \Phi(x)^{1/2}\,\text{dx}$$

Model problem Poisson equation with error representation

$$\eta = \sum_{K\in\mathbb{T}_h} \big\{(f+\Delta u_h, z-I_h z)_K - \tfrac{1}{2}([\partial_n u_h], z-I_h z)_{\partial K}\big\}$$

for estimating the error with respect to functional $J(\cdot)$.

- Convergence of residuals

$$f(x)+\Delta u_h(x_K) \to \Phi_1(x_K), \qquad -\tfrac{1}{2}h_K^{-1}[\partial_n u_h](x_K) \to \Phi_2(x_K)$$

- Convergence of weights

$$h_K^{-2}(z-I_h z)(x_K) \to \Phi_3(x_K)$$

- Error representation formula with $\Phi := (\Phi_1+\Phi_2)\Phi_3$:

$$\eta \approx \sum_{K\in\mathbb{T}_h} h_T^2\{\Phi_1+\Phi_2\}\Phi_3(x_K) \approx \int_\Omega h(x)^2\Phi(x)\,dx =: E(h)$$

  with a distributed mesh-size function $h(x)$.

- Mesh complexity formula:

$$N(h) = \sum_{K\in\mathbb{T}_h} h_K^d h_K^{-d} \approx \int_\Omega h(x)^{-d}\,dx$$

Mesh optimization problems:

$$\begin{aligned} (I) \quad & \eta(u_h) \to \min, \quad N \le N_{max}, \\ (II) \quad & N \to \min, \quad \eta(u_h) \le TOL \end{aligned}$$

Solutions:

$$\begin{aligned} h_{opt}^{(I)}(x) &= \Big(\frac{W}{N_{\max}}\Big)^{1/d}\Psi(x)^{-1/(2+d)}, \\ h_{opt}^{(II)}(x) &= \Big(\frac{TOL}{W}\Big)^{1/d}\Psi(x)^{-1/(2+d)} \end{aligned}$$

$$W = \int_\Omega \Psi(x)^{d/(2+d)}\,dx < \infty \quad (!)$$

Proof for Problem (I): Classical Lagrange approach:

$$L(h,\lambda) = E(h) + \lambda\{N(h) - N_{max}\}$$

with Lagrangian multiplier $\lambda \in \mathbb{R}$. First-order optimality condition

$$\frac{d}{dt} L(h + t\varphi, \lambda + t\mu)_{|t=0} = 0$$

for all admissible variations φ and μ,

$$2h(x)\Psi(x) - d\lambda h(x)^{-d-1} = 0, \qquad \int_\Omega h(x)^{-d}\, dx - N_{\max} = 0$$

and, consequently,

$$h(x) = \Big(\frac{2}{d\lambda}\Psi(x)\Big)^{-1/(2+d)}, \qquad \Big(\frac{2}{d\lambda}\Big)^{d/(2+d)} \int_\Omega \Psi(x)^{d/(2+d)}\, dx = N_{\max}$$

From this, we deduce the desired relations

$$\lambda \equiv \frac{2}{d}\, h(x)^{2+d}\Psi(x)\,, \qquad h_{opt}(x) = \Big(\frac{W}{N_{\max}}\Big)^{1/d} \Psi(x)^{-1/(2+d)}$$

The mesh-optimization problem (II) can be treated in an analogous way.

- The first identity implies that an optimal mesh-size distribution with local cell-widths h_T is characterized by the equilibration of the cell-indicators η_T as assumed by the *error balancing strategy.*

- Even for rather *irregular* functionals $J(\cdot)$ the quantity W is bounded. For example, the evaluation of $J(u) = \partial_i u(P)$ for smooth u in $\mathbb{R}^2$ leads to $\Psi(x) \approx |x_P|^{-3}$ and, consequently,
$$W \approx \int_\Omega |x - P|^{-3/2}\, dx < \infty$$

- The explicit formulas for $h_{opt}(x)$ have to be used with care in designing a mesh as their derivation implicitly assumes that they actually correspond to *scalar* mesh-size functions of *isotropic* meshes, a condition however which is not incorporated into the formulation of the mesh-optimization problems.

- A strong objection against the practical value of this formula is the need of approximating the weighting function $\Phi(x)$ on the current mesh for steering the refinement process which seems a contradiction in itself.

1.4 Use of error estimators for postprocessing

Poisson equation in 2D written in variational form

$$a(u,\varphi) = (f,\varphi) \quad \forall \varphi \in V$$

$$a(u_h,\varphi_h) = (f,\varphi_h) \quad \forall \varphi_h \in V_h$$

The target functional is $J(\cdot)$ and $z \in V$ the corresponding dual solution with $z_h \in V_h$ its Ritz projection. There holds

$$J(u) = a(u,z) = (f,z)$$

We recall the identity

$$J(u) = J(u_h) + a(e,z) = J(u_h) + \rho(u_h)(z - z_h)$$

$$\rho(u_h)(z - z_h) = (f, z - z_h) - a(u_h, z - z_h)$$

With the patchwise biquadratic interpolation $\tilde{z}_h := I_{2h}^{(2)} z_h$ of z_h on the mesh $\mathbb{T}_h$:

$$J(e) \approx \rho(u_h)(\tilde{z}_h - z_h) = \rho(u_h)(\tilde{z}_h)$$

Rewriting this relation as

$$J(u) \approx \tilde{J}_1(u_h) := J(u_h) + (f, \tilde{z}_h) - a(u_h \tilde{z}_h)$$

we obtain a presumably better approximation to $J(u)$ than is $J(u_h)$. Indeed, the error can be written as

$$J(u) - \tilde{J}_1(u_h) = J(e) - \rho(u_h)(\tilde{z}_h) = a(e,z) - a(u_h, \tilde{z}_h) = a(e, z - \tilde{z}_h)$$

which implies

$$|J(u) - \tilde{J}_1(u_h)| \leq \|\nabla e\| \, \|\nabla(z - \tilde{z}_h)\|$$

Since it is not clear whether $\tilde{z}_h$ is a reasonably better approximation to z than z_h, this estimate is of only questionabls value. Further, the two energy-norm errors correspond both to the 'primal' mesh $\mathbb{T}_h$ and can therefore not be minimized independently. It would be desirable to have the possibility of using independent meshes $\mathbb{T}_h$ for u_h and $\mathbb{T}_h^*$ for z_h in constructing an approximation of $J(u)$.

Proposition. *Let $\mathbb{T}_h$ and $\mathbb{T}_h^*$ be two independent meshes and V_h and V_h^* corresponding finite element spaces in which the Ritz projections u_h and z_h^* of u and z are computed. Further, denote by $\tilde{z}_h^*$ the patchwise biquadratic interpolation of z_h^* on the dual mesh $\mathbb{T}_h^*$. Then, for the post-processed approximation*

$$\tilde{J}_2(u_h) := J(u_h) + (f, \tilde{z}_h^*) - a(u_h, \tilde{z}_h^*)$$

there holds the estimate

$$|J(u) - \tilde{J}_2(u_h)| \le \|\nabla e\| \, \|\nabla(z - \tilde{z}_h^*)\|$$

Here, the two energy-norm terms can be minimized independently by optimizing the primal and dual meshes $\mathbb{T}_h$ and $\mathbb{T}_h^*$.

Proof. We have

$$\begin{aligned} J(u) - \tilde{J}_2(u_h) &= J(u) - J(u_h) - \rho(u_h)(\tilde{z}_h^*) \\ &= (f, z) - a(u_h, z) - (f, \tilde{z}_h^*) + a(u_h, \tilde{z}_h^*) \\ &= (f, z - \tilde{z}_h^*) - a(u_h, z - \tilde{z}_h^*) \\ &= a(e, z - \tilde{z}_h^*) \end{aligned}$$

This implies the assertion.

One may hope to obtain an even better approximation by

$$\tilde{J}_3(u_h) := \tilde{J}_2(\tilde{u}_h) = J(\tilde{u}_h) + (f, \tilde{z}_h^*) - a(\tilde{u}_h, \tilde{z}_h^*)$$

where $\tilde{u}_h$ is the patchwise biquadratic interpolation of u_h.

Numerical test.

Model Poisson problem with $\Omega = (-1,1)^2$,

$$u(x) = (1-x_1^2)(1-x_2^2)\exp(1-x_2^{-4})$$

and the error functional

$$J(u) = \int_{-1}^{1} u(x_1,0)\,dx$$

In this example primal and dual solution have irregularitries at different locations such that is is to expected that maximal efficiency is achieved using different meshes for u and z_h.

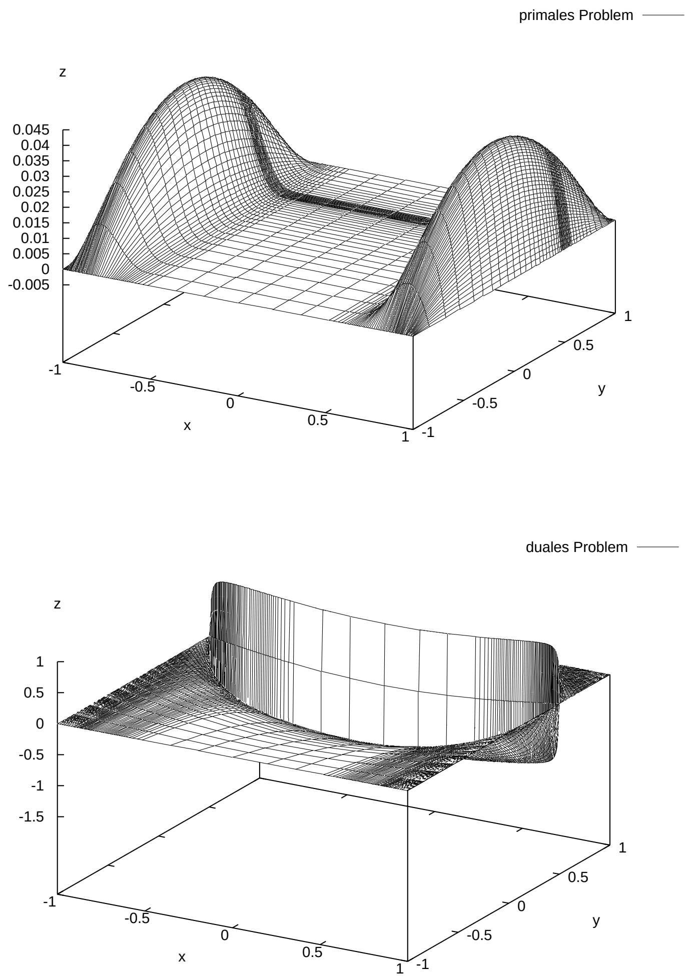


Figure. Primal (left) and dual (right) solution of the model problem

The figure shows the mesh efficiencies of $J(u_h)$ and the three post-processed approximations $\tilde{J}_1(u_h)$, $\tilde{J}_2(u_h)$ and $\tilde{J}_3(u_h)$. The mesh refinements are driven by energy-norm error indicators as derived before separately on the primal and dual meshes $\mathbb{T}_h$ and $\mathbb{T}_h^*$, respectively. We see that $\tilde{J}_1(u_h)$ does not bring significant adventages over the original approximation $J(u_h)$. The two other approximations $\tilde{J}_2(u_h)$ and $\tilde{J}_3(u_h)$ which use different meshes for u and z are clearly superior and show a mesh complexity like $TOL \approx N^{-2}$ $(N = N_{primal} + N_{dual})$.

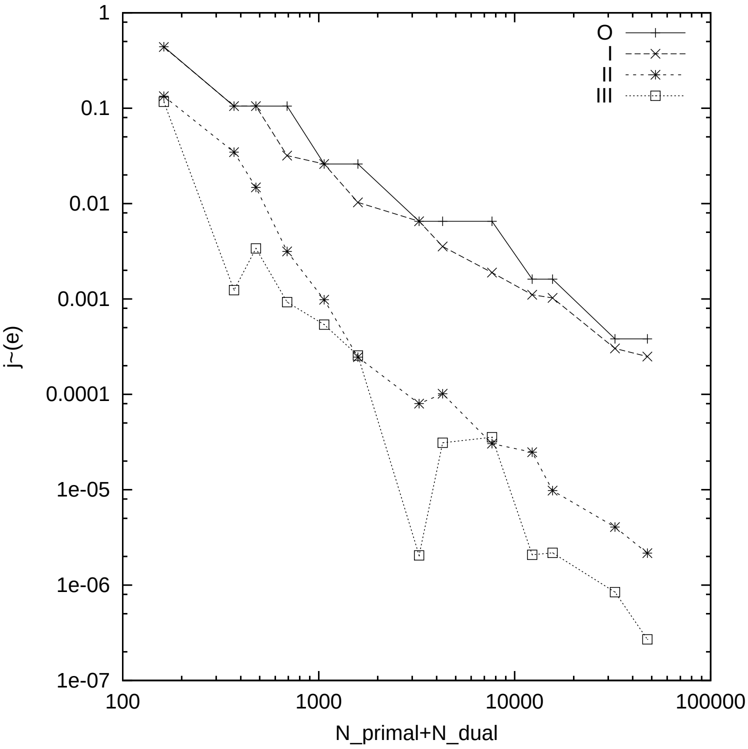


Figure. Mesh efficiencies of postprocessing

1.5 Toward anisotropic mesh adaptation

Sometimes *isotropic* mesh refinement as discussed so far is not efficient for properly resolving certain features of the solution. For example, in singular perturbed problems of the form

$$-\epsilon\Delta u + b\cdot\nabla u = f, \;\text{ in } \Omega, \quad u_{|\partial\Omega} = 0$$

with small coefficient ϵ, boundary layers may occur in which the solution has large derivative in normal direction to the boundary derivative while it varies only slowly in tangential direction.

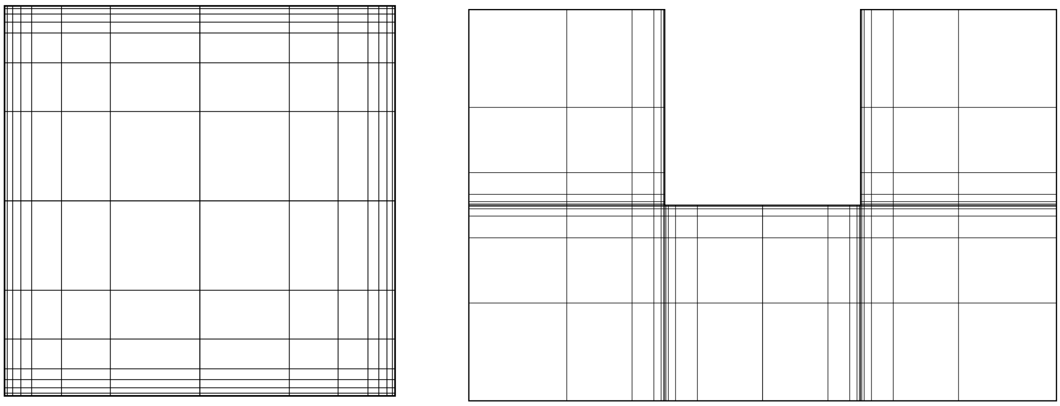


Figure. Locally anisotropic tensor-product meshes

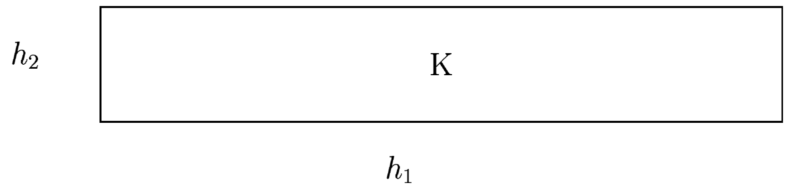
We consider adaptive cell stretching alone and, for simplicity, concentrate on the construction of 'optimal' cartesian tensor-product meshes. Starting form an error representation of the form

$$J(e) = \sum_{K\in\mathbb{T}_h} \big\{(f+\Delta u_h, z-I_h z)_K - \tfrac{1}{2}([\partial_n u_h], z-I_h z)_{\partial K\setminus\partial\Omega}\big\}$$

we have to adress the following questions:

- How can we detect anisotropic behavior of the true solutiopn u?
- How can we detect anisotropic behavior of the dual solution z?
- How can we obtain an indicator for 'optimal' cell stretching?

Suppose that the domain's boundary and the mesh are oriented along the coordinate axes. Then any cell $K \in \mathbb{T}_h$ is characterized by its widths h_i in the x_i-directions.



The edge-residual terms contain on each edge Γ information about second derivatives of the primal and dual solutions u and z:

- jump terms: $[\partial_1 u_h]_\Gamma \approx h_1 \partial_1^2 u_{|\Gamma}$
- weights: $(z - I_h z)_{|\Gamma} \approx h_2^2 \partial_2^2 z_{|\Gamma}$

Assuming the second-order derivatives of u as constant, we obtain

$$\begin{aligned} |([\partial_n u_h], z - I_h z)_{\partial K}| &\approx h_1^3 h_2 |\partial_2^2 u|\,|\partial_1^2 z| + h_1 h_2^3 |\partial_1^2 u|\,|\partial_2^2 z| \\ &= |K|\big\{h_1^2 |\partial_2^2 u|\,|\partial_1^2 z| + |K|^2 h_1^{-2} |\partial_1^2 u|\,|\partial_2^2 z|\big\} \end{aligned}$$

Minimizing this with respect to h_1 yields the necessary condition

$$2h_1 |\partial_2^2 u|\,|\partial_1^2 z| - 2|K|^2 h_1^{-3} |\partial_1^2 u|\,|\partial_2^2 z| = 0 \quad \Rightarrow \quad h_1^4 = |K|^2 \frac{|\partial_1^2 u|\,|\partial_2^2 z|}{|\partial_2^2 u|\,|\partial_1^2 z|}$$

and, consequently,

$$(I) \qquad \frac{h_1^2}{h_2^2} \approx \frac{|\partial_1^2 u|\,|\partial_2^2 z|}{|\partial_2^2 u|\,|\partial_1^2 z|}$$

Remark. This result is counter-intuitive as it does not indicate the optimal cell stretching due to interpolation theory:
Consider the case that u is linear in x_1-direction, i.e., $\partial_1^2 u \equiv 0$, and that z is isotropic. Then, the formula would suggest to refine the cell in x_1 direction which is evidently the wrong decision. It seems that considering only the edge terms in the error estimate, as usually suggested, is not enough.

In view of this observation, we now follow a more heuristic approach and base the anisotropic cell adaptation on an estimate for the interpolation error. We recall the anisotropic interpolation error

$$\|\nabla(u - I_h u)\|_K \le c\left(h_1^2\|\partial_1\nabla u\|_K^2 + h_2^2\|\partial_2\nabla u\|_K^2\right)^{1/2}$$

Hence, assuming the second-order derivatives as constant on K, we have

$$\|\nabla(u - I_h u)\|_K \le c|K|^{1/2}\left(h_1^2|\partial_1\nabla u|^2 + |K|^2 h_1^{-2}|\partial_2\nabla u|^2\right)^{1/2}$$

Minimizing this with respect to h_1 results in the necessary condition

$$2h_1|\partial_1\nabla u|^2 - 2|K|^2 h_1^{-3}|\partial_2\nabla u|^2 = 0 \quad \Rightarrow \quad h_1^2 = |K|\frac{|\partial_2\nabla u|}{|\partial_1\nabla u|}$$

and, consequently,

$$\frac{h_1}{h_2} \approx \frac{|\partial_2\nabla u|}{|\partial_1\nabla u|}$$

In view of this result, we now consider the heuristic error indicator

$$\eta_K := \|\nabla(u - I_h u)\|_K \|\nabla(z - I_h z)\|_K$$

which is minimized for

$$(II) \qquad \frac{h_1^2}{h_2^2} \approx \frac{|\partial_2\nabla u|\,|\partial_2\nabla z|}{|\partial_1\nabla u|\,|\partial_1\nabla z|}$$

This relation simultaneously reflects possible anisotropies in the primal and dual solution.

Numerical tests

Test problem

$$-\Delta u = f \quad \text{in } \Omega = (-1,1)^2, \quad u_{|\partial\Omega} = 0$$

Test case 1. Solution

$$u(x) = (1-x_1^2)^2(1-x_2)^2(kx_1^2+0.1)^{-1}$$

where the parameter $k = 1, 4, 16, 64, \ldots$, determines the strength of the anisotropy. The right hand side is determined as $f := -\Delta u$.
Target quantity:

$$J(u) := |\Omega|^{-1} \int_\Omega u\, dx$$

The anisotropy is only in the primal solution while the dual solution satisfies $-\Delta z = 1$ and is isotropic.

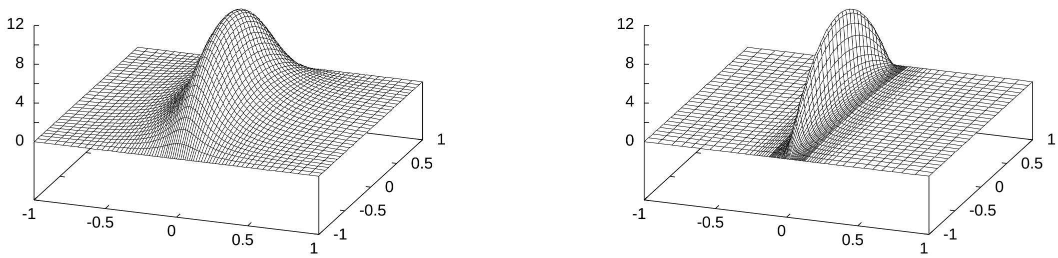


Figure. Anisotropic solutions for $k = 4$ (left) and $k = 64$ (right);

The computation starts from a coarse uniform tensor-product mesh which is then successively adapted on the basis of the relations (I) and (II)

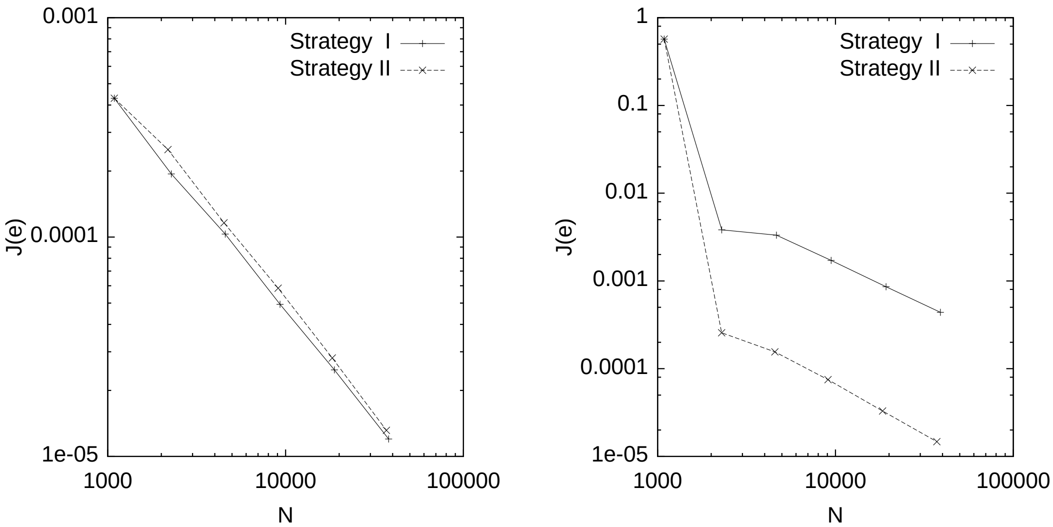


Figure. Mesh efficiencies of anisotropic refinement by Strategy I and Strategy II, for $k = 1$ (left) and $k = 64$ (right)

Test 2. Solution and functional

$$u(x) := (-x_1^2)(1-x_2^2)\exp(-x_1^{-4}), \qquad J(u) := \int_{-1}^{1} \partial_2 u(x_1, 0.5)\, dx_1$$

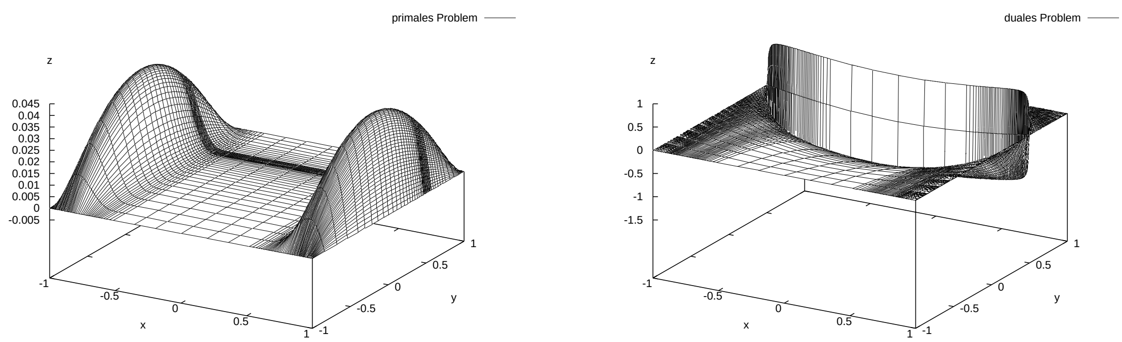


Figure. Primal and dual solution of Test 2

1.6 h/p adaptivity

In the following, we will briefly discuss the extension of our approach to a posteriori error control to higher-order finite elements. Let $V_h^{(p)} \subset V$, be finite element spaces of oder $p+1$. We recall the error representation,

$$J(e) = \sum_{K \in \mathbb{T}_h} \big\{ (R(u_h), z - I_h^{(p)} z)_K + (r(u_h), z - I_h^{(p)} z)_{\partial K} \big\}$$

with the cell- and edge-.residuals $R(u_h)$ and $r(u_h)$ as defined above and some local interpolation $I_h^{(p)} z \in V_h^{(p)}$. The evaluation of this identity for use as an error estimator may be done in a similar way as in the low-order case $p = 1$ employing a patchwise $(p+1)$-degree interpolation $I_{2h}^{(p+2)} z_h$ of the Ritz projection $z_h \in V_h^{(p)}$.

On the basis of the a posteriori error estimator $\eta(u_h)$ and the resulting local error indicators η_K, 'optimal' distributions of h_K and p_K are constructed by a series of adaptation cycles such that at the final stage the equilibration property is achieved:

$$\eta_K \approx \frac{TOL}{N_h}, \qquad N_h = \#\{K \in \mathbb{T}_{\text{opt}}\}$$

Let a tolerance TOL be given. In solving a stationary problem the adaptation process usually starts from a coarse mesh $\mathbb{T}_h^{(0)}$, $k = 1, 2, \ldots$, with mesh-size distribution $h^{(0)}$ and polynomial degree $p^{(0)} \equiv 1$. Then, a sequence of meshes $T_h^{(k)}$, $k = 1, 2, \ldots$, with corresponding distributions $h_K^{(k)}$ and $p_K^{(k)}$ is constructed by the following process:

1. On the current mesh $\mathbb{T}_h^{(k)}$ compute $u_h^{(k)}$ and $z_h^{(k)}$, and evaluate $\eta_\omega(u_h^{(k)})$ and the cell-error indicators η_K. If $\eta(u_h^{(k)}) < TOL$, then STOP.

2. Order cells according to the size of η_K. On each cell $K \in \mathbb{T}_h^{(k)}$ test whether

$$\eta_K < \frac{1}{2}\frac{TOL}{N_h} ? \qquad N_h = \#\{K \in \mathbb{T}_h^{(k)}\}$$

   If YES, procede to next cell. If NO, consider the following three cases:

   - Cell K and the polynomial degree p_K had been left unchanged in the preceding cycle. Then, leave K again unchanged but increase p_K to $p_K + 1$.
   - Cell K had been left unchanged in the preceding cycle, but p_K had been increased. Check whether

$$\eta_K < h_K \eta_K^{\text{old}}$$

     If YES, then again increase p_K to $p_K + 1$. If NO, then refine K into 2^d cells.
   - Cell K had been obtained by refinement of a mother cell $K_m \in \mathbb{T}_h^{(k-1)}$ (without changing the polynomial degree). Check whether

$$\eta_K \leq 2^{-p_K} \eta_{K_m}^{\text{old}} ?$$

     If YES, increase p_K to $p_K + 1$ and keep h_K. If NO, keep p_K and refine K into 2^d cells.

3. Usually the local changes of h_K and p_K cause additional hanging nodes and discontinuity of functions. In order to restore conformity (in the sense defined above), some neighboring cells need to be refined and their polynomial degree raised.

In this adaptation process, we try to raise p_K whenever possible. The philosophy underlying this rule is that usually for enhancing accuracy on a cell K it is more economical to raise p_K rather than to reduce h_K.

Numerical test (V. Heuveline 2003)

The computational domain is $\Omega = (-1,1) \times (-1,3)$ possibly with a vertical slit with tip at $(0,0)$. The error $J(e)$ is estimated by the error estimator $\eta_\omega := \eta_\omega(u_h)$ which is evaluated by the procedure described above. The initial mesh consists of $N_0 = 45$ cells. For the purpose of this particular test, we did not go for maximum efficiency, but rather iterated on each adaptation level as long as necessary in order to achieve satisfactory equilibration of the indicators η_K over the mesh. The quality of the error estimate is expressed in terms of the 'effectivity index'

$$I_{\text{eff}} := |\eta_\omega(u_h)/J(e)|$$

Further, we monitor the number of degrees of freedom $N := \dim V_h^p$, the actual error $J_\omega = J_\omega(e)$ obtained by our method, and the error $J_{\mathrm{E}} = J_{\mathrm{E}}(e)$ obtained by using a standard energy-type error estimator:

$$\eta_{\mathrm{E}}(u_h) := \Big(\sum_{K\in\mathbb{T}_h} h_K^2 p_K^{-2} \|f + \Delta u_h\|_K^2 + \tfrac{1}{2} h_K p_K^{-1} \|[\partial_n u_h]\|_{\partial K}^2 \Big)^{1/2}$$

The semi-singular case

On the domain $\Omega_0 = (-1,1) \times (-1,3)$, we compute the derivative point value $J(u) := \partial_1 u(x_0)$ at $x_0 = (0.5, 2.5)$. The solution is $u(x) = \sin(\pi(x_1+1)/2)\sin(3\pi(x_2+1)/4)$.

The fully singular case

On the slit domain $\Omega_1 = (-1,1) \times (-1,3) \setminus \{x \in \mathbb{R},\, x_1 = 0, -1 < x_2 < 0\}$ defined above, we compute the derivative point value $J(u) := \partial_1 u(x_0)$. The exact solution is $u(x) = r^{1/2}\sin(\theta/2)(x_1-1)(x_1+1)(x_2-3)(x_2+1)$.

The semi-singular case

N	$J_\omega(e)$	$N/\ln(J_\omega)^2$	$\eta_w(u_h)$	I_{eff}	$J_{\text{E}}(e)$	J_{E}/J_ω
45	$4.45e-01$	68	$1.02e+01$	23.2	$4.45e-01$	1
125	$3.50e-01$	113	$7.40e+00$	21.1	$4.33e-01$	1
233	$2.59e-02$	17	$3.42e-01$	13.2	$7.93e-02$	3
297	$4.19e-03$	10	$3.10e-02$	7.4	$3.59e-02$	8
412	$3.21e-04$	6	$6.75e-04$	2.1	$9.98e-03$	3
581	$3.09e-05$	5	$4.02e-05$	1.3	$5.34e-03$	173
812	$4.32e-06$	5	$5.19e-06$	1.2	$1.34e-03$	310
1113	$4.21e-07$	5	$5.06e-07$	1.1	$3.29e-04$	781

Table. Computation of $\partial_1 u(x_0)$ for smooth solution

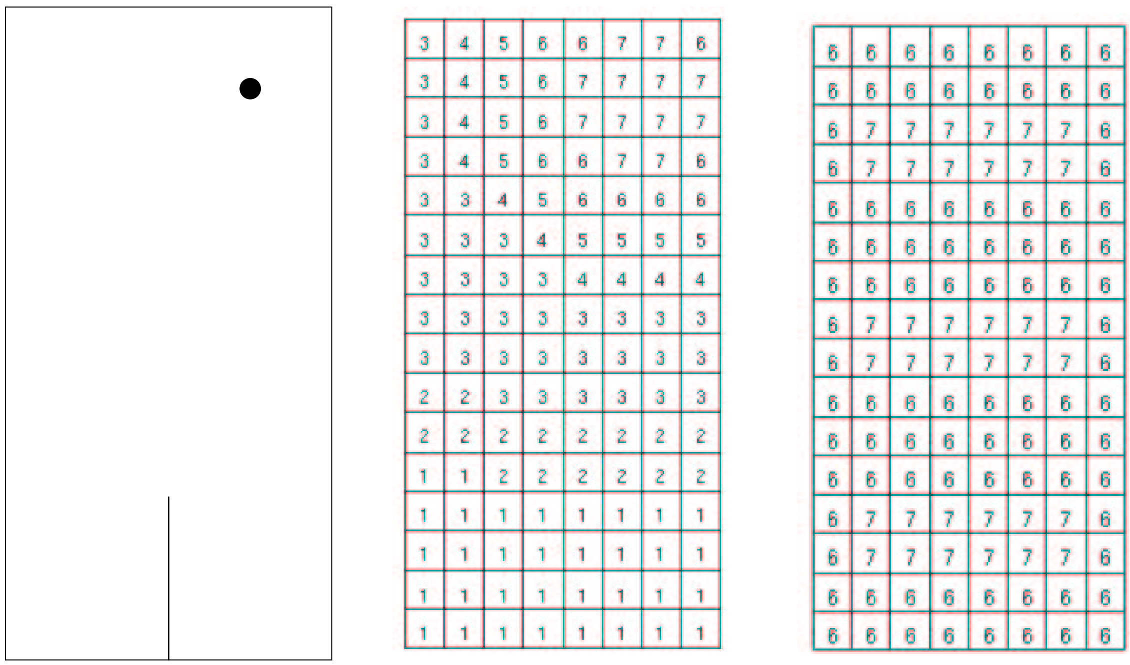


Figure. Configuration of the test problem (left), optimized distribution of p by η_ω (middle), and by η_{E} (right)

As expected the automatic adaptation process keeps the initial mesh unrefined and only raises p. We see that for computing point values, *hp* adaptivity based on the weighted error estimator η_ω is more efficient than that using the energy error estimator η_{E}.

The fully singular case

N	$J_\omega(e)$	$N/\ln(J_\omega)^2$	$\eta_w(u_h)$	I_{eff}	$J_{\text{E}}(e)$	J_{E}/J_ω
740	$2.02e-01$	289	$9.61e+00$	47.6	$8.56e-02$	1
1138	$8.45e-03$	50	$1.96e-01$	23.3	$4.65e-02$	6
1467	$3.45e-03$	45	$4.31e-02$	12.5	$6.48e-03$	2
1736	$8.43e-04$	34	$6.65e-03$	7.9	$8.32e-03$	10
2284	$7.73e-05$	25	$2.47e-04$	3.2	$2.23e-03$	29
2943	$8.72e-06$	22	$2.00e-05$	2.3	$3.32e-04$	38
3752	$8.34e-07$	19	$1.50e-06$	1.8	$8.71e-05$	104
5372	$3.34e-08$	18	$3.60e-08$	1.1	$9.32e-06$	282
6156	$4.43e-10$	15	$4.87e-10$	1.1	$1.37e-07$	309

Table. Computation of $\partial_1 u(x_0)$ with varying h and p

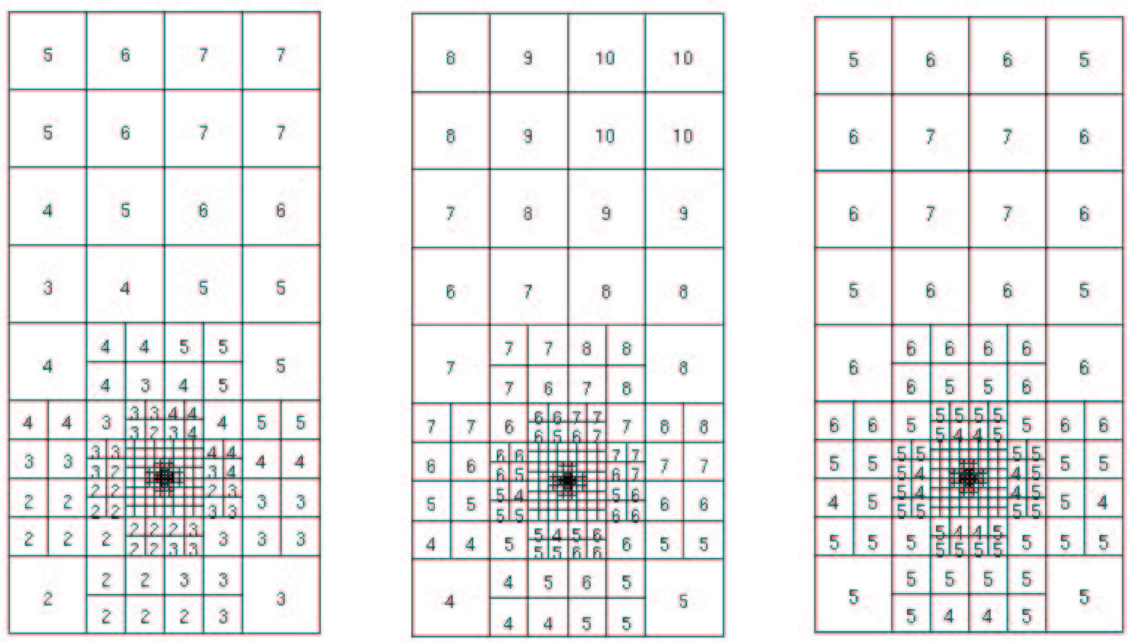


Figure. Optimized h and p by η_ω for $TOL \approx 10^{-6}$ (left) and $TOL \approx 10^{-9}$ (middle), and by η_{E} (right)

The weighted error estimator η_ω is asymptotically sharp and more efficient for computing the point value $\partial_1 u(P)$ than the energy-error estimator η_{E}.

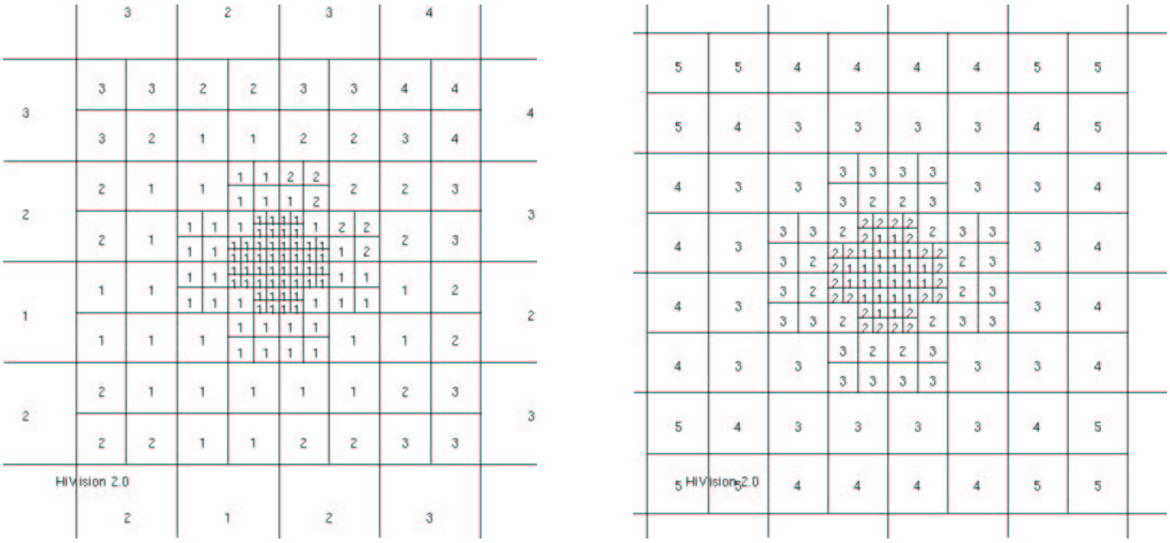


Figure. Zooms into optimized meshes by η_ω (left) and $\eta_{\rm E}$ (right)

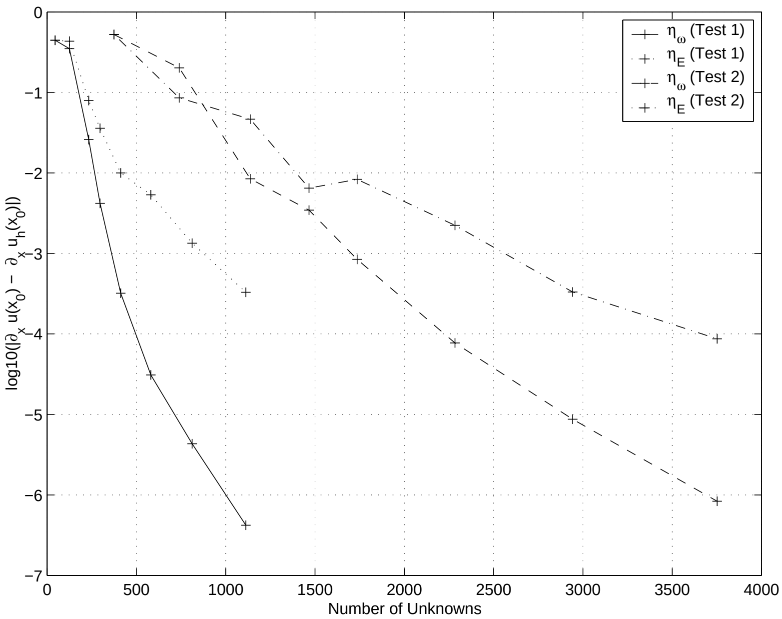


Figure. Efficiency of *hp* adaptation using the weighted and the energy-error estimators η_ω and $\eta_{\rm E}$