4 Applications in Structural Mechanics

We discuss the use of the DWR method for the finite element solution of problems in linear elasticity and in elasto-plasticity. This includes the treatment of incompressible material which prepares for fluid mechanical applications.

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4.1 Lamé-Navier system

Fundamental problem of linear elasticity theory:

$$-\nabla\cdot\sigma = f, \quad \sigma = A\epsilon(u), \quad \epsilon(u) = \tfrac{1}{2}(\nabla u + \nabla u^T) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \Gamma_D\,, \quad n\cdot\sigma = g \quad \text{on } \Gamma_N$$

Describes the (small) deformation of an elastic body occupying a bounded (polyhedral) domain $\Omega\subset\mathbb{R}^d$ ($d=2$ or 3) which is fixed along a part Γ_D ($\text{meas}(\Gamma_D)\neq 0$) of its boundary $\partial\Omega$, under the action of a body force with density f and a surface traction g along $\Gamma_N = \partial\Omega\setminus\Gamma_D$.

Linear–elastic isotropic material law,

$$\sigma = A\epsilon(u) = 2\mu\epsilon^D(u) + \kappa\nabla\cdot uI$$

with constants $\mu > 0$ and $\kappa > 0$, and ϵ^D the deviatoric part of ϵ.

Primal variational formulation:

$$a(u,\psi) := (A\epsilon(u),\epsilon(\psi)) = (f,\psi) + (g,\psi)_{\Gamma_N} \quad \forall\,\psi\in V$$

where $V = \{v\in H^1(\Omega)^d,\ v = 0 \text{ on } \Gamma_D\}$.

Finite element discretization with linear/bilinear elements in subspaces $V_h\subset V$ on meshes matching the decomposition $\partial\Omega = \Gamma_u\cup\Gamma_\sigma$.

$$a(u_h,\psi_h) = (f,\psi_h) + (g,\psi_h)_{\Gamma_\sigma} \quad \forall\,\psi_h\in V_h$$

Galerkin orthogonality relation for error $e = u - u_h$:

$$a(e,\psi_h) = 0, \quad \psi_h\in V_h$$

A posteriori error analysis

For error functional $J(\cdot)$ solve dual problem:

$$a(\varphi, z) = J(\varphi) \quad \forall\, \varphi \in V$$

Taking $\varphi = e$ and using Galerkin orthogonality,

$$J(e) = a(e, z) = a(e, z - \psi_h), \quad \psi_h \in V_h$$

Splitting the global integration over Ω into the contributions of the mesh cells $T \in \mathbb{T}_h$ and integrating cell-wise by parts yields

$$J(e) = \sum_{K \in \mathbb{T}_h} \Big\{ (-\nabla \cdot A\epsilon(e), z - \psi_h)_K + (n \cdot A\epsilon(e), z - \psi_h)_{\partial K} \Big\}$$

Observing $-\nabla \cdot A\epsilon(u) = f$ and the continuity of $n \cdot A\epsilon(u)$ across interelement edges,

$$J(e) = \sum_{K \in \mathbb{T}_h} \Big\{ (R(u_h), z - \psi_h)_K + (r(u_h), z - \psi_h)_{\partial K} \Big\}$$

with cell residuals $R(u_h)_{|K} := f + \nabla \cdot A\epsilon(u_h)$ and the edge residuals:

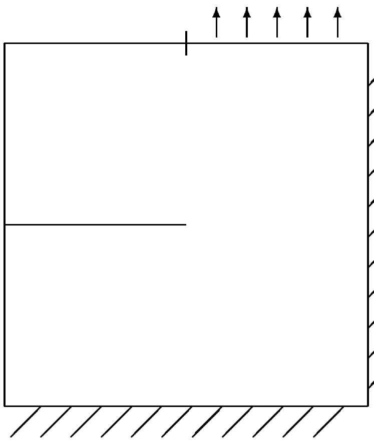
$$r(u_h)_{|\Gamma} := - \begin{cases} \frac{1}{2} n \cdot [A\epsilon(u_h)], & \text{if } \Gamma \subset \partial K \setminus \partial\Omega \\ n \cdot A\epsilon(u_h), & \text{if } \Gamma \subset \Gamma_D \\ n \cdot A\epsilon(u_h) - g, & \text{if } \Gamma \subset \Gamma_N \end{cases}$$

Energy-norm error estimate:

$$\|e\|_E \le \eta_E(u_h) := c_S c_I \Big(\sum_{K \in \mathbb{T}_h} \rho_K^2 \Big)^{1/2}.$$

Numerical test (F.-T. Suttmeier 1997)

A square elastic disc with a crack is subjected to a constant boundary traction acting along half of the upper boundary. Along the right-hand and lower parts of the boundary the disc is clamped and along the remaining part of the boundary (including the crack) it is left free.



The solution has a singularity with a stress singularity (expressed in terms of polar coordinates (r, θ)):

$$\sigma \approx r^{-1/2}$$

The material parameters are chosen as commonly used for aluminium, i.e., $2\mu \sim \lambda \sim 0.16N/m^2$. The surface traction is of size $g \equiv 0.1N/m^2$.

Computation of the mean normal stress over Γ_D,

$$J(u) = \int_{\Gamma_u} n \cdot A\epsilon(u) \cdot n \, ds$$

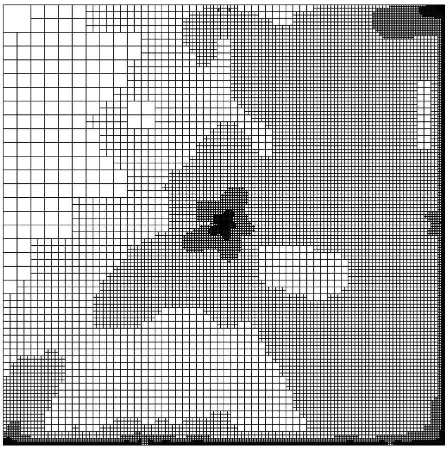
Regularization with $\epsilon = TOL$. The reference solution is σ_{ref}.

$$E^{\mathrm{rel}} := \left| \frac{J_\epsilon(\sigma_h - \sigma_{\mathrm{ref}})}{J_\epsilon(\sigma_{\mathrm{ref}})} \right|, \quad I_{\mathrm{eff}} := \left| \frac{\eta_\omega(u_h, \sigma_h)}{J_\epsilon(\sigma_h - \sigma_{\mathrm{ref}})} \right|$$

L	N	$J(u_h)$	E^{rel}	I_{eff}
1	256	0.017080	0.0283	1.80
2	484	0.019542	0.0180	1.96
3	1060	0.021138	0.0113	1.95
4	2113	0.022157	0.0070	1.96
5	4435	0.022795	0.0044	1.92
6	8830	0.023198	0.0027	1.86
7	15886	0.023428	0.0017	1.79
8	29947	0.023593	0.0010	1.79

L	N	$J(u_h)$	E^{rel}
1	256	0.017080	0.0283
2	544	0.018174	0.0237
3	1180	0.019363	0.0188
4	2659	0.020528	0.0139
5	6193	0.021538	0.0096
6	13423	0.022319	0.0064
7	31336	0.022811	0.0043
8	65332	0.023153	0.0029

Table. Results for $\eta_\omega(u_h)$ (left) and $\eta_E(u_h)$ (right).



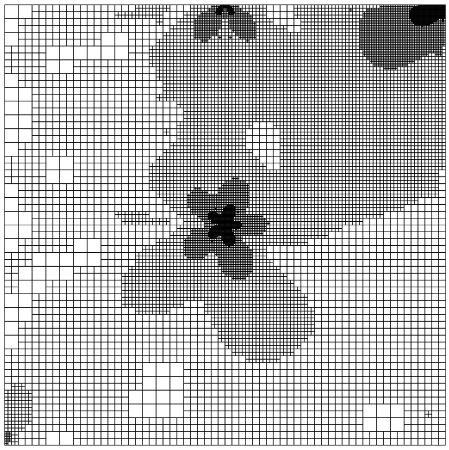


Figure. Results for $\eta_\omega(u_h)$ (left) and $\eta_E(u_h)$ (right).

4.2 A model problem in elasto-plasticity theory (a non-differentiable nonlinearity)

Fundamental problem in the static deformation theory of linear–elastic perfect–plastic material (*Hencky* model):

$$
\begin{aligned}
&\nabla\cdot\sigma = -f, \quad \epsilon(u) = A:\sigma + \lambda, \quad \epsilon(u) = \tfrac{1}{2}(\nabla u + \nabla u^T) \quad \text{in } \Omega \\
&\lambda : (\tau - \sigma) \le 0 \quad \forall\, \tau \quad \text{with} \quad F(\tau) \le 0 \\
&u = 0 \ \text{ on } \Gamma_D, \quad \sigma\cdot n = g \ \text{ on } \Gamma_N
\end{aligned}
$$

λ plastic growth.

This system describes the deformation of an elasto-plastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2$ or $3)$ which is fixed along a part Γ_D $(\text{meas}(\Gamma_D) \neq 0)$ of its boundary $\partial\Omega$, under the action of a body force with density f and a surface traction g along $\Gamma_N = \partial\Omega\setminus\Gamma_D$.

Linear–elastic isotropic material law:

$$\sigma = 2\mu\epsilon^D(u) + \kappa\nabla\cdot uI$$

with constants $\mu > 0$ and $\kappa > 0$, while the plastic behavior follows the von Mises flow rule, with some $\sigma_0 > 0$:

$$F(\sigma) = |\sigma^D| - \sigma_0 \le 0$$

Primal variational formulation:

$$A(u)(\psi) := (C(\epsilon(u)), \epsilon(\psi)) - (f, \psi) - (g, \psi)_{\Gamma_N} = 0 \qquad \forall \psi \in V$$

where $C(\epsilon(u)) = \Pi(2\mu\epsilon^D(u)) + \kappa\nabla\cdot uI$,

$$
\Pi(2\mu\epsilon^D(u)) = \begin{cases} 2\mu\epsilon^D(u) & , \text{ if } \ |2\mu\epsilon^D(u)| \le \sigma_0, \\ \dfrac{\sigma_0}{|\epsilon^D(u)|}\epsilon^D(u) & , \text{ if } \ |2\mu\epsilon^D(u)| > \sigma_0 \end{cases}
$$

This nonlinearity is only Lipschitz continuous.

Finite element approximation (Q_1-elements):

$$A(u_h)(\psi_h) = 0 \qquad \forall \psi_h \in V_h,$$

Associated stress σ_h :

$$\sigma_h = \Pi(2\mu\epsilon^D(u_h)) + \kappa\nabla \cdot u_h I.$$

Given a (linear) error functional $J(\cdot)$, we have the a posteriori error representation, with second-order remainder,

$$J(e) = \rho(u_h)(z - \psi_h) + R^{(2)}, \quad \psi_h \in V_h$$

where

$$\rho(u_h)(\cdot) = -A(u_h)(\cdot)$$

Linear dual problem:

$$(C'(u)\epsilon(\psi), \epsilon(z)) = J(\psi) \quad \forall \psi \in V$$

$$C'(\tau)\epsilon := \begin{cases} C\epsilon, & \text{if } \quad |2\mu\tau^D| \le \sigma_0, \\ \dfrac{\sigma_0}{|\tau^D|}\Big\{I - \dfrac{(\tau^D)^T\tau^D}{|\tau^D|^2}\Big\}\epsilon^D + \kappa\,\mathrm{tr}(\epsilon)I, & \text{if } \quad |2\mu\tau^D| > \sigma_0 \end{cases}$$

The remainder term is $R^{(2)} = \mathcal{O}(e^2)$ in regions where the form $A(\cdot)(\cdot)$ is C^2, i.e. outside the elastic-plastic transition zone $\{|2\mu\tau^D| = \sigma_0\}$. The residual term in the error identity has the form

$$-A(u_h)(z-\psi_h) = \sum_{K\in\mathbb{T}_h} \Big\{(R(u_h), z-\psi_h)_K + (r(u_h), z-\psi_h)_{\partial K}\Big\}$$

with the cell and edge residuals

$$R(u_h)_{|K} = f - \nabla\cdot C(\epsilon(u_h))$$

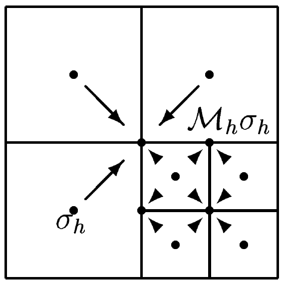
$$r(u_h)_{|\Gamma} = -\begin{cases} \frac{1}{2}n \cdot [C(\epsilon(u_h))], & \text{if } \Gamma \subset \partial K \setminus \partial\Omega \\ n \cdot C(\epsilon(u_h)) - g, & \text{if } \Gamma \subset \Gamma_N, \\ n \cdot C(\epsilon(u_h)), & \text{if } \Gamma \subset \partial\Omega \setminus \Gamma_N \end{cases}$$

Alternative heuristic error indicators for comparison:

(1) *ZZ-error indicator (à la Zienkiewicz/Zhu):*
An approximation $\sigma \approx M_h\sigma_h$ to σ is constructed by local averaging, ,

$$\|e_\sigma\| \;\approx\; \eta_{ZZ}(u_h) \;=\; \Big(\sum_{K\in\mathbb{T}_h} \|M_h\sigma_h - \sigma_h\|_K^2 \Big)^{1/2}$$

The nodal value at a point of the triangulation determining $\mathcal{M}_h\sigma_h$ is obtained by averaging the cell-wise constant values of σ_h of those cells having this point in common.



(2) *An energy-error indicator (à la Johnson/Hansbo):*
This heuristic energy-error estimator is based on decomposing the domain Ω into *discrete* plastic and elastic zones, $\Omega = \Omega_h^p \cup \Omega_h^e$. Accordingly the error estimator has the form

$$\|e_\sigma\| \;\approx\; \eta_E(u_h) \;=\; c_i \Big(\sum_{K\in\mathbb{T}_h} \eta_K^2 \Big)^{1/2}$$

with the local error indicators defined by

$$\eta_K^2 := \begin{cases} h_K^2\{\rho_K + \rho_{\partial K}\}^2, & \text{if } K \subset \Omega_h^e \\ \{\rho_K + \rho_{\partial K}\}\|M_h\sigma_h - \sigma_h\|_K, & \text{if } K \subset \Omega_h^p \end{cases}$$

Numerical tests (F.-T. Suttmeier 1998)

a) Square plate with a slit:

Figure. Plot of $|\sigma^D|$ (plastic regions black) computed on a mesh with $N \approx 64\,000$ cells.

Compute the mean normal stress over the clamped part of the boundary,

$$J(\sigma) = \int_{\Gamma_u} n \cdot \sigma \cdot n \, ds$$

L	N	$J_\epsilon(\sigma_h)$	E^{rel}	I_{eff}
2	484	0.019542	0.0180	1.96
3	1060	0.021138	0.0113	1.95
4	2113	0.022157	0.0070	1.96
5	4435	0.022795	0.0044	1.92
6	8830	0.023198	0.0027	1.86
7	15886	0.023428	0.0017	1.79
8	29947	0.023593	0.0010	1.79
9	52288	0.023697	0.0006	1.86

Table. Results obtained by the *weighted* error estimator $\eta_\omega(u_h)$.

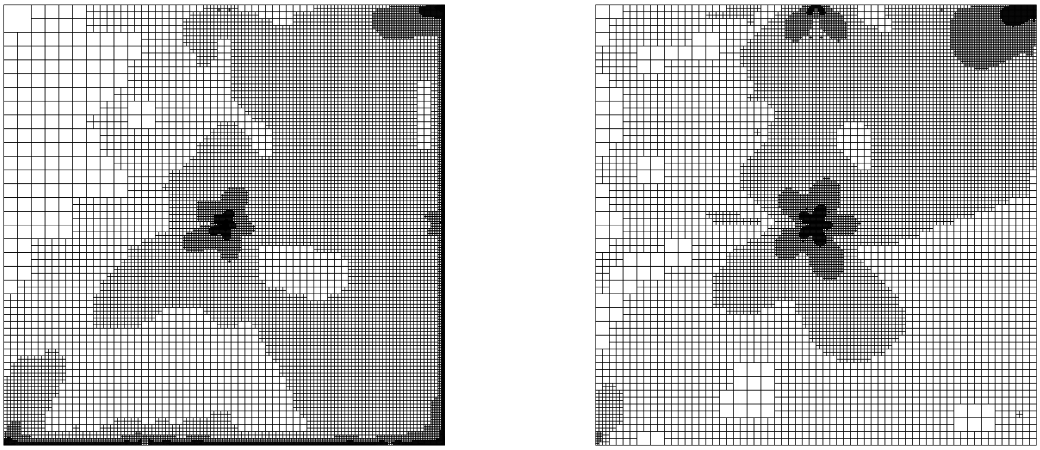


Figure. Finest meshes obtained by $\eta_\omega(u_h)$ (left) and $\eta_E(\sigma_h)$ (right).

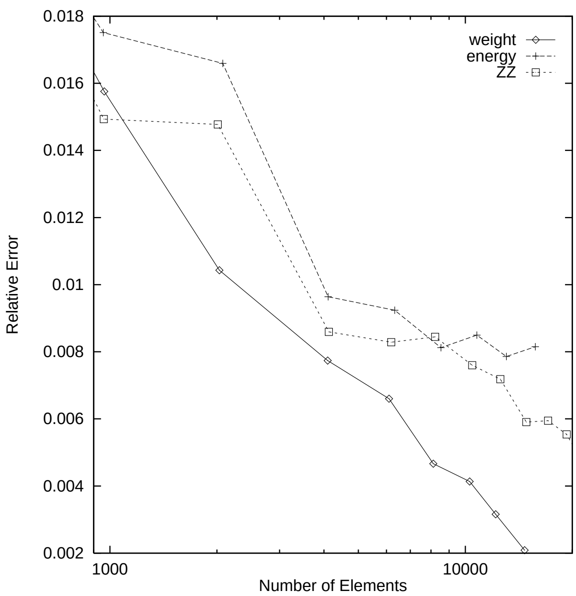


Figure. Relative error for $J(\sigma)$ on grids based on the different error indicators.

The weighted error estimator turns out to be efficient even on coarse meshes. This indicates that the strategy of evaluating the weights ω_T computationally works also for the present irregular nonlinear problem.

b) Benchmark square plate with a hole:

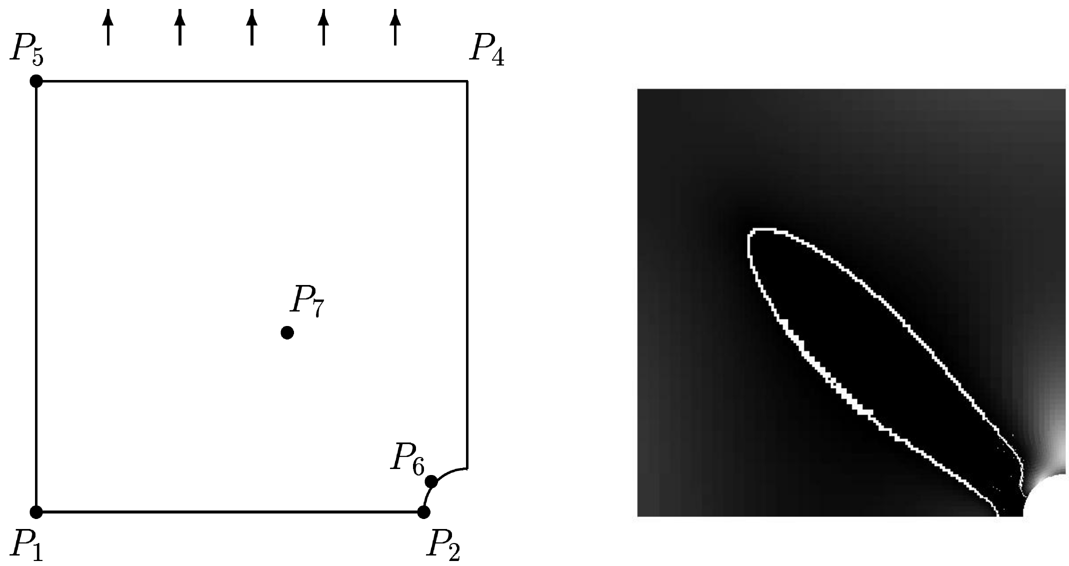


Figure. Geometry of the benchmark problem and plot of $|\sigma^D|$ (plastic region black, transition zone white) computed on a mesh with $N \approx 10\,000$ cells.

Geometrically two-dimensional model (restriction to a quarter-domain) with plane-strain approximation, i.e., $\epsilon_{i3} = 0$, and perfectly plastic material behavior. The material parameters are chosen as those of aluminium, $\kappa = 164,206\,N/mm^2$, $\mu = 80,193.80\,N/mm^2$, $\sigma_0 = \sqrt{2/3}\,450$. The boundary traction is given in the form $g(t) = tg_0$, $g_0 = 100$, $t \in [0,6]$. For the stationary Hencky model, the calculations are performed with one load step from $t=0$ to $t=4.5$.

The quantities to be computed are:

- Displacements u_1 and u_2 at various points and stress $\sigma_{22}(P_2)$.

The solutions on very fine (adapted) meshes with about 200,000 cells are taken as reference solutions u_{ref} for determining the relative errors E^{rel} and the effectivity indices I_{eff} of the error estimator.

N	$u_1(P_5)$	E^{rel}	I_{eff}
1000	6.5991e-02	7.7403e-02	0.64
2000	6.3462e-02	3.6121e-02	0.83
4000	6.2159e-02	1.4846e-02	1.04
8000	6.1554e-02	4.9704e-03	1.55
16000	6.1389e-02	2.2746e-03	1.74

Table. Results for $u_1(P_5)$ based on the error estimator $\eta_\omega(u_h)$.

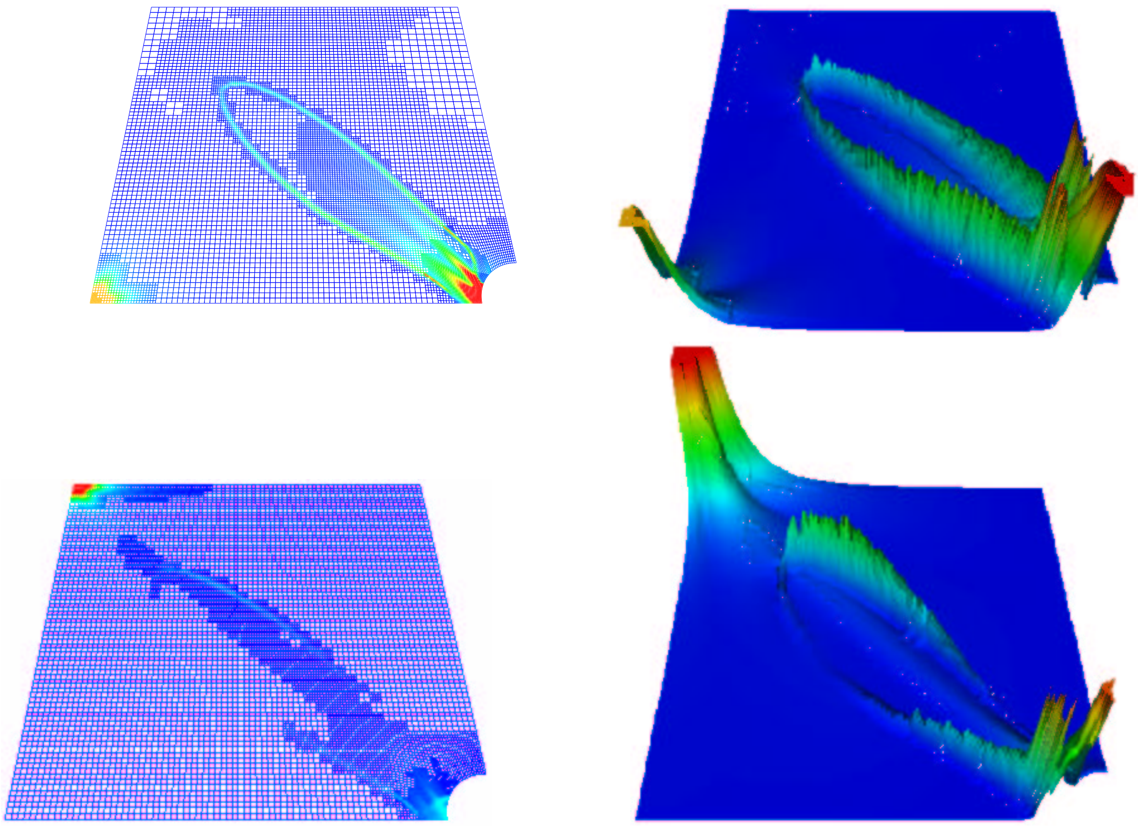


Figure. Optimized meshes for computing $u_1(P_1)$ (top) and $u_1(P_5)$ (bottom) together with corresponding weight distributions ω_T .

4.3 Displacement-pressure discretization

In the plastic region the material behavior is almost incompressible which can cause stability problems. In order to cope with this problem, one may use a stabilised finite element discretization using an auxiliary "pressure" variable. We consider again the Hencky model. The finite element subspaces $V_h \subset V$ and $W_h \subset W$ are supplemented by a subspace $Q_h \subset Q$ for the discrete "pressure".

Discrete problem: Find $\{u_h, \sigma_h, p_h\} \in V_h \times W_h \times Q_h$, such that

$$\begin{aligned}(\sigma_h - \Pi C\epsilon(u_h), \tau_h) + (\sigma_h, \epsilon(\varphi_h)) - (p_h, \nabla \cdot \varphi_h) \;&=\; F(\varphi_h)\\ (\nabla \cdot u_h, \chi_h) + (\kappa^{-1} p_h, \chi_h) \;&=\; 0\end{aligned}$$

for all $\{\varphi, \tau, \chi\} \in V_h \times W_h \times Q_h$. Here, we choose Q_h of "equal-order" as the "displacement space" V_h , i.e., it consists also of continuous, piecewise (isoparametric) bilinear functions **(stability problem)**.

Stabilized scheme:

$$\begin{aligned}(\sigma_h - C(\epsilon(u_h)), \tau_h) + (\sigma_h, \epsilon(\varphi_h)) - (p_h, \nabla \cdot \varphi_h) \;&=\; F(\varphi_h)\\ (\nabla \cdot u_h, q_h) + \kappa^{-1}(p_h, q_h) + \alpha \sum_{K \in \mathbb{T}_h} h_K^2 (\nabla p_h, \nabla q_h)_K \;&=\; 0\end{aligned}$$

for all $\{\varphi_h, \tau_h, q_h\} \in V_h \times W_h \times Q_h$.

Stability estimate:

$$\sup_{u_h \in V_h} \frac{(p_h, \nabla \cdot u_h)}{\|\epsilon(u_h)\|} + \Big(\alpha \sum_{K \in \mathbb{T}_h} \delta_K \|\nabla p_h\|_K^2\Big)^{1/2} \geq \gamma \|p_h\|, \quad p_h \in Q_h$$

Numerical test (F.-T. Suttmeier 2000)

Model problem *"square disc with crack"* with material values $\kappa = 2\mu = 160000$, and boundary traction $g = tg_0$, with $g_0 = 100$ and $t = 2.2340$. Target quantity:

$$J(u) := \int_S u \cdot n \, ds = \int_{\Omega_S} \nabla \cdot u \, dx$$

where S is a suitable circular path around the tip of the crack.

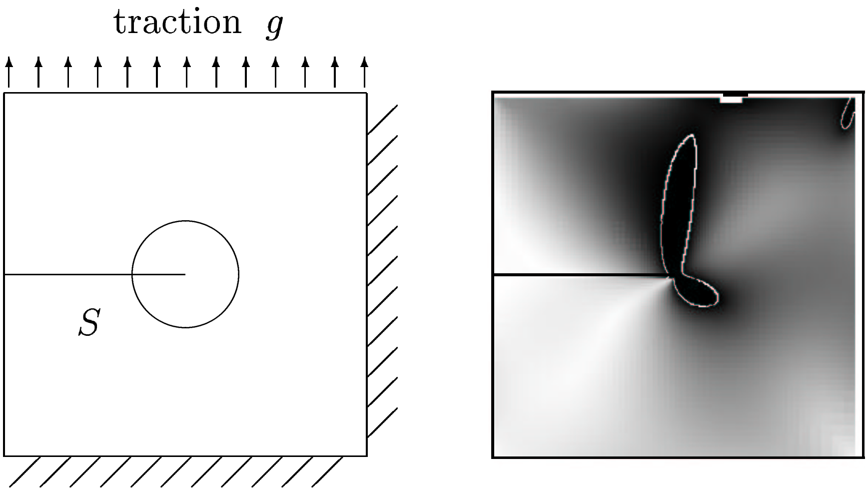


Figure. Geometry of the square disc test problem and plot of $|\sigma^D|$ (plastic regions black) computed on a mesh with $N \approx 64\,000$ cells

	$J(u_h)$	
N	u-form	u/p-form
1000	1.6760e-04	1.693630e-04
2000	1.6817e-04	1.695619e-04
4000	1.6875e-04	1.696680e-04
8000	1.6926e-04	1.699004e-04
16000	1.6963e-04	1.699354e-04
32000	1.6986e-04	1.700872e-04

Table. Results for computing $J_S(u)$ on adaptive grids by the primal and the displacement/pressure discretization ($J_S(u) \approx 1.7020\text{e-}04$).