Adaptive FEM for Inverse Acoustic Scattering

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Introduction: Outline

- Adaptive FEM for time-dependent inverse scattering
- Find wave speed coeff. from meas. wave-re¤ on bdy
- Hybrid £nite element/difference meth: cG(1)cG(1) expl
- Optimization problem: Lagrangian stationary
- Optimality: Jacobian $= 0$
- Optimization: Quasi-Newton BFGS lim stor (< 10 it.)
- Ill-posed: Regularization
- A posteriori est. of reconstruction error
- Hessian: dual weights in a post. (cg)

Acoustic Wave Equation

$$\alpha \frac{\partial^2 p}{\partial t^2} - \triangle p = f, \qquad \text{in } \Omega \times (0,T),$$

$$p(\cdot,0) = 0, \quad \frac{\partial p}{\partial t}(\cdot,0) = 0, \quad \text{in } \Omega,$$

$$p\big|_\Gamma = 0, \quad \text{on } \Gamma \times (0,T),$$

- $p(x,t) \in \mathbf{R}$ is the pressure ($+$ hom bound initial cond),
- $\alpha(x) = 1/c(x)^2$, with $c(x)$ the variable wave speed,
- Forward problem: Given $\alpha(x)$ find $p(x,t)$.
- Inverse problem: Given $p(x,t)$ on boundary, find $\alpha(x)$.

Inverse Scattering

Find the function $\alpha(x)$ which minimizes the cost

$$E(p,\alpha) = \frac{1}{2}\int_0^T \int_\Omega (p-\tilde{p})^2 \delta_{obs} dx dt + \frac{1}{2}\gamma \int_\Omega (\alpha^2 + |\nabla \alpha|^2) \; dx dt,$$

- $p(x,t)$ solves the wave equation with $\alpha(x)$,
- $\tilde{p}$ is observed data at x_{obs} on part of the body,
- $\delta_{obs} = \sum \delta(x_{obs})$ is a sum of multiples of delta-functions,
- $\gamma > 0$ is a regularization parameter.

Lagrangian

$$L(u) = E(p,\alpha) - ((\alpha Dp, D\varphi)) + ((\nabla p, \nabla \varphi)) - ((f,\varphi)),$$

where

- $u = (p, \varphi, \alpha)$
- $D = \frac{\partial}{\partial t}$
- $\varphi(x,t)$ dual variable $\varphi(\cdot,T) = D\varphi(\cdot,T) = 0$
- $((\cdot,\cdot))$ space-time L_2 inner product.

Search for a stationary point u satisfying: $L'(u;\cdot) = 0$, where $L'(u;\cdot)$ is the Jacobian of L at u.

$L'(u,\cdot) = 0, \quad p(0) = \varphi(T) = 0$

In strong form:

$$\begin{aligned}
\alpha D^2 p - \triangle p &= f \\
\alpha D^2 \varphi - \triangle \varphi &= -(p - \tilde{p})\delta_{obs} \\
-\gamma \triangle \alpha + \gamma \alpha - \int_0^T \frac{\partial p}{\partial t}\frac{\partial \varphi}{\partial t}\, dt &= 0 \\
p(0) = Dp(0) = \varphi(T) = D\varphi(T) &= 0
\end{aligned}$$

Solve by Adaptive Galerkin cG(1)cG(1) in space-time. Mass-lumping gives explicit time-stepping.

FEM for forward problem

$$((p,q)) = \int_\Omega \int_0^T pq\,dx\,dt, \quad \|p\|^2 = ((p,p)),$$

Trial space:

$$W_h^p := \{w \in W^p : w|_{K\times J} \in P_1(K) \times P_1(J),\},$$

where $P_1(K)$ and $P_1(J)$ denote the set of linear functions on K and J, respectively, with $\{K\}$ space triangulation and $\{J\}$ tim intervals.

$$W^p := \{w \in H^1(\Omega \times I) : w(\cdot,0) = 0, w|_\Gamma = 0\}.$$

FEM for forward problem

Test space:

$$W_h^\varphi := \{w \in W^\varphi : w|_{K\times J} \in P_1(K) \times P_1(J)\},$$

where

$$W^\varphi := \{w \in H^1(\Omega \times I) : w(\cdot, T) = 0, w|_\Gamma = 0\}.$$

Find $p_h \in W_h^p$ such that $\forall \bar{\varphi} \in W_h^\varphi$,

$$-((\alpha D p_h, D\bar{\varphi})) + ((\nabla p_h, \nabla \bar{\varphi})) = ((f, \bar{\varphi})). \qquad (1)$$

where the initial condition $Dp(0) = 0$ is imposed in weak form through the variational formulation.

FEM for inverse problem

Define

$$V_h := \{v \in L_2(\Omega) : v \in P_0(K), \forall K \in K_h\},$$

$$U_h = W_h^p \times W_h^\varphi \times V_h$$

FEM: Find $u_h \in U_h$, such that

$$L'(u_h; \bar{u}) = 0 \ \ \forall \bar{u} \in U_h,$$

Solve by Quasi-Newton BFGS with limited storage.

A posteriori for Lagrangian

$$\begin{aligned} L(u) - L(u_h) &= \int_0^1 \frac{d}{d\epsilon} L(\epsilon u + (1-\epsilon) u_h) d\epsilon \\ &= \int_0^1 L'(\epsilon u + (1-\epsilon) u_h; u - u_h) d\epsilon \\ = L'(u_h; u - u_h) + R &= L'(u_h; u - u_h^I) + R \end{aligned}$$

where R is a second order remainder term, and we used the Galerkin orthogonality writing $u - u_h = (u - u_h^I) + (u_h^I - u_h)$ where $u_h^I \in U_h$ denotes an interpolant of u. Neglecting the term R, we get the following a post error estimate:

A posteriori for Lagrangian

$$
\begin{aligned}
|L(u) - L(u_h)| \leq & \int_0^T \int_\Omega R_{p_1} \sigma_\varphi \; dxdt + \int_0^T \int_\Omega R_{p_2} \sigma_\varphi \; dxdt \\
+ & \int_0^T \int_\Omega R_{p_3} \sigma_\varphi \; dxdt + + \int_0^T \int_\Omega R_{\varphi_1} \sigma_p \; dxdt \\
+ & \int_0^T \int_\Omega R_{\varphi_2} \sigma_p \; dxdt + \int_0^T \int_\Omega R_{\varphi_3} \sigma_p \; dxdt \\
+ & \int_0^T \int_\Omega R_\alpha \sigma_\alpha \; dx
\end{aligned}
$$

where the different residuals R are defined as

Residuals

$$R_{p_1} = \big|f\big|, \qquad R_{p_2} = \max_{S\subset\partial K} h_k^{-1}\big|\big[\partial_s p_h\big]\big|,$$

$$R_{p_3} = \alpha_h \tau^{-1}\big|\big[\partial_t p_h\big]\big|, \qquad R_{\varphi_1} = \big|p_h - \tilde{p}\big|,$$

$$R_{\varphi_2} = \max_{S\subset\partial K} h_k^{-1}\big|\big[\partial_s \varphi_h\big]\big|, \qquad R_{\varphi_3} = \alpha_h \tau^{-1}\big|\big[\partial_t \varphi_h\big]\big|,$$

$$R_\alpha = \left|\frac{\partial \varphi_h}{\partial t}\right| \cdot \left|\frac{\partial p_h}{\partial t}\right|,$$

Weights

The different weights σ have the following form:

$$
\begin{aligned}
\sigma_\varphi &= C_1\tau\,|[\partial_t\varphi_h]| + C_1 h\,|[\partial_s\varphi_h]|\,, \\
\sigma_p &= C_1\tau\,|[\partial_t p_h]| + C_1 h\,|[\partial_s p_h]|\,, \\
\sigma_\alpha &= C_2\big|[\alpha_h]\big|,
\end{aligned}
$$

A posteriori for reconstruction

Dual problem: Find $\tilde{u}$ such that

$$-L''(u_h;\bar{u},\tilde{u}) = (\psi,\bar{u}) \ \ \forall \bar{u},$$

where ψ acts as given data, and $L''(u;\cdot,\cdot)$ is the Hessian of the Lagrangian $L(u)$ at u, which expresses the sensitivity of the Jacobian $L'(u;\cdot)$ with respect to changes in u. Choosing here $\bar{u} = u - u_h$ and using the fact that $L''(u;\bar{u},\tilde{u})$ is symmetric in $\bar{u}$ and $\tilde{u}$, the following error representation:

$$\begin{aligned}((\psi, u - u_h)) &= -L''(u_h; u - u_h, \tilde{u}) \\ &= -L'(u;\tilde{u}) + L'(u_h;\tilde{u}) + R \\ &= L'(u_h;\tilde{u}) + R = L'(u_h;\tilde{u} - \tilde{u}^I) + R\end{aligned}$$

A posteriori for reconstruction

$$((\psi, u - u_h)) \approx L'(u_h; \tilde{u} - \tilde{u}^I),$$

In part. the term

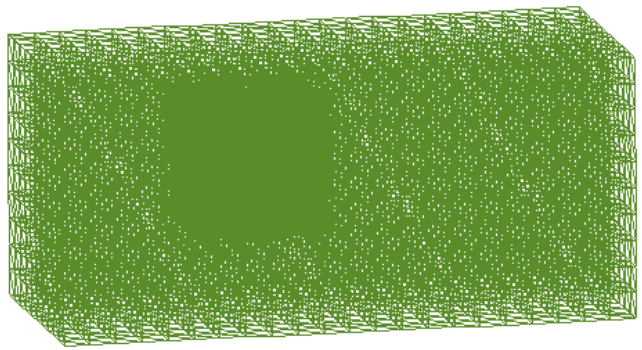
$$\int_0^T \int_\Omega R_\alpha \sigma_{\tilde{\alpha}} \; dx$$

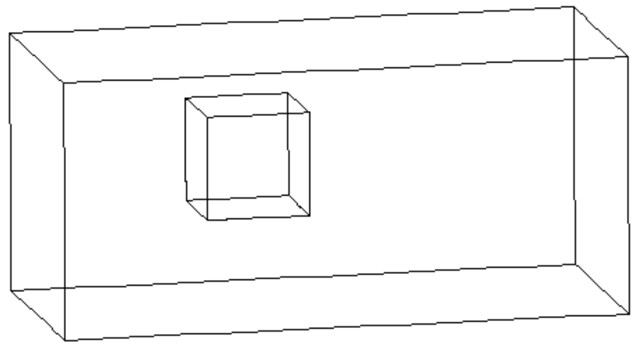
with

$$\sigma_{\tilde{\alpha}} = C_2 \big| [\tilde{\alpha}_h] \big|,$$

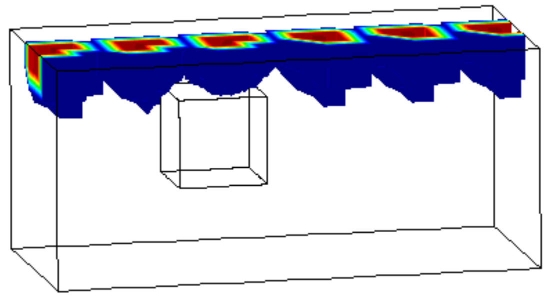
Note: the jump $[\tilde{\alpha}]$ of a computed $\tilde{\alpha}$, appears as a weight in the equation expressing stationarity with respect to the coef£cient α: sensitivity in the identification of α.

Cubic Scatterer

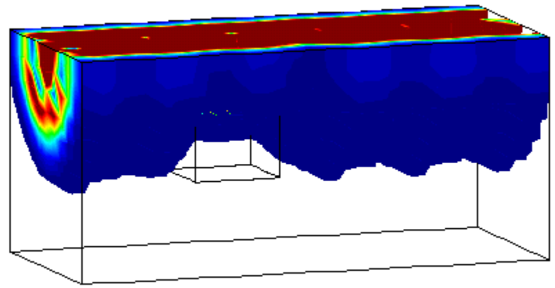




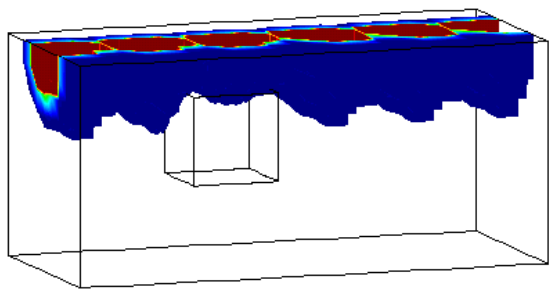
Experiment: Six pulses



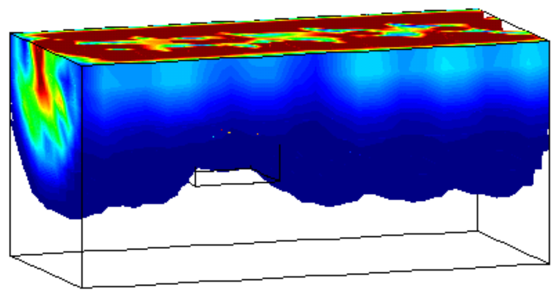
t=0.2



t=0.7

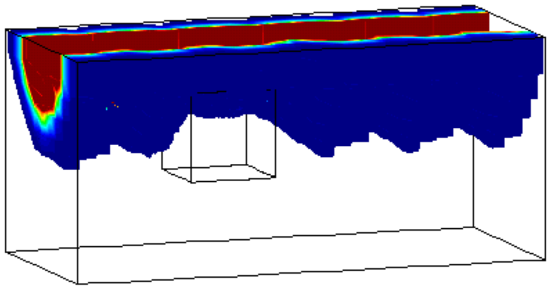


t=0.3

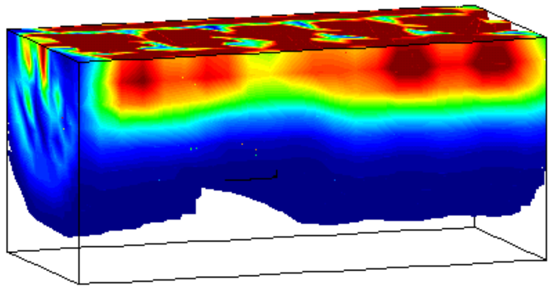


t=0.9

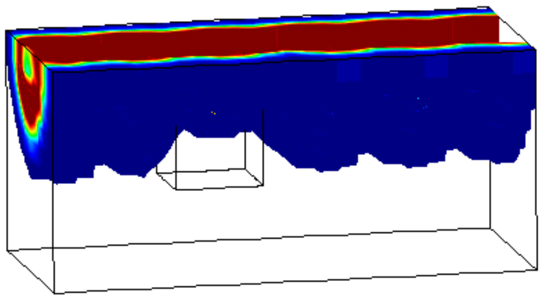
Six pulses



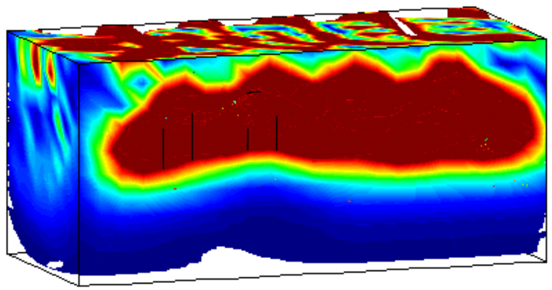
t=0.5



t=1.1

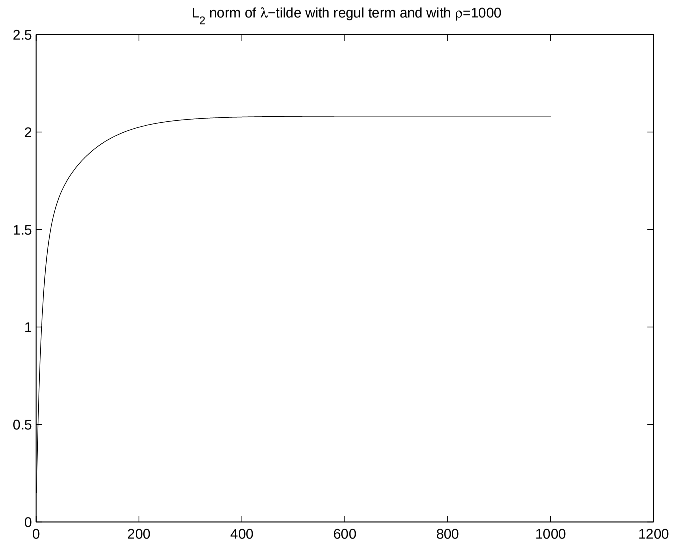


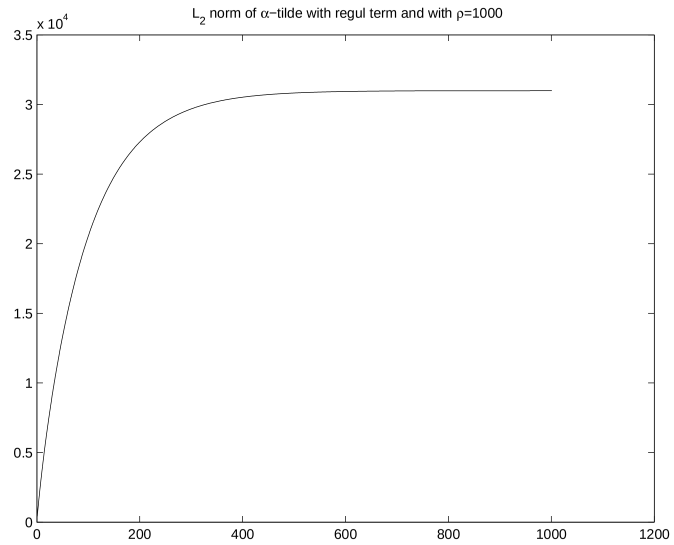
t=0.6



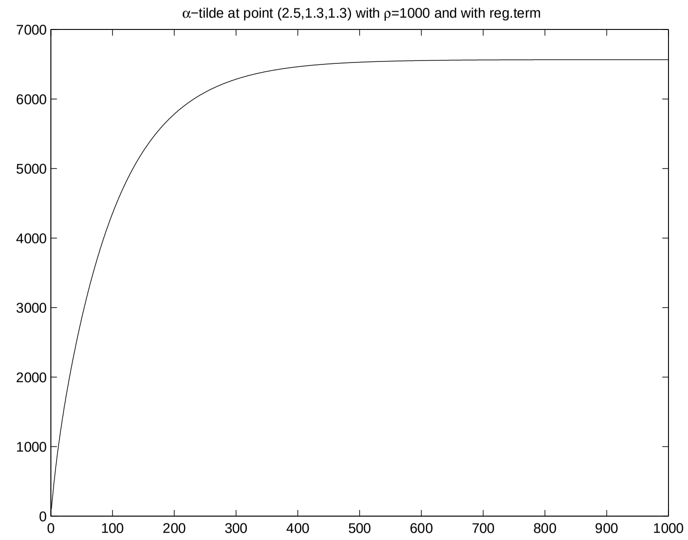
t=1.5

Hessian sol. $\gamma = 0.0001$, $\rho = 1000$

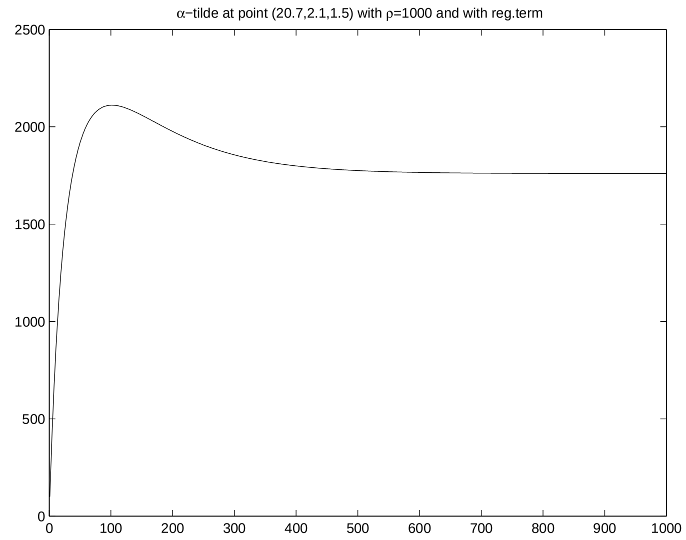




Hessian sol $\gamma = 0.0001$, $\rho = 1000$



a)

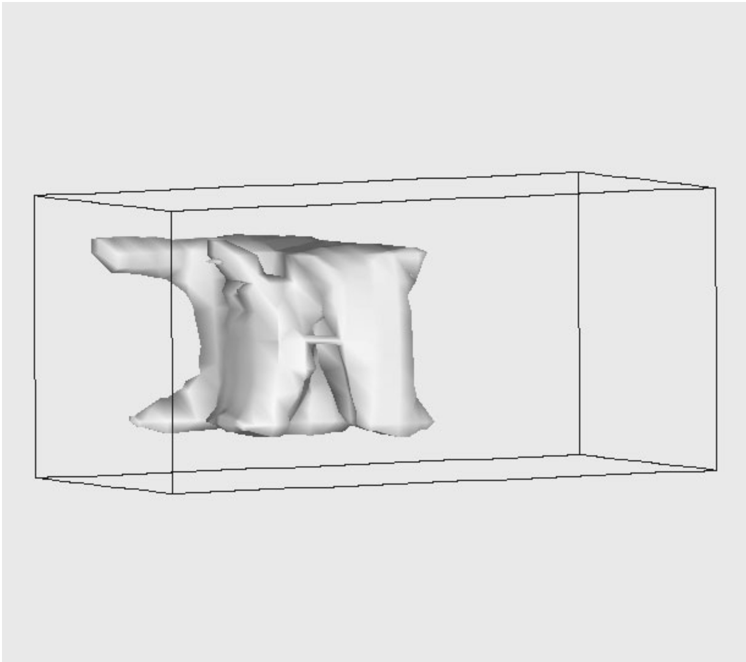


b)

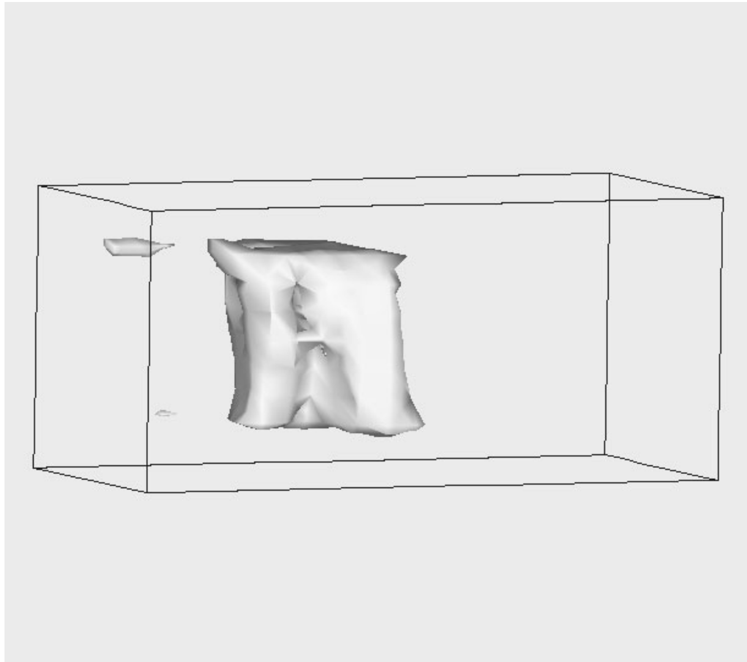
Cost: $\gamma = 0.0001$

opt.it.	2783 nodes	2847 nodes	3183 nodes	3771 nodes	4283 nodes	6613 nodes
1	0.0493302	0.0516122	0.051569	0.0529257	0.0535081	0.0537523
2	0.0405683	0.0423093	0.0419412	0.0428817	0.0433272	0.0439134
3	0.0235056	0.0239327	0.0245081	0.0271383	0.0285571	0.031920
4	0.0191902	0.0192185	0.0187792	0.0205331	0.0221997	0.0239426
5	0.0115005	0.0110448	0.0174202		0.0205711	0.0104240
6			0.0156732		0.0112331	0.0101503
7			0.0121359		0.0102246	

Reconstruction of cube

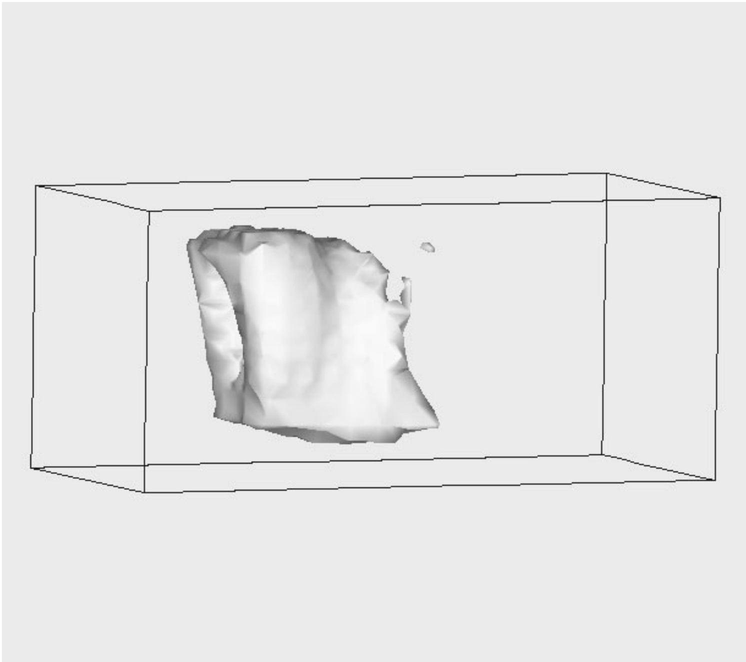


coarse mesh, 9 q.N. it., $\alpha \approx 1.21$

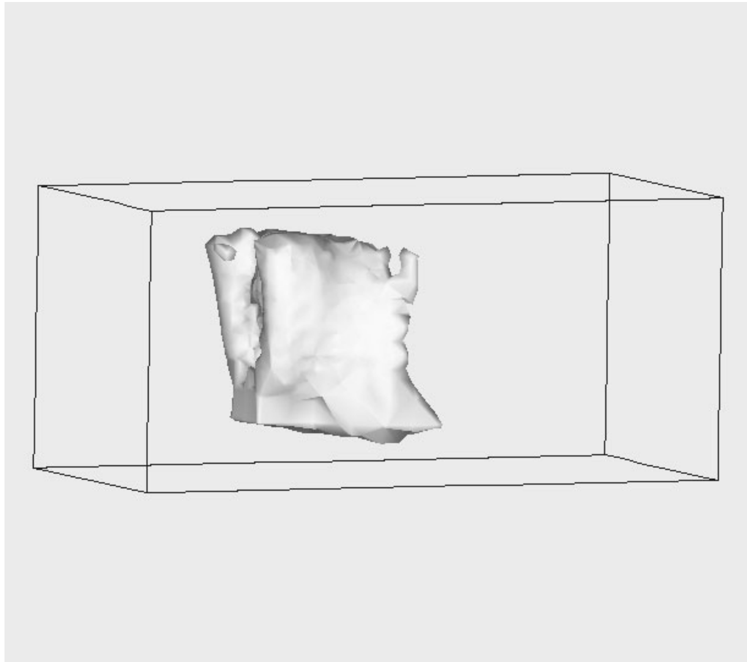


1 time ref., 8 q.N.it., $\alpha \approx 1.27$

Reconstruction of cube

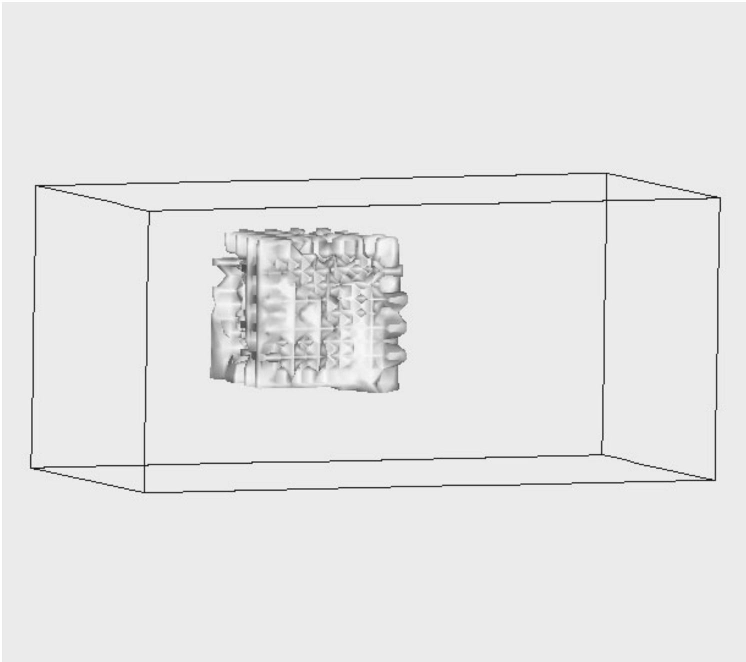


3 times ref.mesh, 4 q.N.it, $\alpha \approx 1.23$

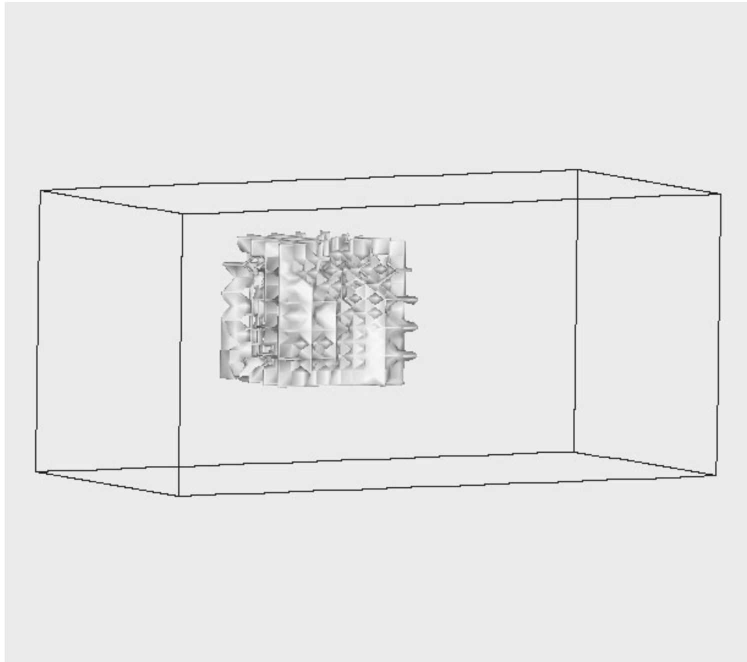


4 times ref.mesh, 9 q.N.it, $\alpha \approx 1.23$

Reconstruction of cube



5 times ref.mesh, 8 q.N.it, $\alpha \approx 1.74$



5 times ref.mesh, 9 q.N.it, $\alpha \approx 1.91$

A posteriori estimate for reconstruction

$$E_\alpha \leq 0.28$$

Notice that the jump $[\tilde{\alpha}]$ across inter-element edges of $\tilde{\alpha}$ occurs as a weight in the a posteriori error estimate, not the value of $\tilde{\alpha}$ itself, and that the jump $[\tilde{\alpha}]$ is less sensitive to the smallness of the regularization parameter γ than $\tilde{\alpha}$ itself.

Conclusion

- 3d time-dep inverse scattering problems on a PC
- 100.000 mesh points in space in 1 day (10 it)
- Upscaling: With 1000 PC 100 milj mesh points in 1 day?
- Adaptive error control
- Sensitivity in reconstruction from Hessian.
- Adaptive choice of regularization.

Adaptive DNS/LES: A New Approach to Computational Turbulence Modeling

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Overview

- Introduction to Computational Turbulence Modeling: Adaptive DNS/LES vs traditional turbulence modeling
- Solving benchmark problems with Adaptive DNS/LES
- Subgrid modeling, dissipation, and eddy viscosity: Adaptive DNS/LES vs traditional turbulence modeling
- Structure of Adaptive DNS/LES: discretization, a posteriori error estimates,...
- Mathematical aspects: uniqueness in output of weak solutions to the Navier-Stokes equations

Mathematical Modeling of Fluid Flow

Navier-Stokes Equations NSE (1821-1845)

$$\dot{u} + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0$$

(u velocity, p pressure, and ν kinematic viscosity)

Reynolds number Re $= \dfrac{UL}{\nu}$

(for many applications of industrial importance Re $\geq 10^6$)

Full resolution in a DNS may require Re^3 space-time points

- Pointwise approx. of NSE in general impossible (today)
- Compute instead (mean) output $M(u,p)$ from NSE: mean values, forces (drag, lift),...

Traditional approach: Averaging

(1) Averaging (filtering) of NSE $\Rightarrow$ RANS, LES, etc. modi£ed equations for a mean value $(\bar{u}, \bar{p})$ of the exact NSE solution (u, p), including the Reynolds Stresses τ $(\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j)$

(2) Compute $(\bar{u}, \bar{p})$ from averaged equations (including some type of subgrid modeling of τ)

(3) Approximate output $M(u, p) \approx M(\bar{u}, \bar{p})$

- Closure Problem: How to model τ in a subgrid model?

  A: Some dissipative (eddy viscosity) subgrid model??
  (typically $\nabla \cdot \tau = \nu_T \Delta \bar{u}$, with ν_T the eddy viscosity)

New approach: Adaptive DNS/LES

- No averaging: compute output $M(u,p)$ directly from NSE
  $\Rightarrow$ No Reynolds stresses to model!!
- Stabilized Galerkin G^2 FEM discretization for NSE (introduces dissipation in turbulent part of domain)
- Quality assessement: A posteriori error estimation
- Output sensitivity information: Dual problems
- Ef£ciency: Adaptivity with respect to output $M(u,p)$
  - Adaptive discretization (mesh refinement)
  - Adaptive DNS/LES in laminar/turbulent flow

New approach: Adaptive DNS/LES

A posteriori error estimate (using duality arguments):

$$|M(u,p) - M(U_h,P_h)| = |\sum_{K\in\mathcal{T}} \mathcal{E}_K| \leq \sum_{K\in\mathcal{T}} |\mathcal{E}_K|$$

Adaptive algorithm: From a coarse mesh $\mathcal{T}^0$ and $k=0$ do

(1) compute approximation to the primal problem on $\mathcal{T}^k$

(2) compute approximation to the dual problem on $\mathcal{T}^k$

(3) if $\sum_{K\in\mathcal{T}^k} |\mathcal{E}_K^k| <$ TOL then STOP, else

(4) re£ne elements $K \in \mathcal{T}^k$ with largest $\mathcal{E}_K^k \quad \rightarrow \quad \mathcal{T}^{k+1}$

(5) set $k = k+1$, then goto (1)

Ex: Generic Bluff Body Problems

Compute output: $M(u,p) = \bar{c}_D = \frac{1}{T}\int_0^T c_D(t)\,dt$

(mean drag coef£cient c_D over a time interval $[0,T]$)

Standard benchmark problems:

(Rodi, Ferziger, Breuer, and Pourquié, 1997)

Ex 1: surface mounted cube, $Re = 40.000$, $T = 40H$

Domain: $15H \times 2H \times 7H$,   Cube: $H \times H \times H$

Ex 2: square cylinder, $Re = 22.000$, $T = 100D$

Domain: $21D \times 14D \times 4D$,   Cylinder: $D \times D \times 4D$

Ex: Surface mounted cube

- No experimental reference values for $\bar{c}_D$?!
- Krajnović and Davidson (LES, 2002): $\bar{c}_D \approx 1.12 - 1.24$

Adaptive DNS/LES cG(1)cG(1): $\bar{c}_D \approx 1.5$

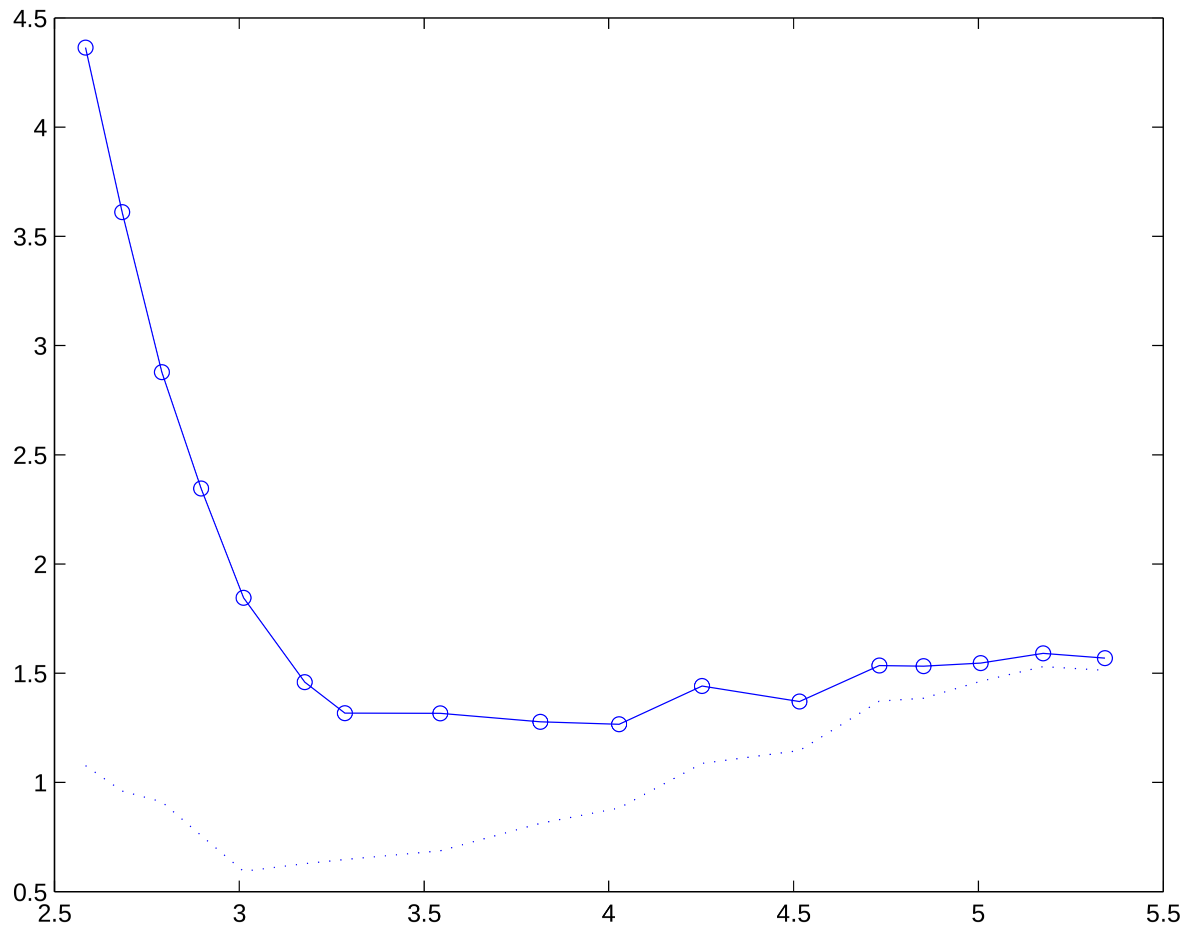
Movie: primal velocity $|u|$ (adaptive step 17)

Movie: primal velocity $|u|$ (adaptive step 17)

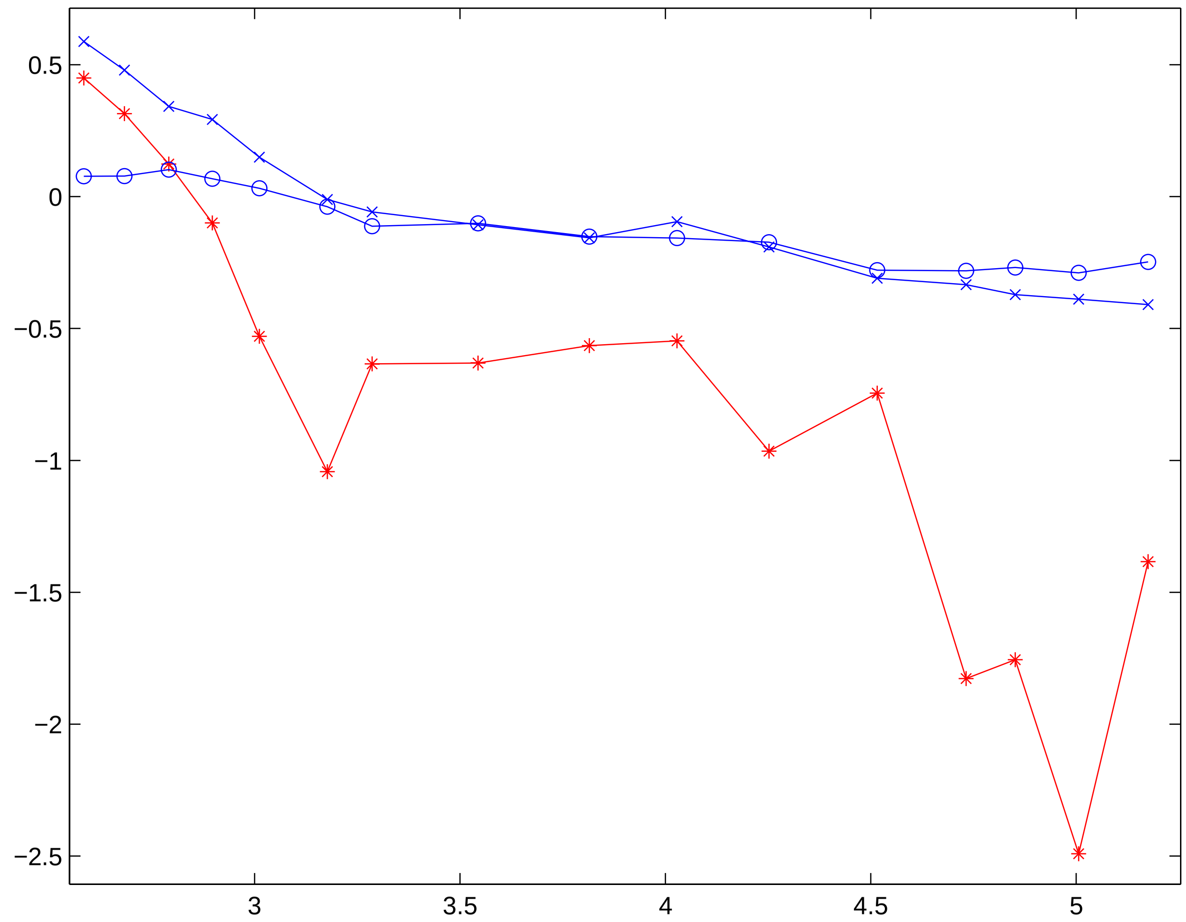
Movie: primal pressure $|p|$ (adaptive step 17)

Movie: primal pressure $|p|$ (adaptive step 17)

Ex: Surface mounted cube: $\bar{c}_D$



Ex: Surface mounted cube: error



Ex: Surface mounted cube: dual

$$-\dot{\varphi}_h - (u \cdot \nabla)\varphi_h + \nabla U_h \cdot \varphi_h - \nu \Delta \varphi_h + \nabla \theta_h = \psi, \quad \nabla \cdot \varphi_h = 0$$

Linear dual problem (data ψ coupling to output $\bar{c}_D = (u, \psi)$)

Movie: dual velocity $|u|$ *(adaptive step 16)*

Movie: dual velocity $|u|$ *(adaptive step 16)*

Movie: dual pressure $|p|$ *(adaptive step 16)*

Movie: dual pressure $|p|$ *(adaptive step 16)*

Movie: initial mesh

Movie: initial mesh

Movie: computational mesh (adaptive step 17)

Movie: computational mesh (adaptive step 17)

Ex: Square cylinder: transitional flow

- Experiments: $\bar{c}_D \approx 1.9 - 2.1$
  (longer cylinder, 2% turbulence in in¤ow, $Re \approx 21.400$)
- LES: $\bar{c}_D \approx 1.66 - 2.77$
- RANS: $\bar{c}_D \approx 1.637 - 2.004$

Adaptive DNS/LES cG(1)cG(1): $\bar{c}_D \approx 2.0 - 2.4$ (prel.)

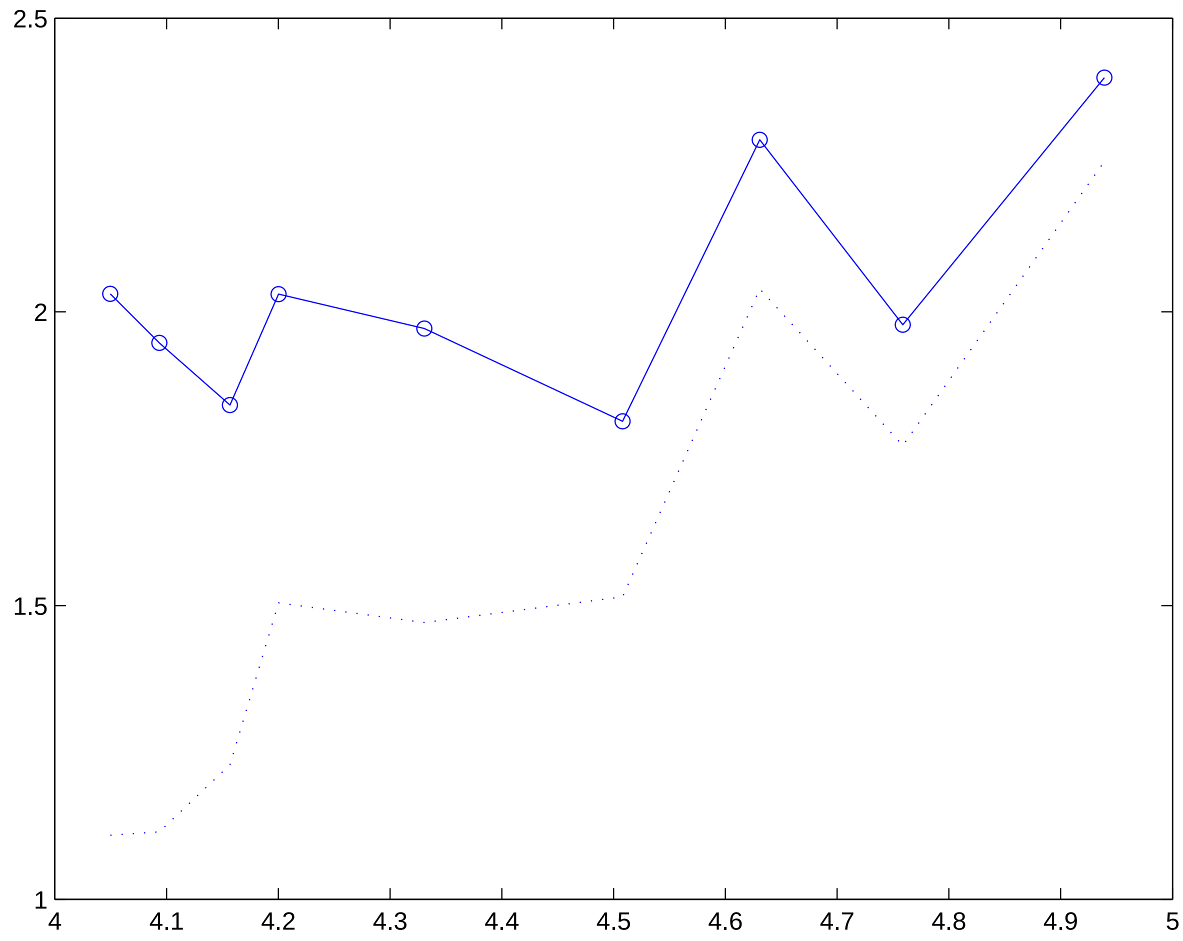
Movie: primal velocity $|u|$ (adaptive step 9)

Movie: primal velocity $|u|$ (adaptive step 9)

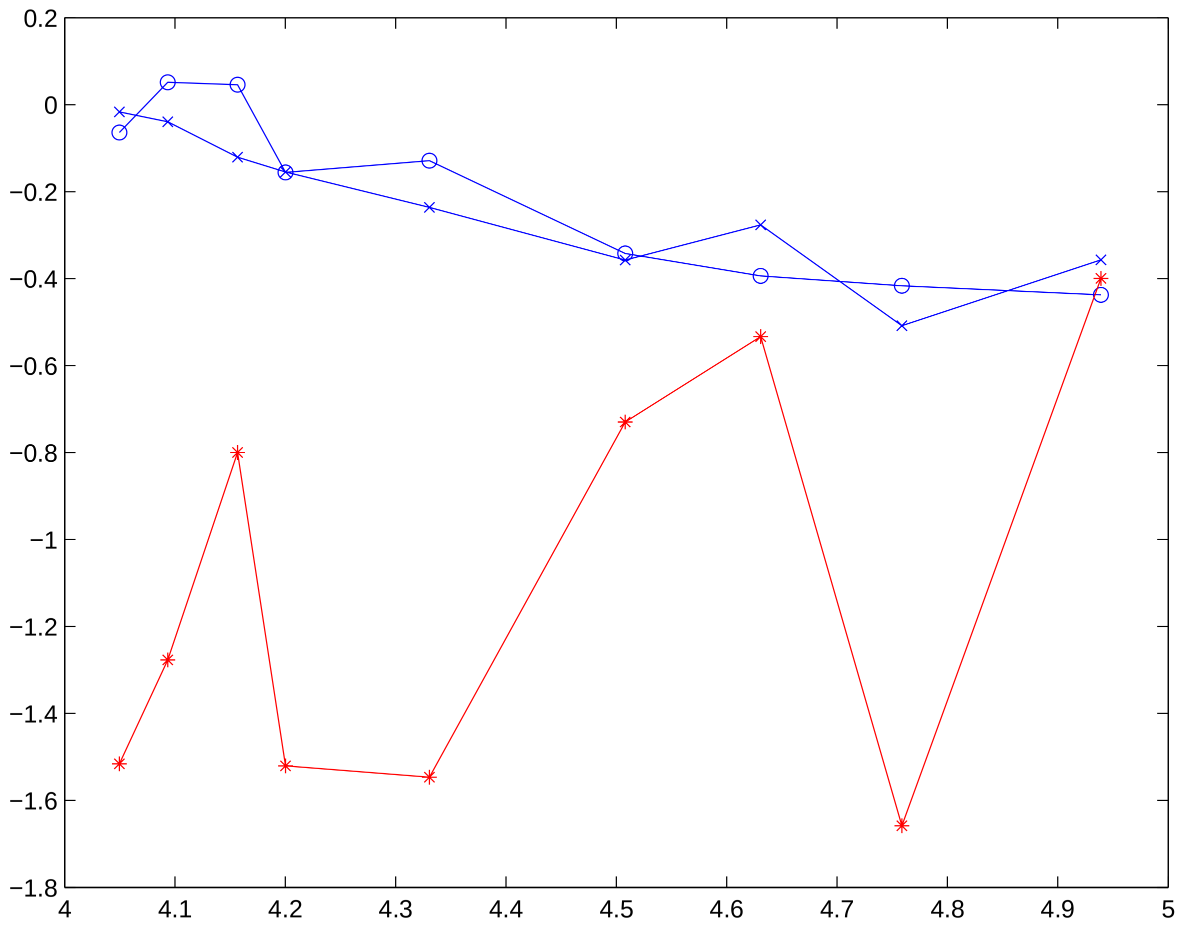
Movie: primal pressure $|p|$ (adaptive step 9)

Movie: primal pressure $|p|$ (adaptive step 9)

Ex: Square cylinder: $\bar{c}_D$



Ex: Square cylinder: error



Ex: Square cylinder: dual

$$-\dot{\varphi}_h - (u \cdot \nabla)\varphi_h + \nabla U_h \cdot \varphi_h - \nu \Delta \varphi_h + \nabla \theta_h = \psi, \quad \nabla \cdot \varphi_h = 0$$

Linear dual problem (data ψ coupling to output $\bar{c}_D = (u, \psi)$)

Movie: dual velocity $|u|$ (adaptive step 9)

Movie: dual velocity $|u|$ (adaptive step 9)

Movie: dual pressure $|p|$ (adaptive step 9)

Movie: dual pressure $|p|$ (adaptive step 9)

Movie: initial mesh

Movie: initial mesh

Movie: computational mesh (adaptive step 9)

Movie: computational mesh (adaptive step 9)

Computational cost

We obtain (reasonably) “accurate” approximation of $\bar{c}_D$

- using less than 10^5 mesh points in space!
- using less than 1Gb memory!
- using the computational power of a PC!

Could it be possible to compute mean quantities in 3D turbulent ¤ow on a PC (or a laptop)?! For any Re !?

Subgrid modeling and dissipation

- ■ RANS, LES, etc. : Reynolds stress subgrid models
- ■ Adaptive DNS/LES: No Reynolds stresses to model!!
- □ Stabilization act as adaptive subgrid model, introducing dissipation of size h in turbulent parts of the ¤ow
- □ Too large stabilization $\Rightarrow$ large estimated error
- □ Too small stabilization $\Rightarrow$ numerics explode

For accurate drag, we need to capture the total dissipation:

(1) Correct volume of turbulent wake

(2) Correct dissipation intensity $\epsilon(x, t)$ in turbulent wake

Subgrid modeling and dissipation

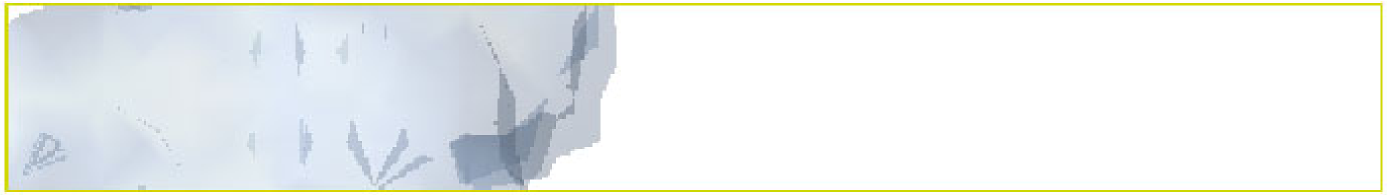
Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 1

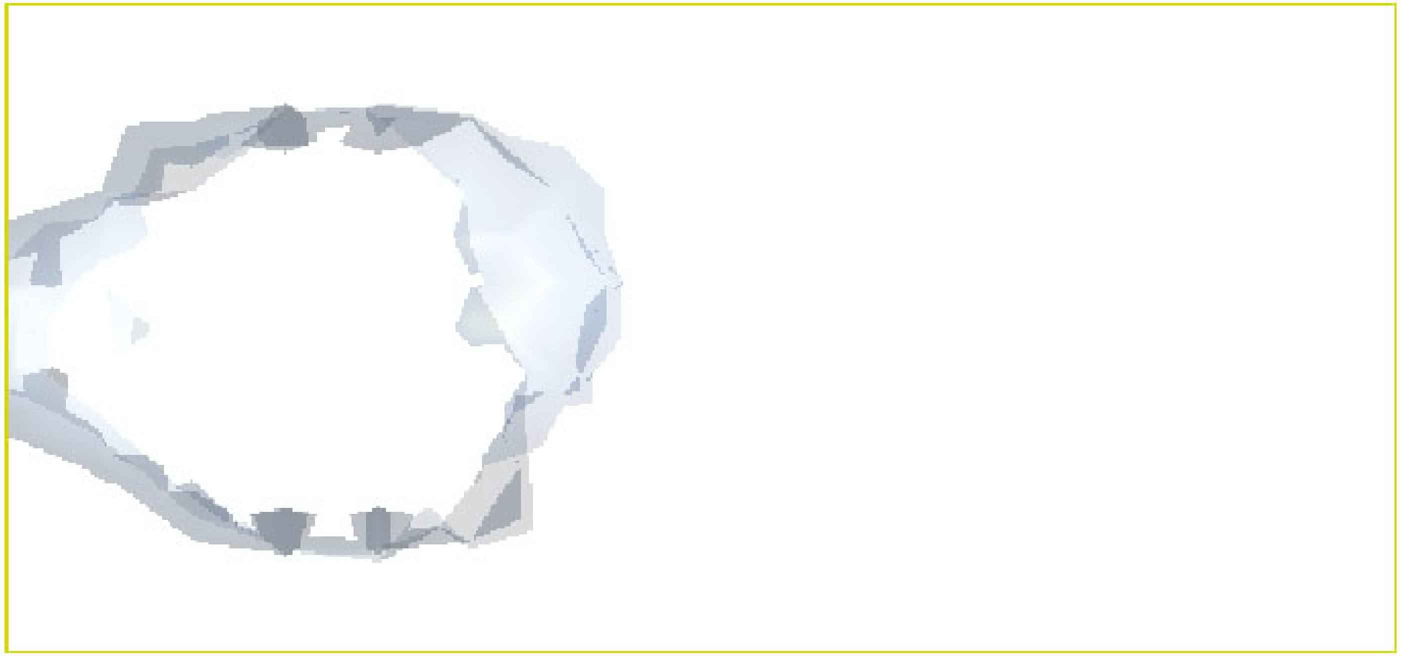
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 2

Subgrid modeling and dissipation

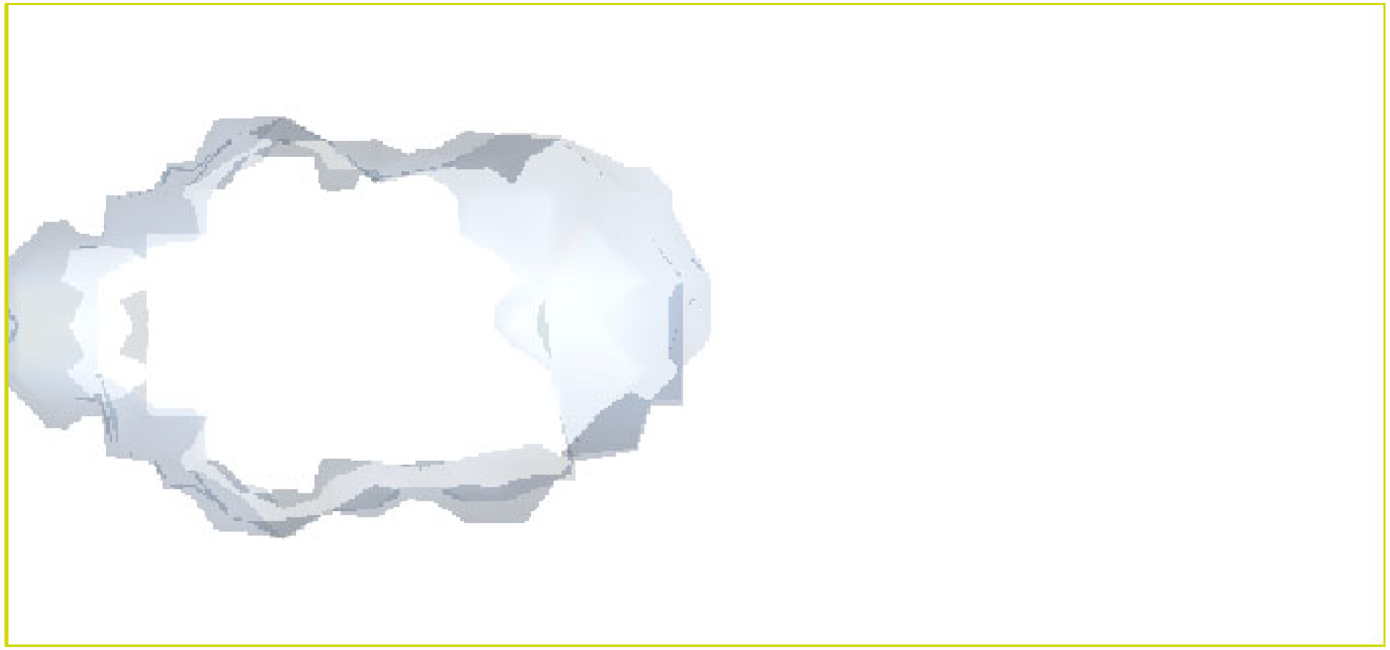
Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 3





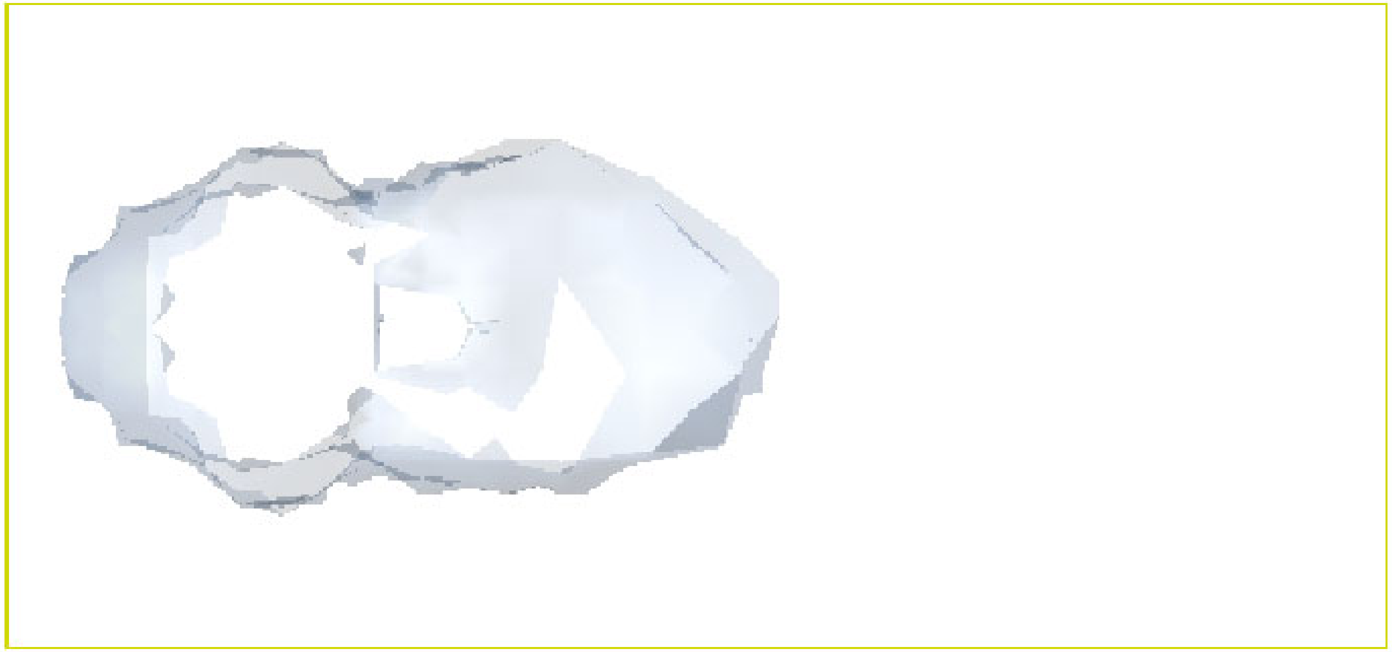
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 4



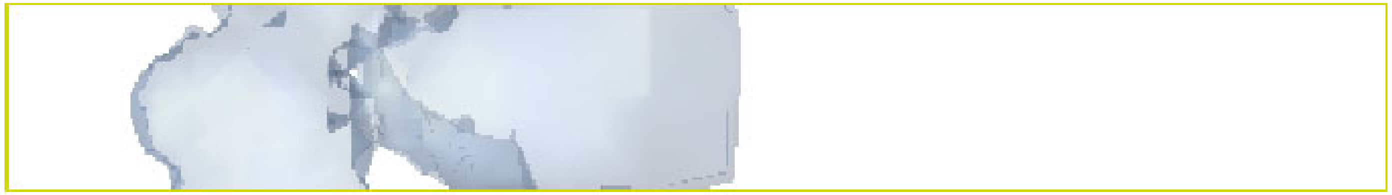
Subgrid modeling and dissipation

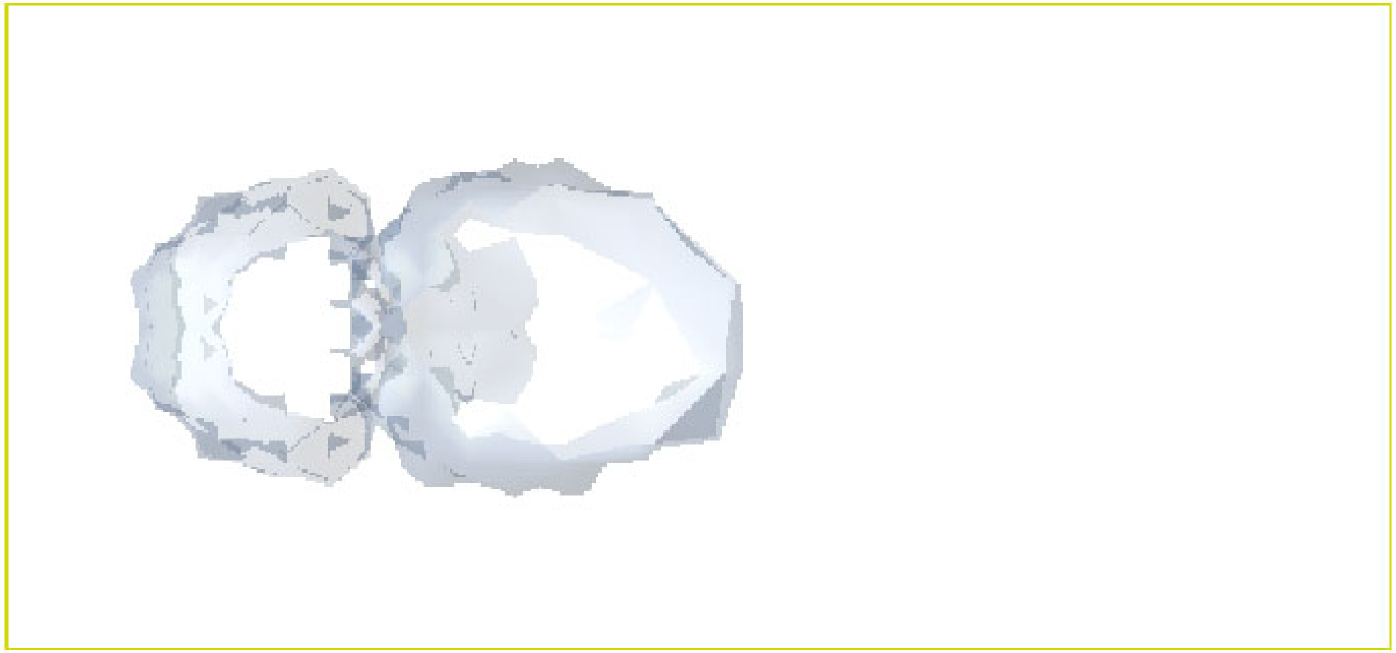
Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 5



Subgrid modeling and dissipation

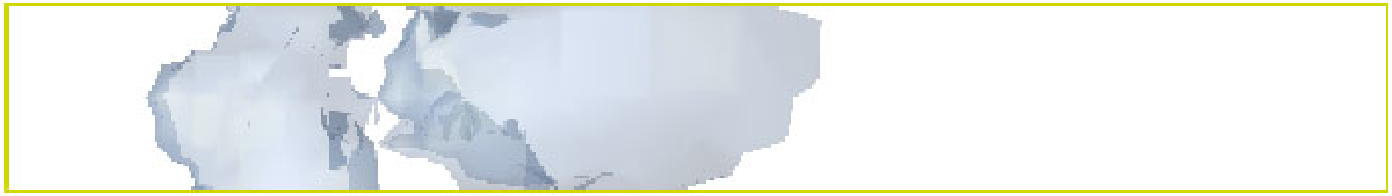
Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 6

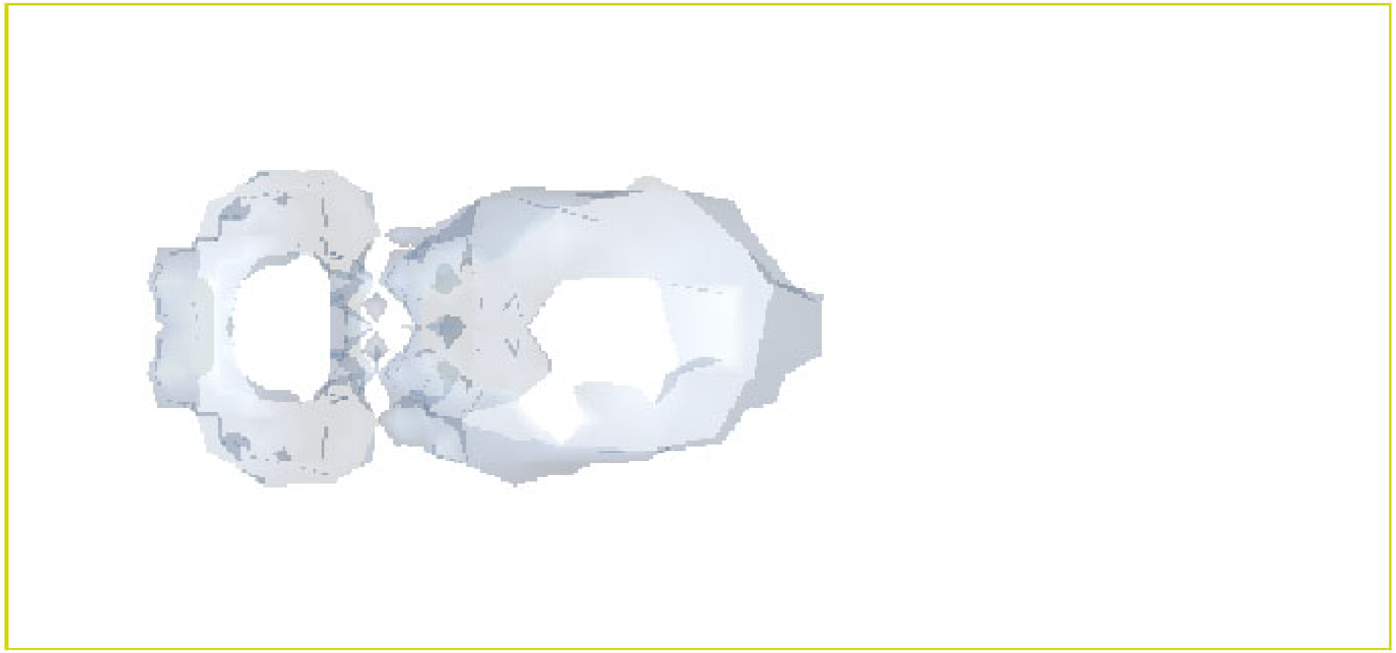




Subgrid modeling and dissipation

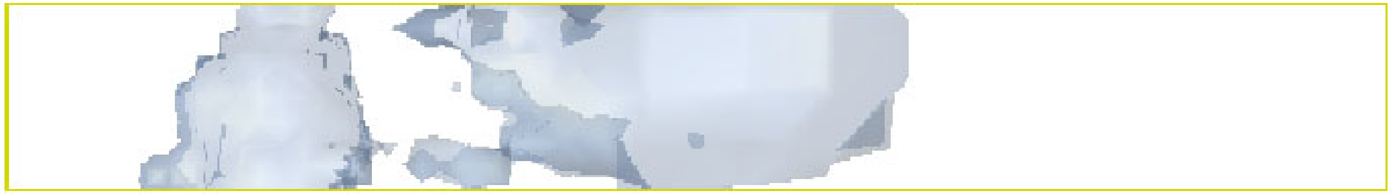
Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 7

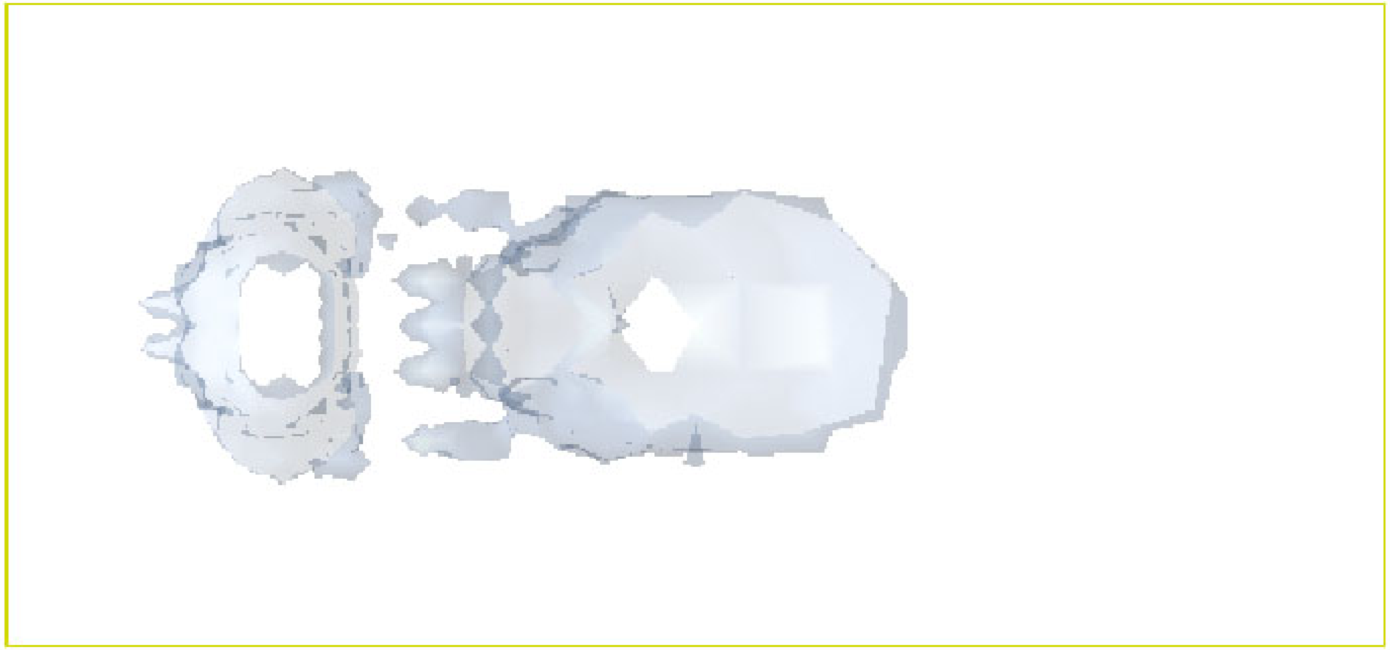




Subgrid modeling and dissipation

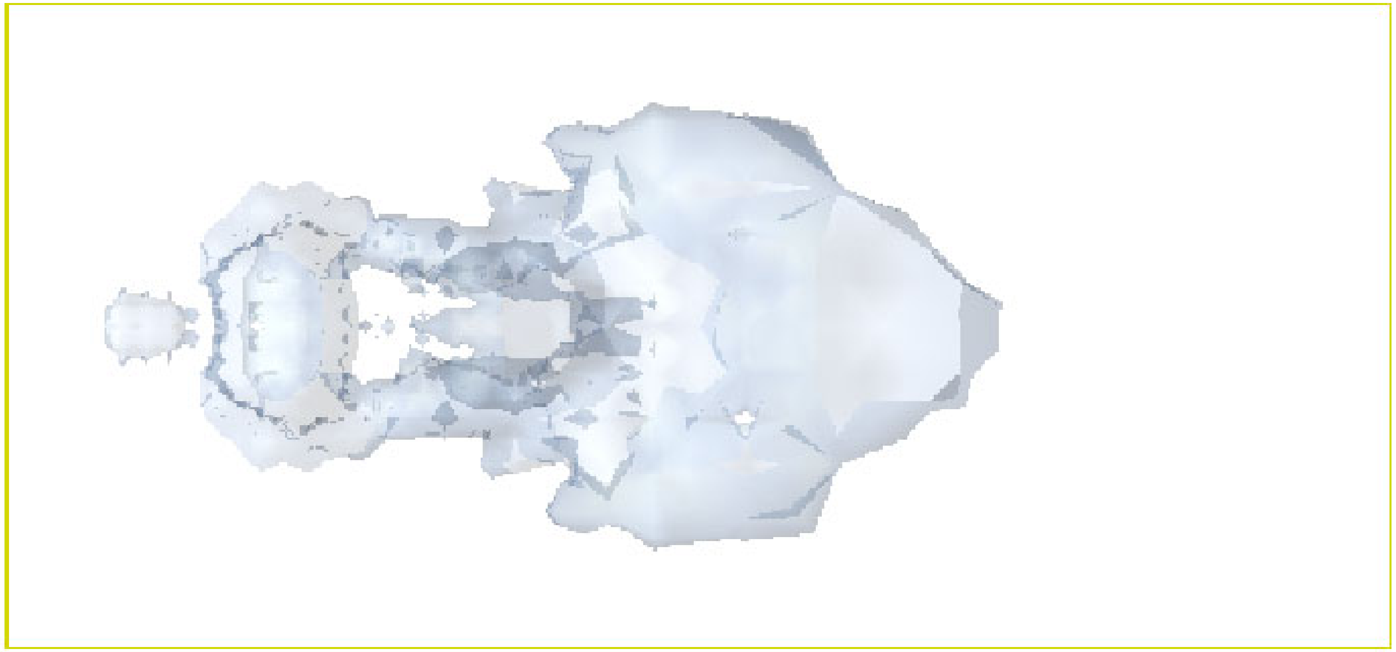
Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 8





Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 9

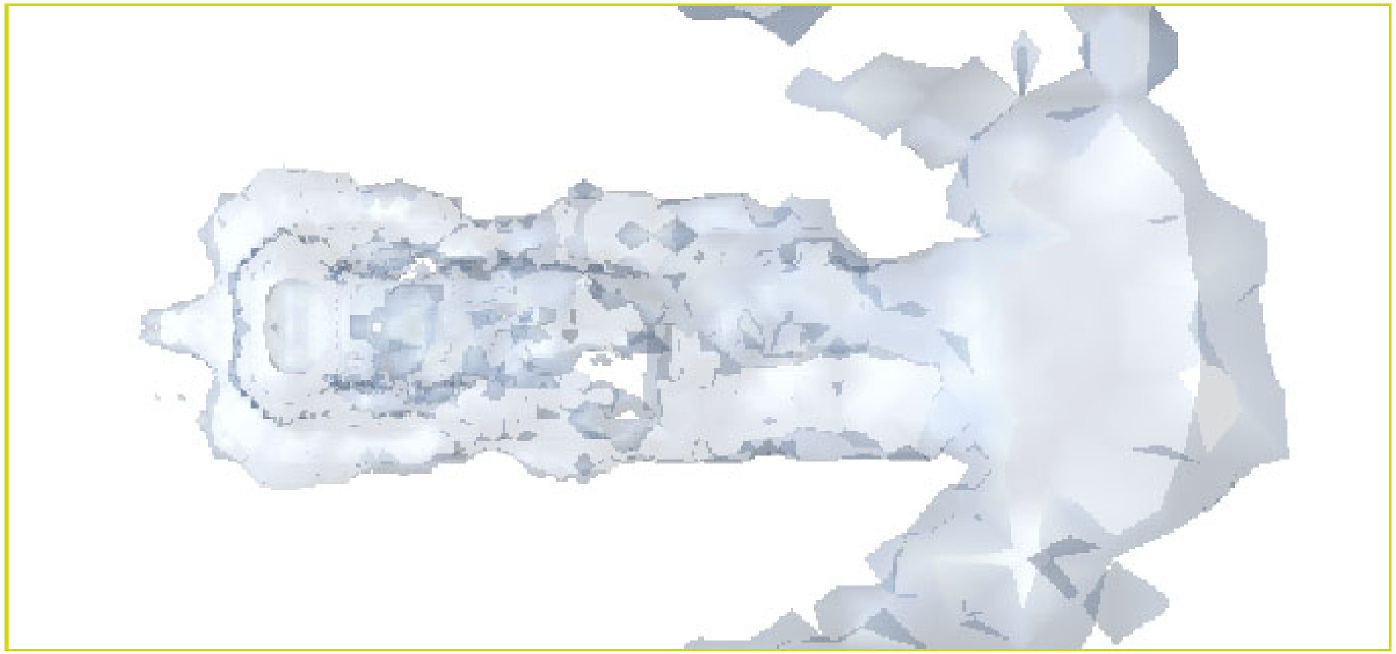


Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 10

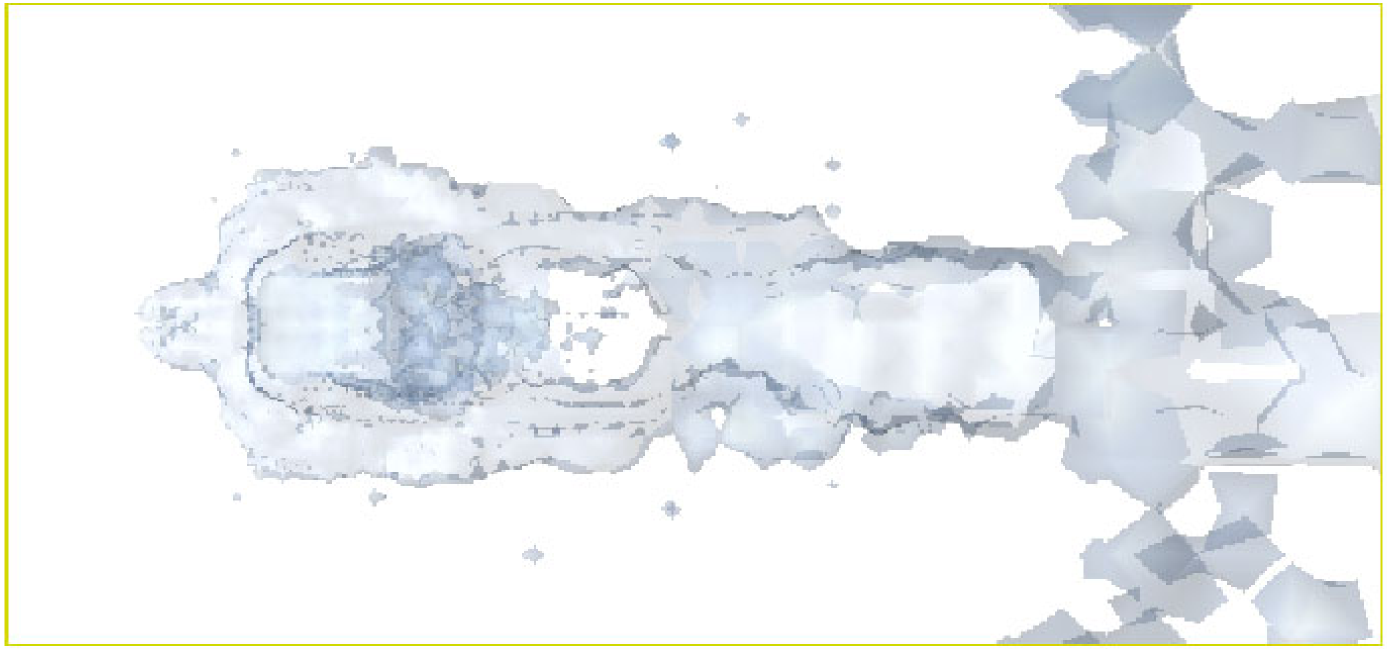
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 11



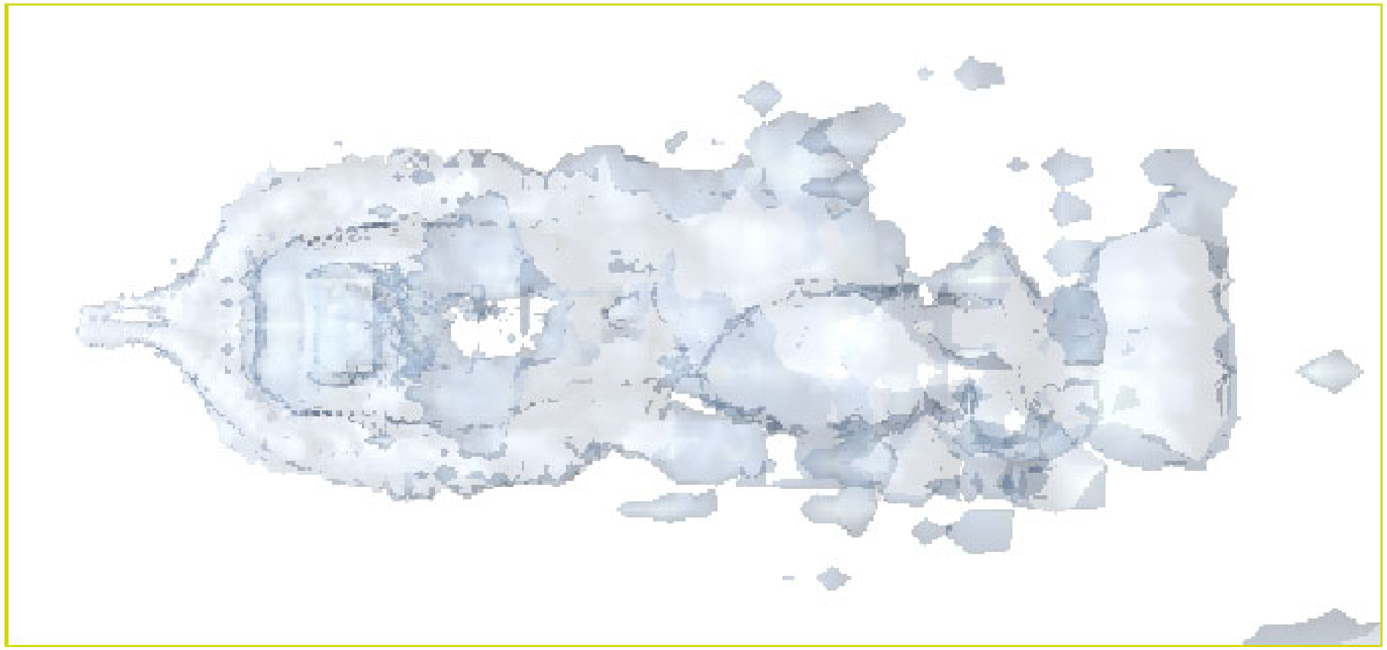
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 12



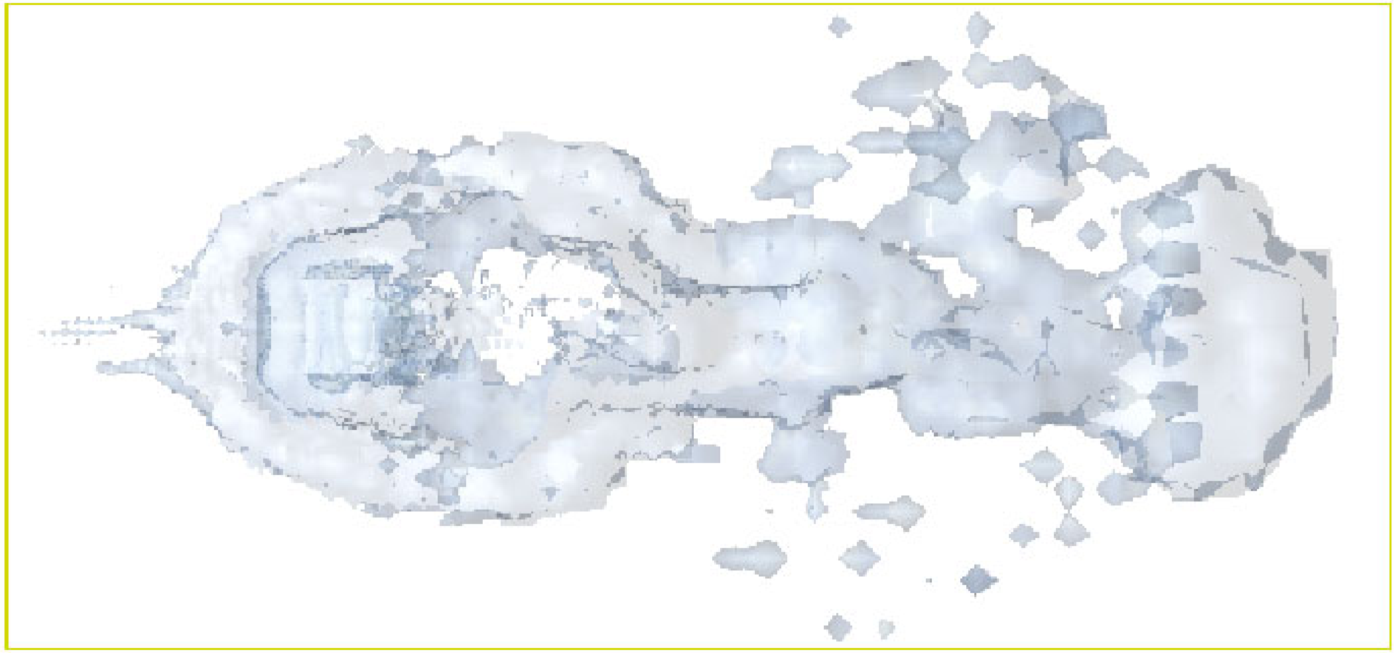
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 13



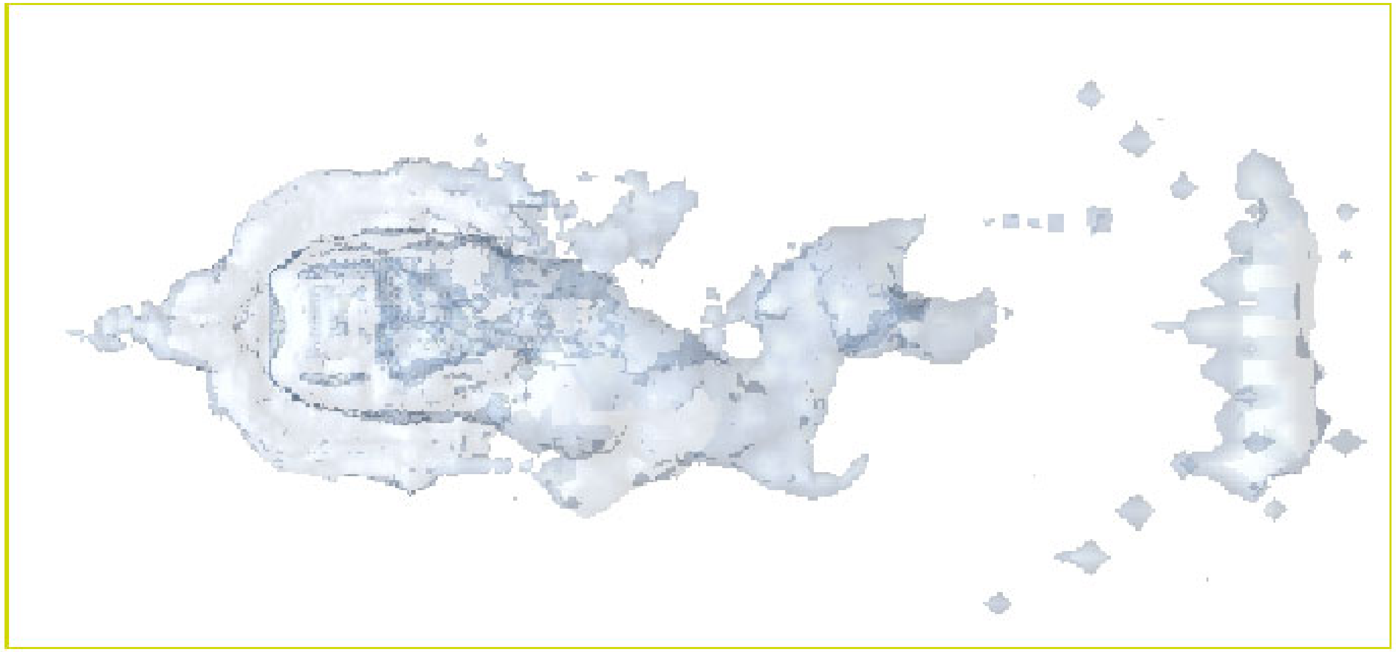
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 14



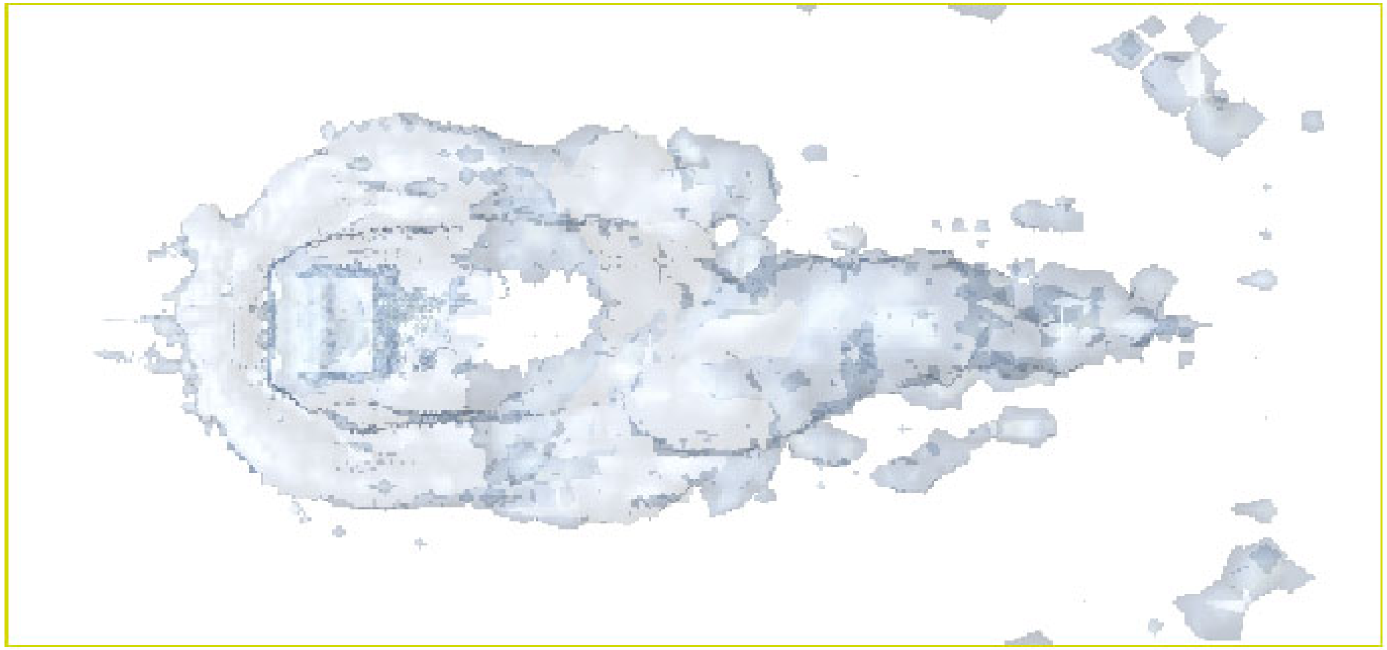
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 15



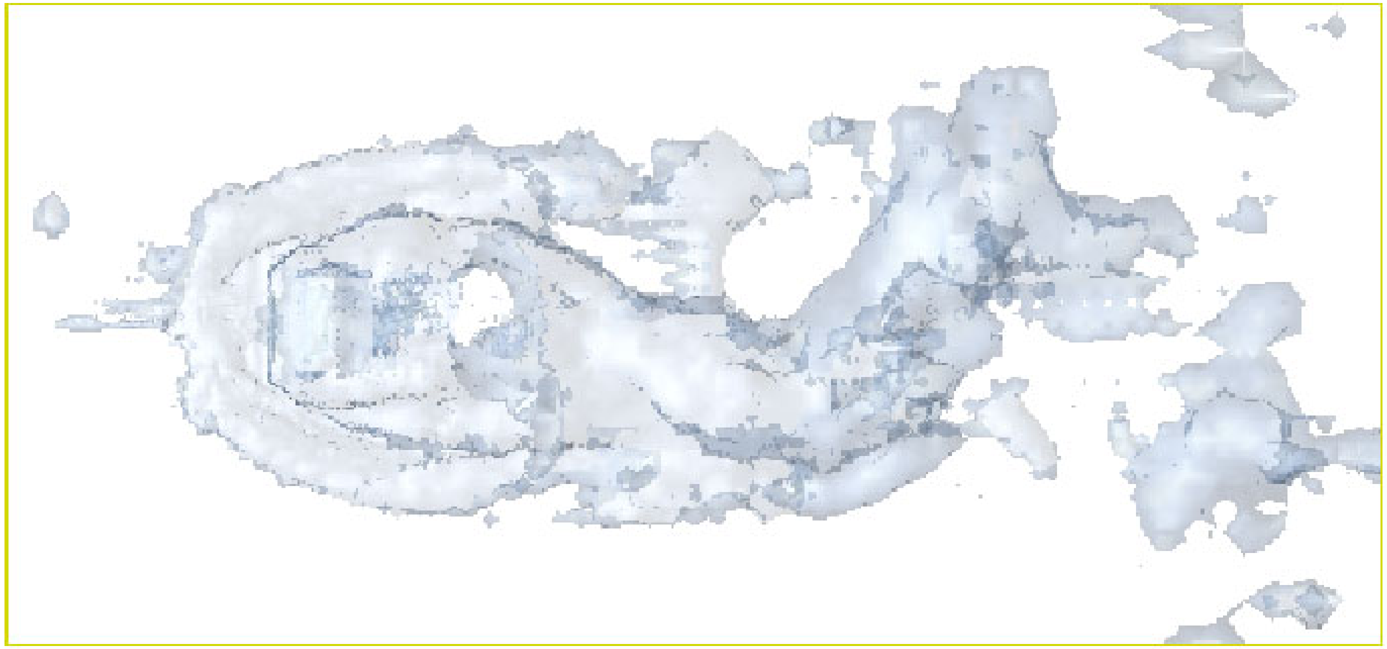
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 16



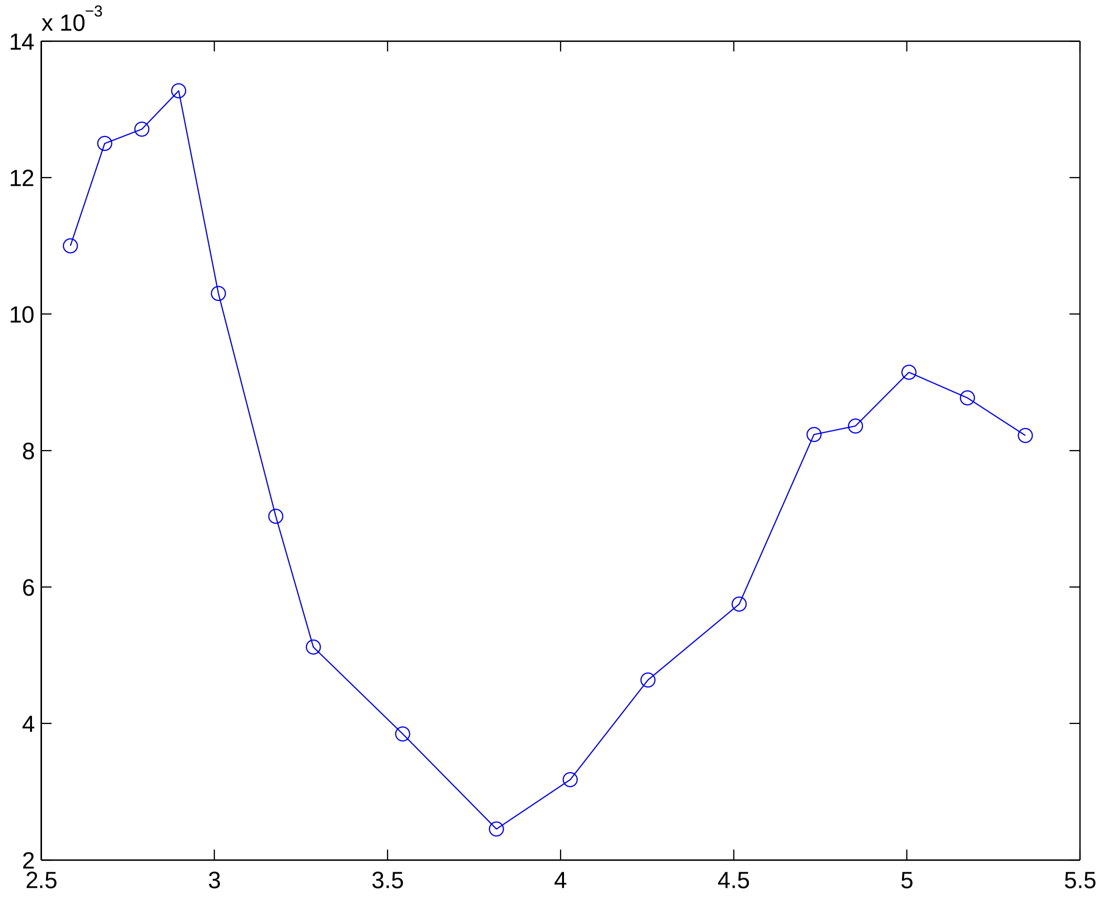
Subgrid modeling and dissipation

Isosurface for time average of $\epsilon(x) = 0.2$ on mesh no 17



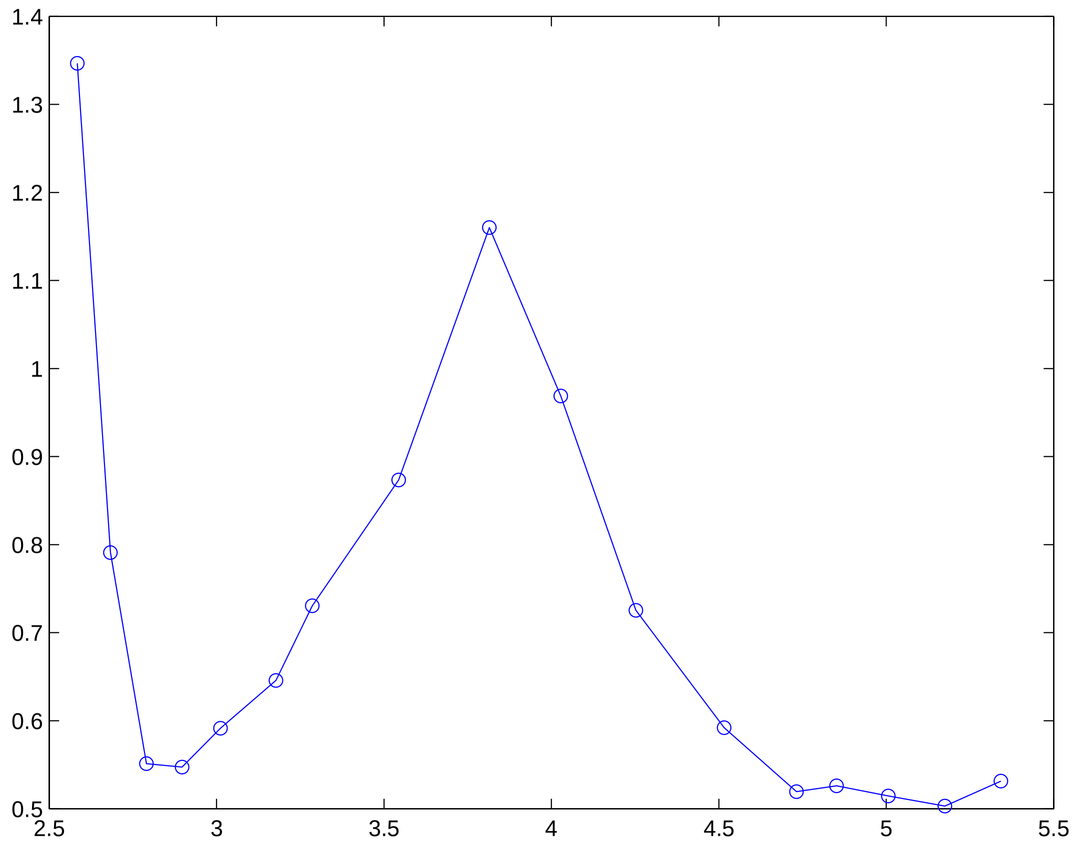
Subgrid modeling and dissipation

Volume of $\Omega_{0.2} = \{x : \text{time average of } \epsilon(x) \geq 0.2\}$



Subgrid modeling and dissipation

Space-time mean value of ϵ in $\Omega_{0.2}$: reach lower limit



Adaptive DNS/LES: Discretization

Weak Navier-Stokes equations WNSE:

$$R_\nu(u,p;v,q) \equiv ((\dot{u},v))+((u\cdot\nabla u,v))+((\nu\nabla u,\nabla v))-((p,\nabla\cdot v))$$
$$+((\nabla\cdot u,q))-((f,v))=0 \quad \forall(v,q)\in V\times Q$$

Galerkin: $R_\nu(U_h,P_h;v,q)=0 \quad \forall(v,q)\in V_h\times Q_h\subset V\times Q$

(unstable for h “large” with respect to ν)

General Galerkin G^2:

$$R_\nu(U_h,P_h;v,q)+SD_h(U_h,P_h;v,q)=0 \quad \forall(v,q)\in V_h\times Q_h$$

SD_h-term corresponds to a least squares stabilization of NSE residuals: $\|\sqrt{h}R_i\|\le C$ (cf. WNSE $\|\sqrt{\nu}\nabla u\|\le C$)

Adaptive DNS/LES: Error Estimate

A posteriori error estimate (using duality arguments):

$$|M(u,p) - M(U_h,P_h)| = |\sum_{K\in\mathcal{T}} \mathcal{E}_K| \leq \sum_{K\in\mathcal{T}} |\mathcal{E}_K|$$

$\mathcal{E}_K = e_D + e_M$ (error indicator for element K)

$$e_D = ((R_1, \varphi_h - \Phi)) + ((R_2, \theta_h - \Theta))$$
$$e_M = SD_h(U_h, P_h; \Phi, \Theta)$$

R_i momentum & continuity eqn. residuals, $(\Phi, \Theta) \in V_h \times Q_h$

$$-\dot{\varphi}_h - (u\cdot\nabla)\varphi_h + \nabla U_h \cdot \varphi_h - \nu\Delta\varphi_h + \nabla\theta_h = \psi, \quad \nabla\cdot\varphi_h = 0$$

Linear dual problem (data ψ coupling to output $\bar{c}_D = (u,\psi)$)

Adaptive DNS/LES: Stability

General Galerkin G^2: $\|\sqrt{h}R_i\| \leq C$

$$|((R_1, \varphi_h - \Phi))| \leq \|hR_1\| \|hD^2\varphi_h\| \leq C\sqrt{h}\, \|hD^2\varphi_h\|$$
$$|((R_2, \theta_h - \Theta))| \leq \|hR_2\| \|hD^2\theta_h\| \leq C\sqrt{h}\, \|hD^2\theta_h\|$$
$$|SD_h(U_h, P_h; \Phi, \Theta)| \leq \ldots \leq C\sqrt{h}\, \|\nabla\varphi_h\| + C\sqrt{h}\, \|\nabla\theta_h\|$$

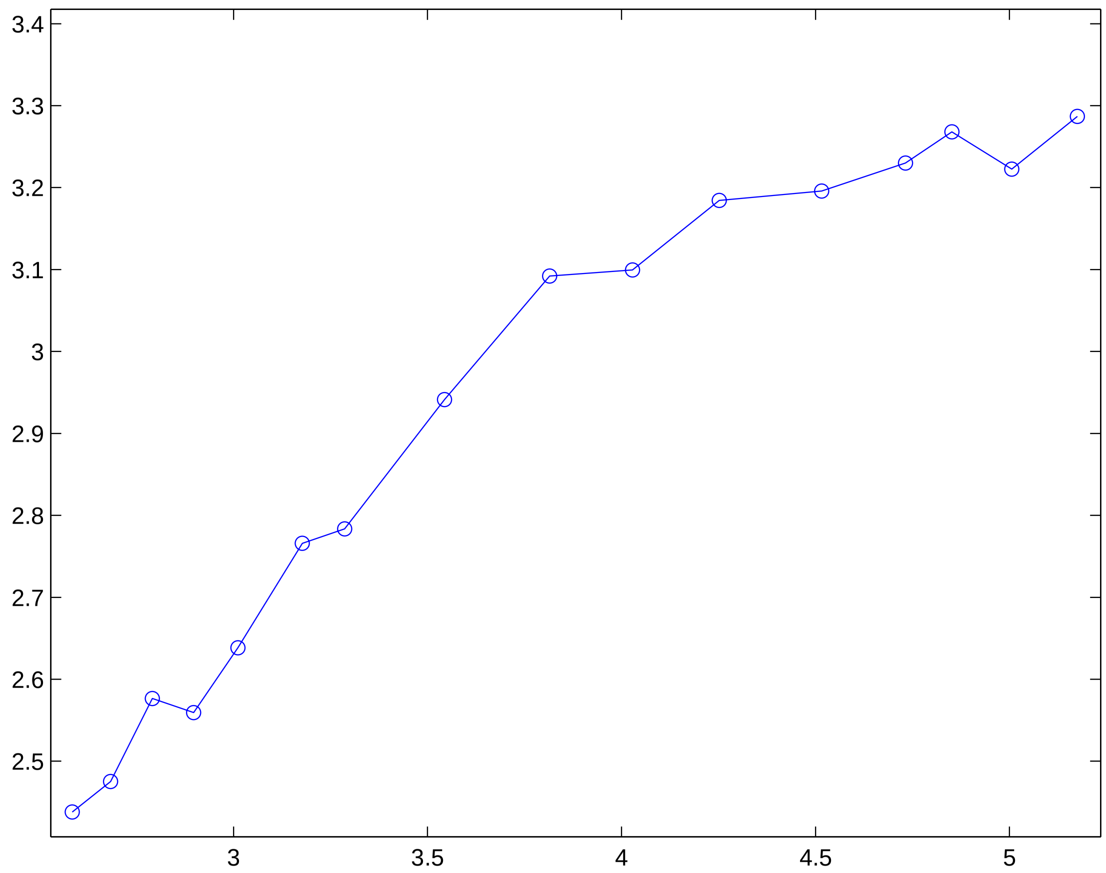
Stability of $\|hD^2\varphi_h\|$, $\|hD^2\theta_h\|$, $\|\nabla\varphi_h\|$, and $\|\nabla\theta_h\|$ crucial!

Perturbations of dual problem for (φ_h, θ_h), from:
(1) coef£cient $u \approx U_h$, (2) numerical approximation

Estimate stability computationally; we have convergence if growth of stability factors is slow ($< \sqrt{h}$)

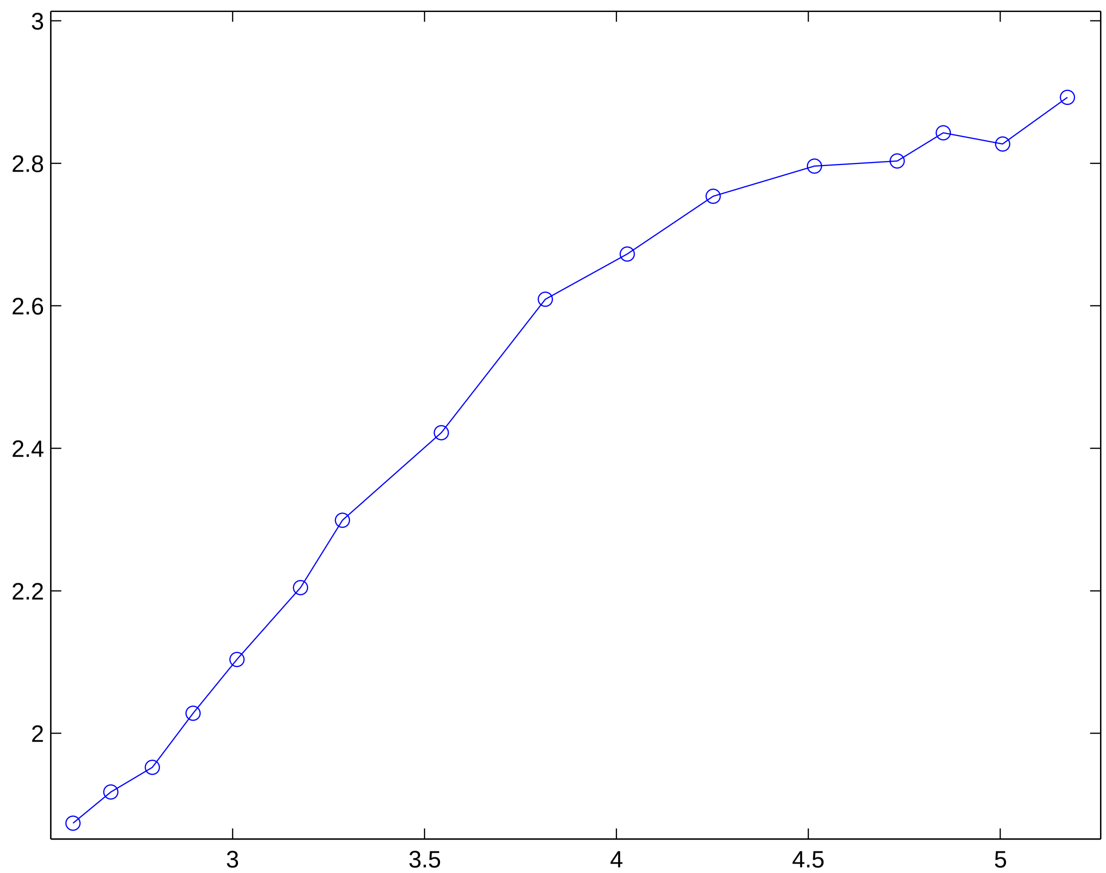
$\|hD^2\varphi_h\|$ vs no mesh points ($\log_{10}$)

Slow logarithmic growth of $\|hD^2\varphi_h\|$



$\|\nabla \varphi_h\|$ vs no mesh points ($\log_{10}$)

Slow logarithmic growth of $\|\nabla \varphi_h\|$



Mathematical Aspects of NSE

Existence & uniqueness of smooth solutions NOT known!
(Clay Institute $1 million Prize Problem)

- Could we regard a turbulent solution as a smooth solution with very large derivatives and with extreme sensitivity in pointwise values to small perturbations?
- Simple (Gronwall) estimates for a proof of uniqueness of smooths solutions would involve constants e^{KT},
  K measuring £rst order derivatives (turb: $K \sim$ Re$^{1/2}$):
  Re = 10^6 and $T = 1 \Rightarrow$ ampli£cation factors $\sim e^{1000}$
- Is a smooth solution to a turbulent ¤ow reasonable?!

Mathematical Aspects of NSE

Weak NSE: $R_\nu(u; v) = 0 \quad \forall v \in H_0^1$

- Existence of weak solutions known!
  (Leray, 1934: using Galerkin approximations)
- Uniqueness of weak solutions NOT known.

(W) Alt: Is the OUTPUT of a weak solution unique?

$$|M(v) - M(w)| \;=\; R_\nu(v; \varphi_\nu) \;=\; R_\nu(w; \varphi_\nu)$$

Weak solutions have zero residual in H^{-1}:

$\varphi_\nu \in H_0^1$ $(\|\varphi_\nu\|_{H_0^1} < \infty)$ $\Rightarrow$ weak solution unique in output

Mathematical Aspects of NSE

Computational formulation of the uniqueness problem (W):

(C) For a given ¤ow, what output can be computed to what tolerance to what cost?

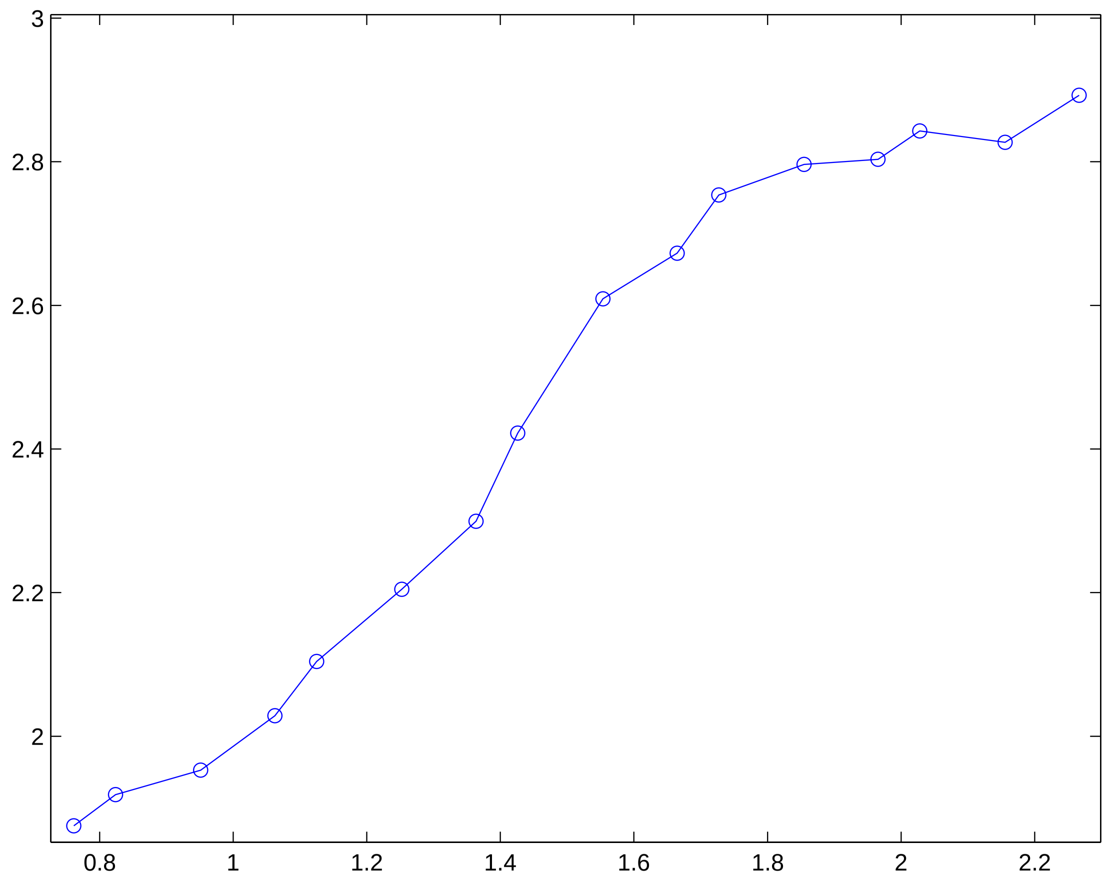
We can test (C) computationally, case by case, by studying the size of $\|\varphi_h\|_{L_2(H^1)}$ as we re£ne the mesh (decrease h)

Extrapolation of h to ν, gives approximation of $\|\varphi_\nu\|_{L_2(H^1)}$

If $\|\varphi_\nu\|_{L_2(H^1)}$ for an output $M(\cdot)$ is not "too large" ($\|\varphi_\nu\|_{L_2(H^1)} >> \nu^{-1/2}$) we have computability of $M(\cdot)$

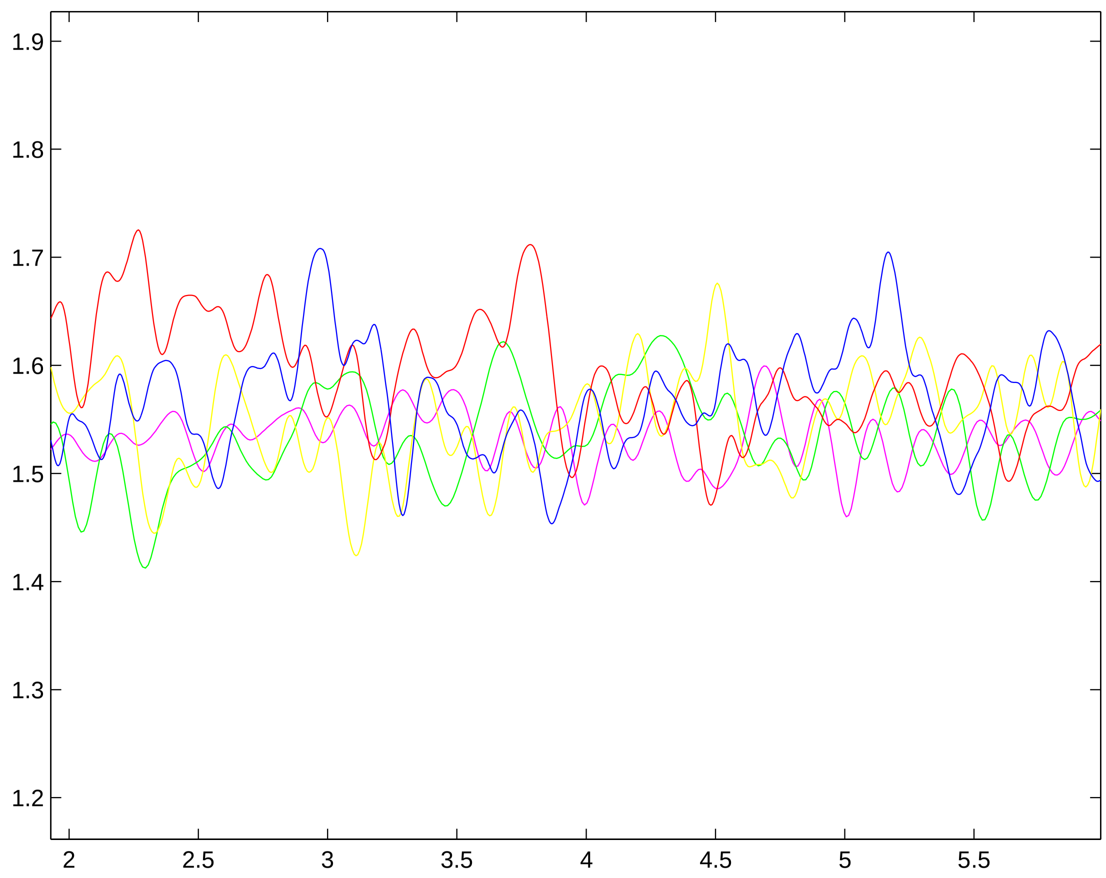
Ex: Mean drag: $\|\varphi_h\|_{L_2(H^1)}$ vs $1/h$

Slow logarithmic growth of $\|\varphi_h\|_{L_2(H^1)} \Rightarrow \|\varphi_\nu\|_{L_2(H^1)} \approx \nu^{-1/2}$



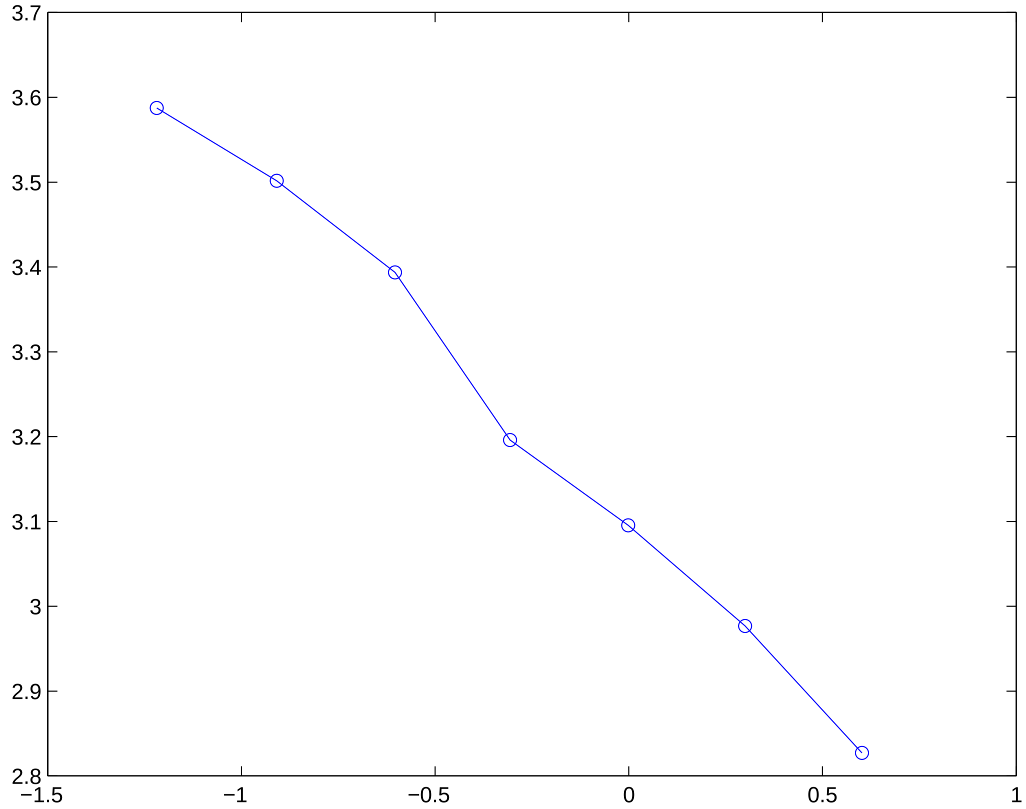
Ex: Pointwise drag vs time

Trajectories for drag $c_D(t)$ converge slowly or not at all



Ex: Pointwise drag: $\|\varphi_h\|_{L_2(H^1)}$ vs ΔT

$c(T) \approx \frac{1}{\Delta T}\int_{T-\Delta T}^{T} c_D(t)\,dt$: $\|\varphi_h\|_{L_2(H^1)}$ grows like $\Delta T^{-1/2}$



Mathematical Aspects of NSE

Computational evidence (benchmark bluff body problem)

- uniqueness (computability) of mean drag:
  - slow logarithmic growth of $\|\varphi_h\|_{L_2(H^1)}$
  - extrapolation gives that $\|\varphi_\nu\|_{L_2(H^1)} \approx \nu^{-1/2}$
- non-uniquness (non-computability) of pointwise drag:
  - for frozen (£ne) h: $\|\varphi_h\|_{L_2(H^1)}$ grows like $\Delta T^{-1/2}$

WNSE and Adaptive DNS/LES

- Existence (using Galerkin) known, by Leray 1934
- Uniqueness/Computability in output: case by case
- No averaging: compute output $M(u,p)$ directly from WNSE $\Rightarrow$ No Reynolds stresses to model!!
- Stabilized Galerkin discretization of WNSE (introduces dissipation in turbulent part of domain)
- Quality assessement: A posteriori error estimation
- Output sensitivity information: Dual problems
- Ef£ciency: Adaptivity with respect to $M(u,p)$

Future directions

- Fluid-structure interaction (ALE methods)
  - *Movie: moving object in 3d NSE*
  - *Movie: corresponding mesh*
- Mixing and reactive ¤ow
  - *Movie: production-consumption in Couette flow*
- Compressible flow
- Applied Mathematics Body and Soul V: Fluid Dynamics (Barth, Hoffman, Johnson), Springer, 2004
- FEniCS/Dol£n PDE software project