A posteriori error estimation for timedependent problems.

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Outline - ODE

- cG(q)
- cG(1)
- The dual problem
- Error estimation
- Stability factors

Objective

Solve the ODE initial value problem

$$\begin{cases} \dot{u}(t) &= f(u(t),t), \quad t \in (0,T], \\ u(0) &= u_0, \end{cases}$$

for $u : [0,T] \to \mathbb{R}^N$.
Example:

$$\begin{cases} \dot{u}(t) &= [\dot{u}_1(t); \dot{u}_2(t)] = f(u), \\ u(0) &= [0;1], \end{cases}$$

Solution $u(t) = (\sin(t), \cos(t))$.

cG(q)

Variational formulation:

$$\int_0^T (\dot{u}, v)\, dt = \int_0^T (f(u,\cdot), v)\, dt.$$

The cG(q)-method for $\dot{u} = f$ then reads:
find $U \in V$ such that:

$$\int_0^T (\dot{U}, v)\, dt = \int_0^T (f(U,\cdot), v)\, dt \quad \forall v \in W,$$

where the trial and test spaces V and W defined as

$$\begin{aligned} V &= \{v \in [\mathcal{C}([0,T])]^N : v_i|_{I_j} \in \mathcal{P}^q(I_j)\}, \\ W &= \{v : v_i|_{I_j} \in \mathcal{P}^{q-1}(I_j)\}. \end{aligned}$$

cG(1)

Now $q = 1$. The trial space consists of linear polynomials and the the test space of constant ploynomials.

cG(1):

$$\int_{t_{n-1}}^{t_n} (\dot{U}, v)\, dt = \int_{t_{n_1}}^{t_n} (f, v)\, dt.$$

Take $v = (0, \ldots, 0, 1, 0, \ldots, 0)$ (i:th position), then

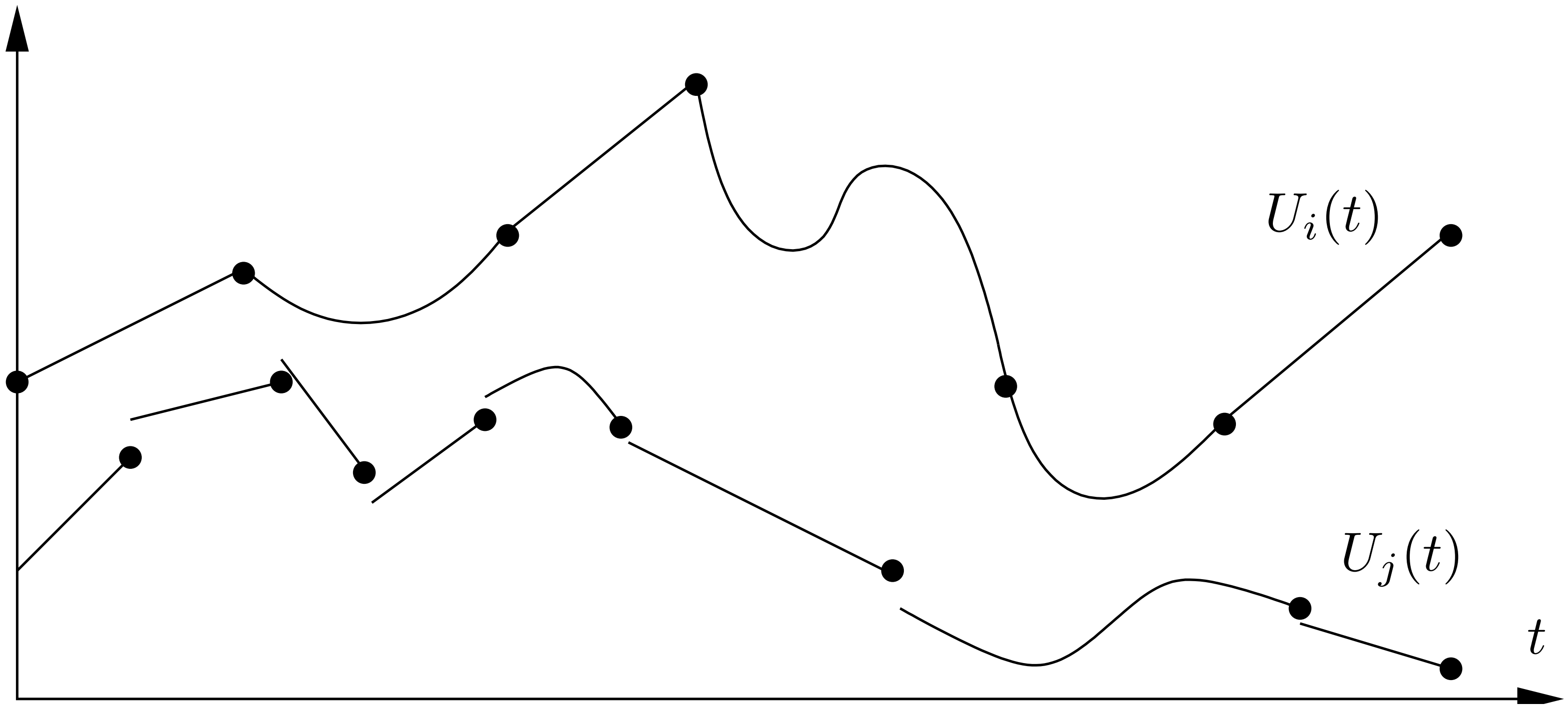
$$\int_{t_{n-1}}^{t_n} \dot{U}_i\, dt = \int_{t_{n-1}}^{t_n} f_i\, dt.$$

or equivalently

$$U_i(t_n) - U_i(t_{n-1}) = \int_{t_{n-1}}^{t_n} f_i\, dt \approx k_n f_i \left(\frac{U_i(t_{n-1}) + U_i(t_n)}{2}, \frac{t_{n-1} + t_n}{2} \right),$$

which is solved with fixpointiteration for $U(t_n)$.

Piecewise polynomials



The dual problem

The dual problem is given by

$$\begin{cases} -\dot{\phi}(t) & = & J^\top(u, U, t)\phi(t) + g(t), \quad t \in [0, T), \\ \phi(T) & = & \psi, \end{cases}$$

where

$$J(v_1, v_2, \cdot) = \int_0^1 \frac{\partial f}{\partial u}(sv_1 + (1-s)v_2, \cdot)\, ds.$$

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By choosing ψ and g, different functionals $L_{\psi,g}(e)$ can be estimated. Two basic examples:

- $\psi \approx e(T)/\|e(T)\|$ and $g = 0$ gives $L_{\psi,g}(e) \approx \|e(T)\|$
- $\psi = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $g = 0$ gives $L_{\psi,g}(e) \approx e_i(T)$

Error estimation - 1

Take $\psi = e(T)/||e(T||)$. Then

$$
\begin{aligned}
||e(T)|| &= (e(T), \psi) + \int_0^T (-\dot{\phi} - J^T \phi, e) dt \\
&= (e(T), \psi) + \int_0^T (-\dot{\phi}, e) - \int_0^T (J^T \phi, e) dt \\
&= (e(T), \psi) - [(\phi, e)]_0^T + \int_0^T (\phi, \dot{e}) - \int_0^T (\phi, Je) dt \\
&= \int_0^T (\phi, \dot{e} - Je) dt = \int_0^T (\phi, \dot{u} - \dot{U} - f(U) + f(u)) dt
\end{aligned}
$$

Error estimation - 2

Take $\psi = e(T)/||e(T||)$. Then

$$
\begin{aligned}
||e(T)|| &= \int_0^T (\phi, \dot{u} - \dot{U} - f(U) + f(u))dt \\
&= \int_0^T (\phi, R(U))dt = \int_0^T (\phi - \pi\phi, R(U))dt \\
&= \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (\phi - \pi\phi, R(U))dt \leq \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} ||\phi - \pi\phi|| ||R(U))||dt \\
&\leq C \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} k^q ||\phi^{(q)}|| ||R||dt \\
&\leq CS(T) \max ||h^q R||,
\end{aligned}
$$

where

$$S(T) = \int_0^T ||\phi^{(q)}||dt.$$

Stability factors

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3. Lorenz, $S(T) \approx e^T$.

Outline - PDE

- Heat equation
- BS equation - Finance
- Wave equation

The heat equation

The heat equation reads

$$\begin{cases} u_t - \Delta u = f, & in \quad \Omega \times [0, T], \\ u(0, x) = u_0, & on \quad \Omega, \\ u(t, .) = 0, & on \quad \partial\Omega. \end{cases}$$

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Multiplying the by a test function $v \in \mathcal{W} = L^2([0,T], H^1(\Omega))$ and integrating on $\Omega \times [0,T]$ we obtain

$$\int_0^T \Big((u_t, v) - (\Delta u, v) \Big) dt = \int_0^T (f, v) dt.$$

Variational formulation

Integration by parts using the boundary condition gives the problem we wish to solve

Problem: Find $u \in \mathcal{W}$ such that

$$\begin{cases} \int_0^T \big((u_t, v) + a(u,v)\big)\, dt = 0, \\ u(0,x) = u_0, \end{cases}$$

for every $v \in \mathcal{W}$, where

$$a(u,v) = (\nabla u, \nabla v).$$

The FE approximation

- Let $\mathcal{V}^p \subset H^1(\Omega)$ denote the space of piecewise continuous functions of order p in space.

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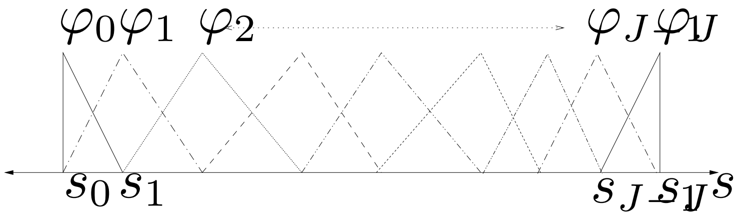
  The standard nodal basis of $\mathcal{V}^1$:



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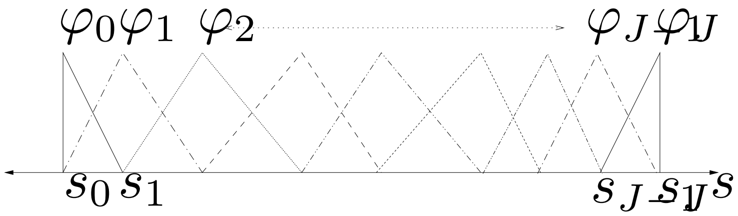

- On each space-time slab $S_n = I_n \times \Omega$, we define

$$\mathcal{W}_n^q = \{w(t,s) : w(t,s) = \sum_{j=0}^{q} t^j v_j(s), v_j \in \mathcal{V}^p, (t,s) \in S_n\}.$$

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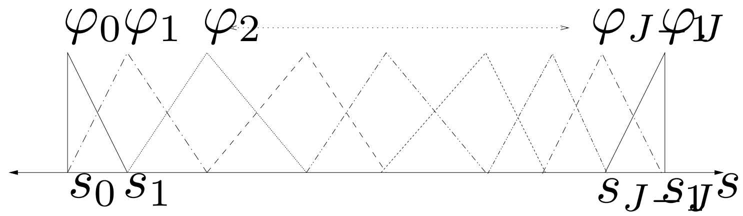

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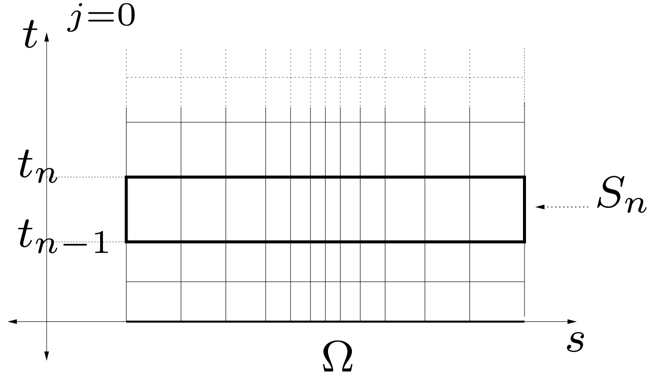

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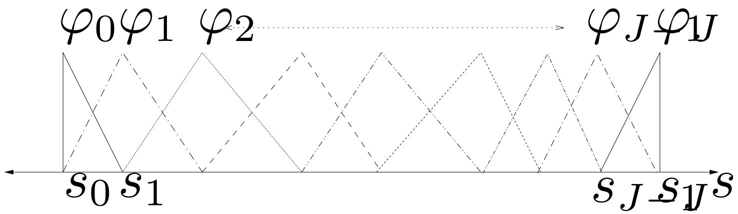

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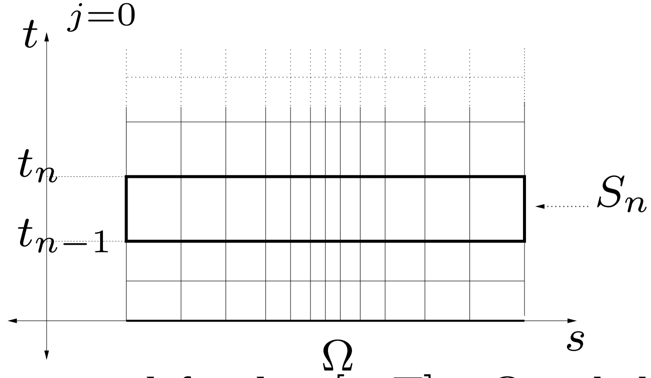

- Let $\mathcal{W}^q \subset \mathcal{W}$ denote the space of functions defined on $[0,T] \times \Omega$ such that $v\,|_{S_n} \in \mathcal{W}_n^q$ for $1 \leq n \leq N$.

The finite element problem

FE problem: Find $U \in \mathcal{W}^q$ such that for $1 \le n \le N$

$$\begin{cases} \int_{I_n} \big((U_t, v) + a(U, v)\big)\, dt = 0 \quad \text{for all } v \in \mathcal{W}_n^{q-1}, \\ U^+(t_n) = U^-(t_n), \quad n = 1, \dots, N, \\ U^+(t_0) = u_0, \end{cases}$$

where $U^{\pm}(t_n) = \lim_{\epsilon \to 0, \epsilon > 0} U(t_n \pm \epsilon)$ and

$$a(U, v) = (\nabla U, \nabla v).$$

A posteriori error estimation

Why?

- To validate the solution and to approximate the error.

A posteriori error estimation

Why?

- To validate the solution and to approximate the error.
- To choose a suitable mesh.

The dual problem

To represent the error in a linear functional,

$$(\psi_2, e(T,x)) + \int_0^T (e, \psi_1),$$

where $e = u - U$, we introduce the continuous dual problem for the heat equation:

Dual problem: Find $\phi \in \mathcal{W}$ such that

$$\begin{cases} -\phi_t - \Delta\phi = \psi_1, & in \quad \Omega \times [0,T], \\ \phi = 0, & on \quad \partial\Omega, \\ \phi(T,x) = \psi_2, & on \quad \Omega. \end{cases}$$

Error representation formula

Multiplying with the error $e = u - U \in \mathcal{W}$ and integrating on $\Omega \times [0, T]$

$$\int_0^T (e, \psi_1) dt = \int_0^T \Big(-(e, \phi_t) - (e, \Delta\phi) \Big) dt.$$

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Using integration by parts and Green's formula we obtain

$$(\psi_2, e(T, x)) + \int_0^T (e, \psi_1) dt = \int_0^T \Big((e_t, \phi) + (\nabla e, \nabla \phi) \Big) dt.$$

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Since $e = u - U$ and u solves the FE problem we get the error representation formula

$$\boxed{(\psi_2, e(T, x)) + \int_0^T (e, \psi_1) dt = -\int_0^T \Big((U_t, \phi) + a(U, \phi) - (f, \phi) \Big) dt}$$

Estimating the error - 1

We now proceed estimating the error when $f = 0$. Let $\pi : \mathcal{W} \to \mathcal{W}^{q-1}$ be the L_2 projection in time, and let P be a suitable interpolation operator into $\mathcal{V}^p$ in space. Then using Galerkin orthogonality we can replace ϕ by $\phi - \pi P\phi = \phi - P\phi + P\phi - \pi P\phi$

$$
\begin{aligned}
(\psi_2, e(T,x)) + \int_0^T (e, \psi_1) dt &= - \int_0^T \Big((U_t, \phi - P\phi) + a(U, \phi - P\phi) \Big)\, dt \\
&\quad - \int_0^T \Big((U_t, P\phi - \pi P\phi) + a(U, P\phi - \pi P\phi) \Big)\, dt \\
&= - \sum_n \sum_j \int_{I_n} \Big(r^x_{\kappa_j}(U), \phi - P\phi \Big)\, dt \\
&\quad - \sum_n \int_{I_n} \Big(r^t(U), P\phi - \pi P\phi \Big)\, dt,
\end{aligned}
$$

Estimating the error - 2

where

$$(r^x_{\kappa_j}(U), \phi - P\phi) = ([U_x], \phi - P\phi)_{\partial\kappa_j} + (U_t - \Delta U, \phi - P\phi)_{\kappa_j}$$

is the space residual, and

$$(r^t(U), P\phi - \pi P\phi) = (U_t - \Delta U, P\phi - \pi P\phi)$$

is the time residual. Here we used the notation $[U_x]$ to denote the jump in U_x over element interfaces.

Estimating the error - 3

Rewriting we see that

$$
\begin{aligned}
|(\psi_2, e(T,x)) + \int_0^T (e,\psi_1)dt| \leq \\
&\sum_n \sum_j |\int_{I_n} ([U_x], \phi - P\phi)_{\partial\kappa_j} + (U_t - \Delta u, \phi - P\phi)_{\kappa_j}| \\
&+ \sum_n |\int_{I_n} (U_t - \Delta u, P\phi - \pi P\phi)| \\
\leq &\sum_n \sum_j \|h_j^{-1/2}[U_x]\|_{\partial\kappa_j \times I_n} \|h_j^{1/2}(\phi - P\phi)\|_{\partial\kappa_j \times I_n} \\
&+ \|U_t - \Delta u\|_{\kappa_j \times I_n} \|\phi - P\phi\|_{\kappa_j \times I_n} \\
&+ \sum_n \|U_t - \Delta u\|_{S_n} \|P\phi - \pi P\phi\|_{S_n}
\end{aligned}
$$

Error estimation algorithm

- Compute an approximation Φ of ϕ using an enriched finite element space, for instance higher order approximation.
- Compute $P\Phi$.
- Compute $\int_{I_n} \left(r^x_{\kappa_j}(U), \phi - P\phi \right) dt$ using quadrature in space and time for each element κ_j and time step.
- Compute $\pi P\Phi$.
- Compute $\int_{I_n} \left(r^t(U), P\phi - \pi P\phi \right) dt$ using quadrature in space and time for each time step.

Finance applications - Outline

- A brief introduction to option pricing

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- Extension of the framework to
  - barrier options
  - lookback options
  - Asian options

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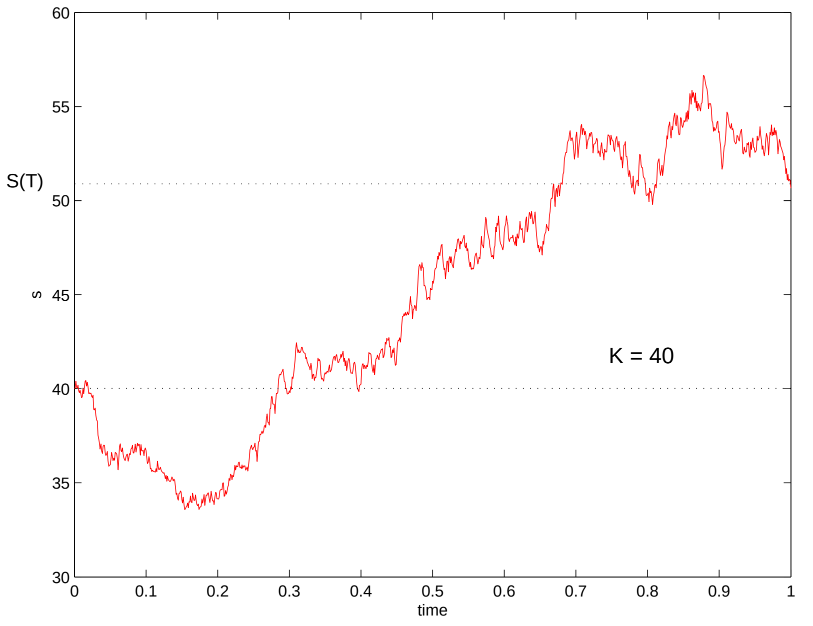

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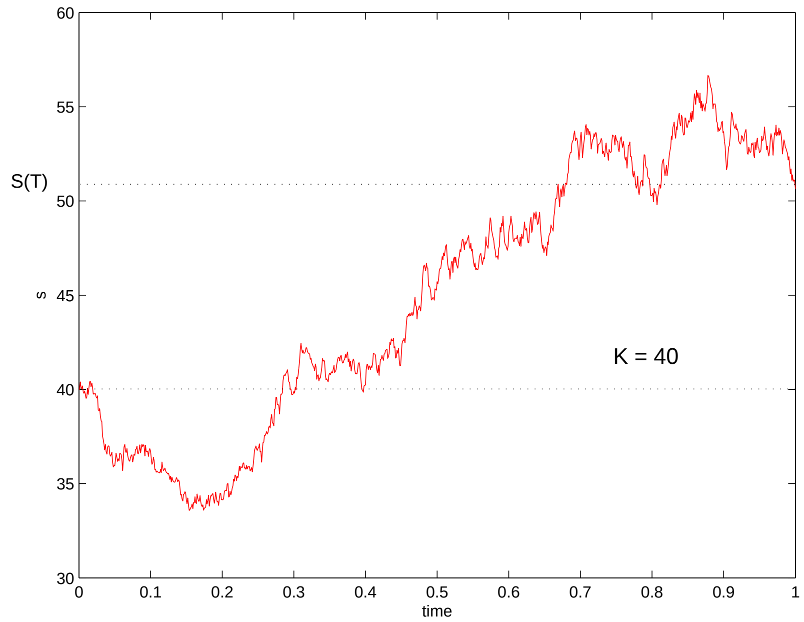

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  Payoff: $\max(0, K - S(T))$,
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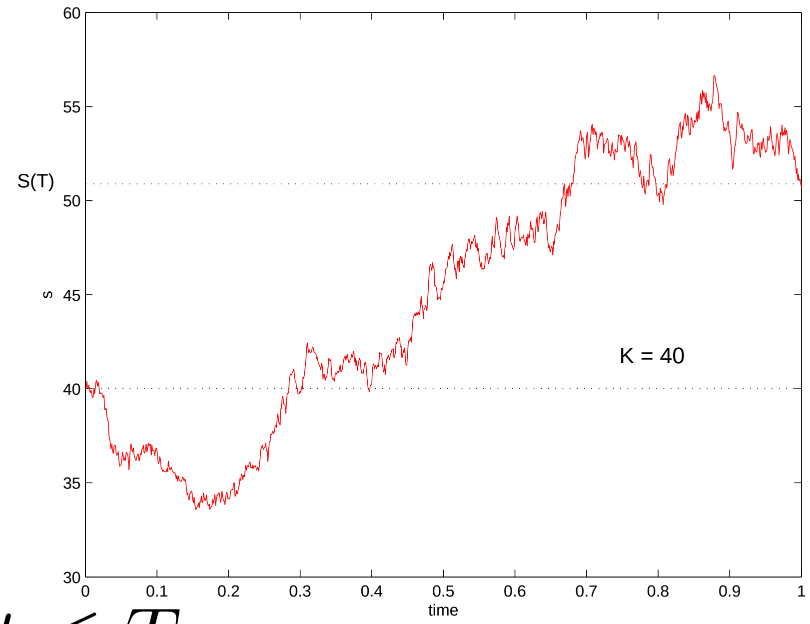
  Payoff: $\max(0, S(t) - K), t \leq T$,



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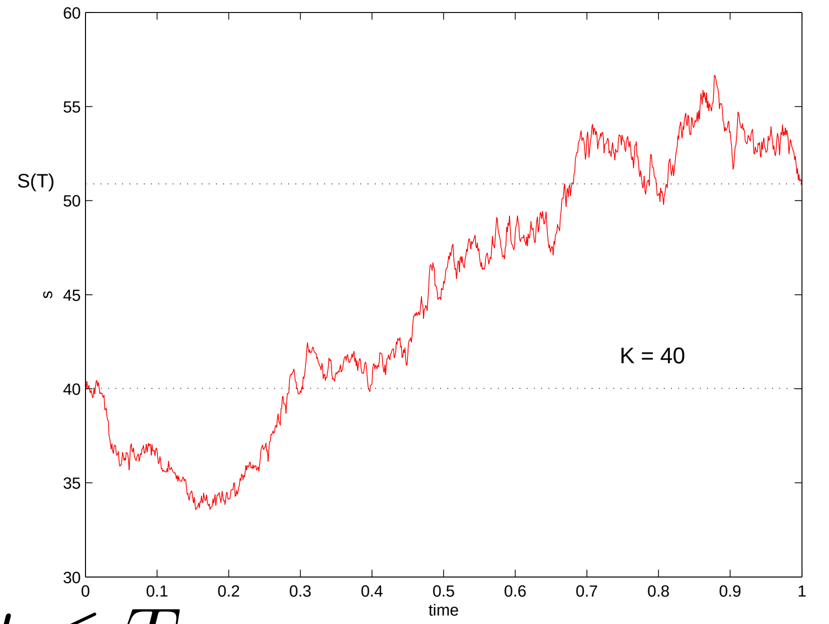

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Model

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- An asset $S(t)$, solving the SDE

$$
\begin{aligned}
dS(t) &= (r - \nu)S(t)dt + \sigma S(t)dW(t)), \\
S(0) &= S_0,
\end{aligned}
$$

where
r is the constant interest rate,
ν is the constant continuous dividend yield,
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The solution to the equation above is

$$S(t) = S(0)e^{(r-\nu-\frac{\sigma^2}{2})t+\sigma W(t)}.$$

Black-Scholes PDE

Let $v(t, S(t))$ denote the value of a portfolio at time t, then

$$v'_t(t, S(t)) + \frac{\sigma^2 S(t)^2}{2} v''_{ss}(t, S(t)) + rS(t)v'_s(t, S(t)) - rv(t, S(t)) = 0,$$
$$t < T,\ S(t) > 0.$$

Together with the terminal condition $v(T, S(T)) = g(S(T))$, the equation above has the following solution,

$$v(t, S(t)) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau + \sigma W(\tau)}\right],$$

where $s = S(t)$ and $\tau = T - t$.

Black-Scholes formula

A European call option with payoff $g(S(T)) = (S(T) - K)^+$, maturity date T and strike price K has the value $c(t, S(t), K)$ at time $t < T$ where

$$c(t, s, K) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

$$d_1 = \frac{\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

where Φ is the probability distribution function for a $N(0, 1)$ distributed stochastic variable.

Variational formulation

Multiplying the Black-Scholes equation by a test function $v \in \mathcal{W} = L^2([0,T], H^1(\Omega))$ and integrating on $\Omega \times [0,T]$ we obtain

$$\int_0^T \Big((u_t, v) + (r - \nu)\big(su_s, v\big) + \frac{\sigma^2}{2}\big(s^2 u_{ss}, v\big) - r\big(u, v\big)\Big) dt = 0.$$

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Integration by parts using the artifi cial boundary condition $u_{ss} = 0$, or equivalently by the Black-Scholes equation

$$u_s = \frac{r}{s(r-\nu)} u - \frac{1}{s(r-\nu)} u_t,$$

gives the problem we wish to solve.

Variational formulation

Problem: Find $u \in \mathcal{W}$ such that

$$\begin{cases} \int_0^T \big(m(u_t, v) + a(u, v)\big) dt = 0, \\ u(T, s) = \max(s - K, 0), \end{cases}$$

for every $v \in \mathcal{W}$, where

$$\begin{cases} m(u_t, v) & = (u_t, v) - \frac{\sigma^2}{2(r-\nu)} (s u_t, v)_{\partial\Omega}, \\ a(u, v) & = (r - \nu - \sigma^2)\big(s u_s, v\big) - \frac{\sigma^2}{2}\big(s^2 u_s, v_s\big) \\ & \quad + \frac{\sigma^2 r}{2(r-\nu)} (s u, v)_{\partial\Omega} - r(u, v). \end{cases}$$

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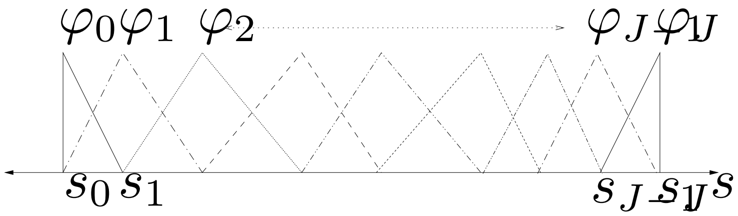

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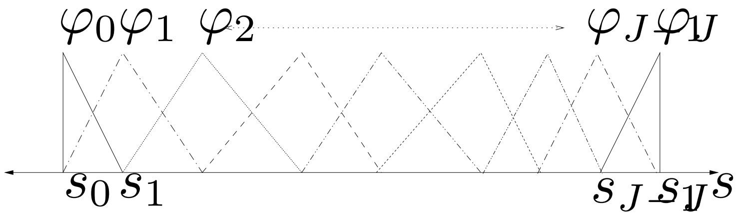

- On each space-time slab $S_n = I_n \times \Omega$, we define

$$\mathcal{W}_n^q = \{w(t,s) : w(t,s) = \sum_{j=0}^{q} t^j v_j(s), v_j \in \mathcal{V}^p, (t,s) \in S_n\}.$$

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- Let $\mathcal{V}^p \subset H^1(\Omega)$ denote the space of piecewise continuous functions of order p in space.
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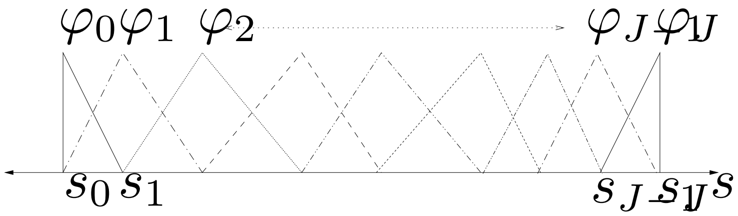

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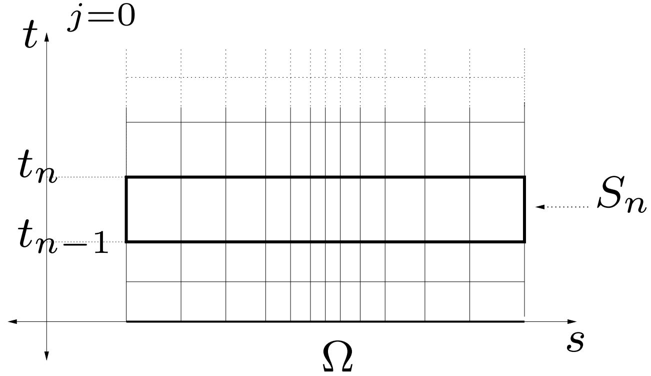

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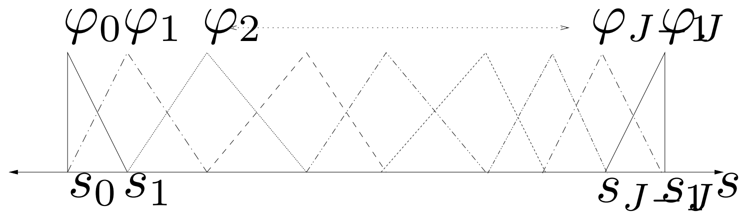

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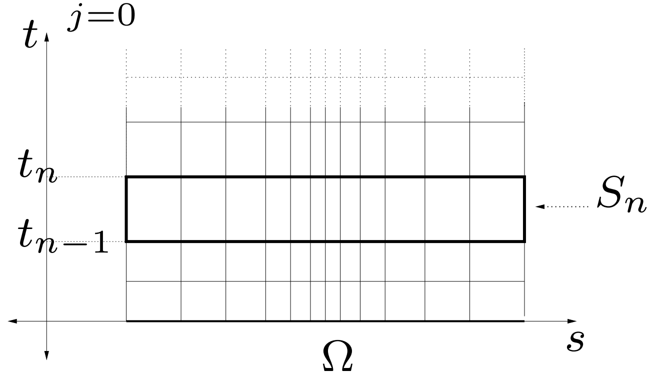


- Let $\mathcal{W}^q \subset \mathcal{W}$ denote the space of functions defined on $[0,T] \times \Omega$ such that $v\,|_{S_n} \in \mathcal{W}_n^q$ for $1 \leq n \leq N$.

The FE problem

FE problem: Find $U \in \mathcal{W}^q$ such that for $1 \leq n \leq N$

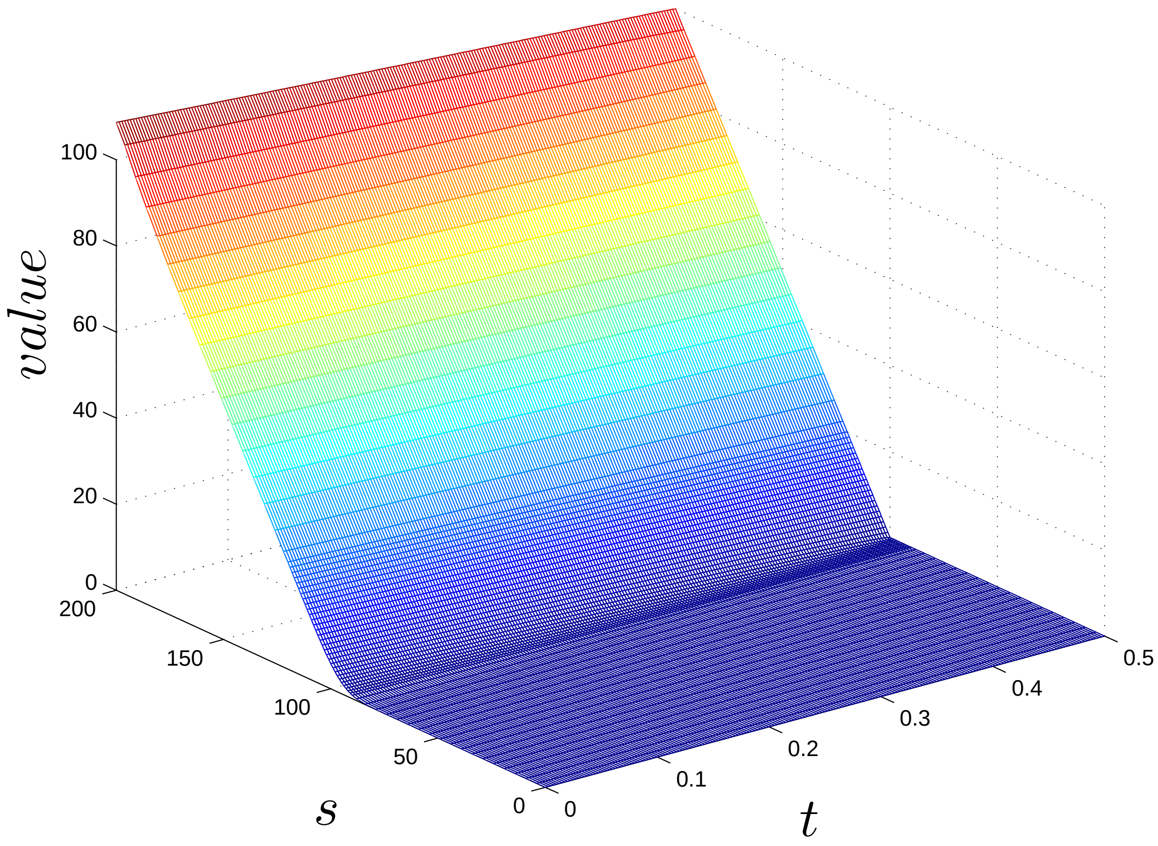
$$
\begin{cases}
\int_{I_n} \big(m(U_t, v) + a(U, v)\big) dt = 0 \quad \text{for all } v \in \mathcal{W}_n^{q-1}, \\
U^-(t_n) = U^+(t_n), \quad n = N-1, \dots, 1, \\
U^-(t_N) = u_T,
\end{cases}
$$

where $U^\pm(t_n) = \lim_{\epsilon \to 0, \epsilon > 0} U(t_n \pm \epsilon)$ and

$$
\begin{cases}
m(u_t, v) = (u_t, v) - \frac{\sigma^2}{2(r-\nu)} (s u_t, v)_{\partial\Omega}, \\
a(u, v) = (r - \nu - \sigma^2)\big(s u_s, v\big) - \frac{\sigma^2}{2}\big(s^2 u_s, v_s\big) \\
\qquad + \frac{\sigma^2 r}{2(r-\nu)} (s u, v)_{\partial\Omega} - r(u, v).
\end{cases}
$$

The European call option

Parameter values are $K = 100$,$\sigma = 0.1$, $q = 0.0$, $r = 0.10$, $T = 0.5$, and $t = 0$.



A posteriori error estimation

Why?

- To validate the solution and to approximate the error.

A posteriori error estimation

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- To validate the solution and to approximate the error.
- To choose a suitable mesh.

A posteriori error estimation

Why?

- To validate the solution and to approximate the error.
- To choose a suitable mesh.
  - we only need the solution in one or a few points
  - if we are interested in the derivative of the solution

The dual problem

To represent the error in a linear functional,

$$(u - U, \psi),$$

we introduce the continuous dual problem for the Black-Scholes equation:

Dual problem: Find $\phi \in \mathcal{W}$ such that

$$\begin{cases} -\phi_t + (\sigma^2 + \nu - 2r)\phi - (r - \nu - 2\sigma^2)s\phi_s + \frac{\sigma^2}{2}s^2\phi_{ss} = 0, \\ \phi(0, s) = \psi. \end{cases}$$

A posteriori error estimation

Multiplying with the error $e = u - U \in \mathcal{W}$ and integrating in space and time we get

$$\int_0^T \Big(- (\phi_t, e) + (\sigma^2 + \nu - 2r)(\phi, e) - (r - \nu - 2\sigma^2)\big(s\phi_s, e\big) + \frac{\sigma^2}{2}\big(s^2\phi_{ss}, e\big)\Big) dt = 0.$$

Error representation formula

Using integration by parts and neglecting the boundary terms we get

$$(\psi, e(0,s)) = -\int_0^T \Big(m(e_t, \phi) + a(e, \phi) \Big) dt.$$

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Since $e = u - U$ and u solves the FE problem we get the error representation formula

$$\boxed{(\psi, e(0, s)) = \int_0^T \Big(m(U_t, \phi) + a(U, \phi) \Big) dt}$$

Estimating the error

Let $\pi : \mathcal{W} \to \mathcal{W}^{q-1}$ be the L_2 projection in time, and let P be a suitable interpolation operator into $\mathcal{V}^p$ in space. Then using Galerkin orthogonality we can replace ϕ by $\phi - \pi P\phi = \phi - P\phi + P\phi - \pi P\phi$

$$
\begin{aligned}
(\psi, e(0,s)) &= \int_0^T \Big(m(U_t, \phi - P\phi) + a(U, \phi - P\phi) \Big)\, dt \\
&\quad \int_0^T \Big(m(U_t, P\phi - \pi P\phi) + a(U, P\phi - \pi P\phi) \Big)\, dt \\
&= \sum_n \sum_j \int_{I_n} \Big(r^s_{\kappa_j}(U), \phi - P\phi \Big)\, dt \\
&\quad \sum_n \int_{I_n} \Big(r^t(U), P\phi - \pi P\phi \Big)\, dt,
\end{aligned}
$$

where

Estimating the error

$$(r^s_{\kappa_j}(U), \phi - P\phi) = -\frac{\sigma^2}{2}(s^2[U_s], \phi - P\phi)_{\partial\kappa_j},$$

$$(U_t + (r-\nu)sU_s + \frac{\sigma^2}{2}s^2U_{ss} - rU, \phi - P\phi)_{\kappa_j}$$

is the space residual, and

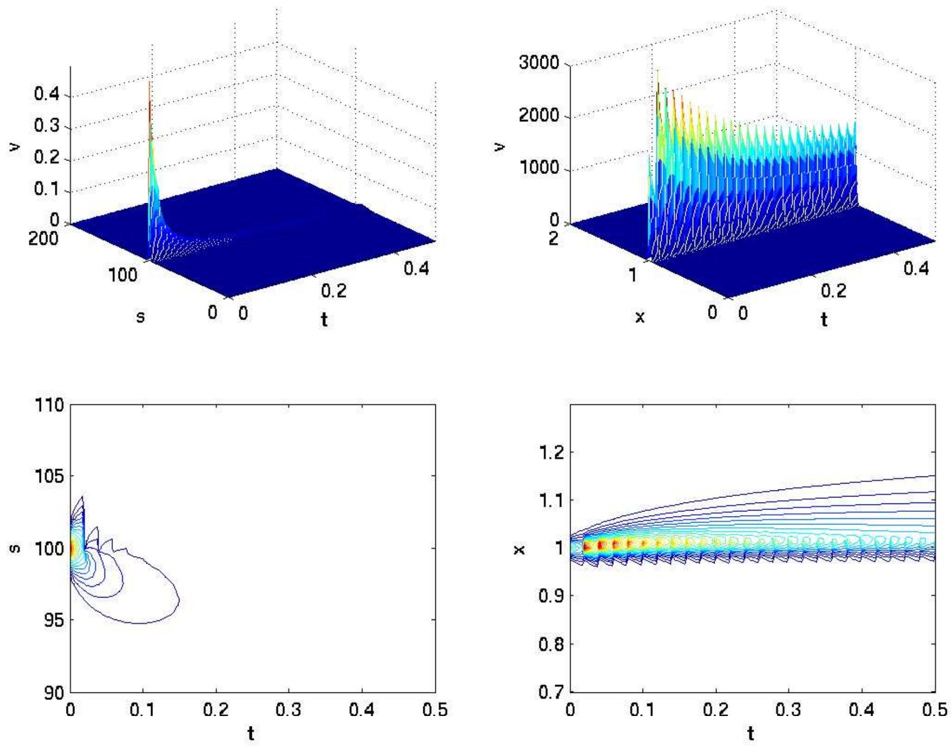
$$(r^t(U), P\phi - \pi P\phi) = (U_t + (r-\nu)sU_s + \frac{\sigma^2}{2}s^2U_{ss} - rU, P\phi - \pi P\phi)$$

is the time residual. Here we used the notation $[U_s]$ to denote the jump in U_s over element interfaces.

Error estimation algorithm

- Compute an approximation Φ of ϕ using an enriched finite element space, for instance higher order approximation.
- Compute $P\Phi$.
- Compute $\int_{I_n} \left(r^s_{\kappa_j}(U), \phi - P\phi \right) dt$ using quadrature in space and time for each element κ_j and time step.
- Compute $\pi P\Phi$.
- Compute $\int_{I_n} \left(r^t(U), P\phi - \pi P\phi \right) dt$ using quadrature in space and time for each time step.

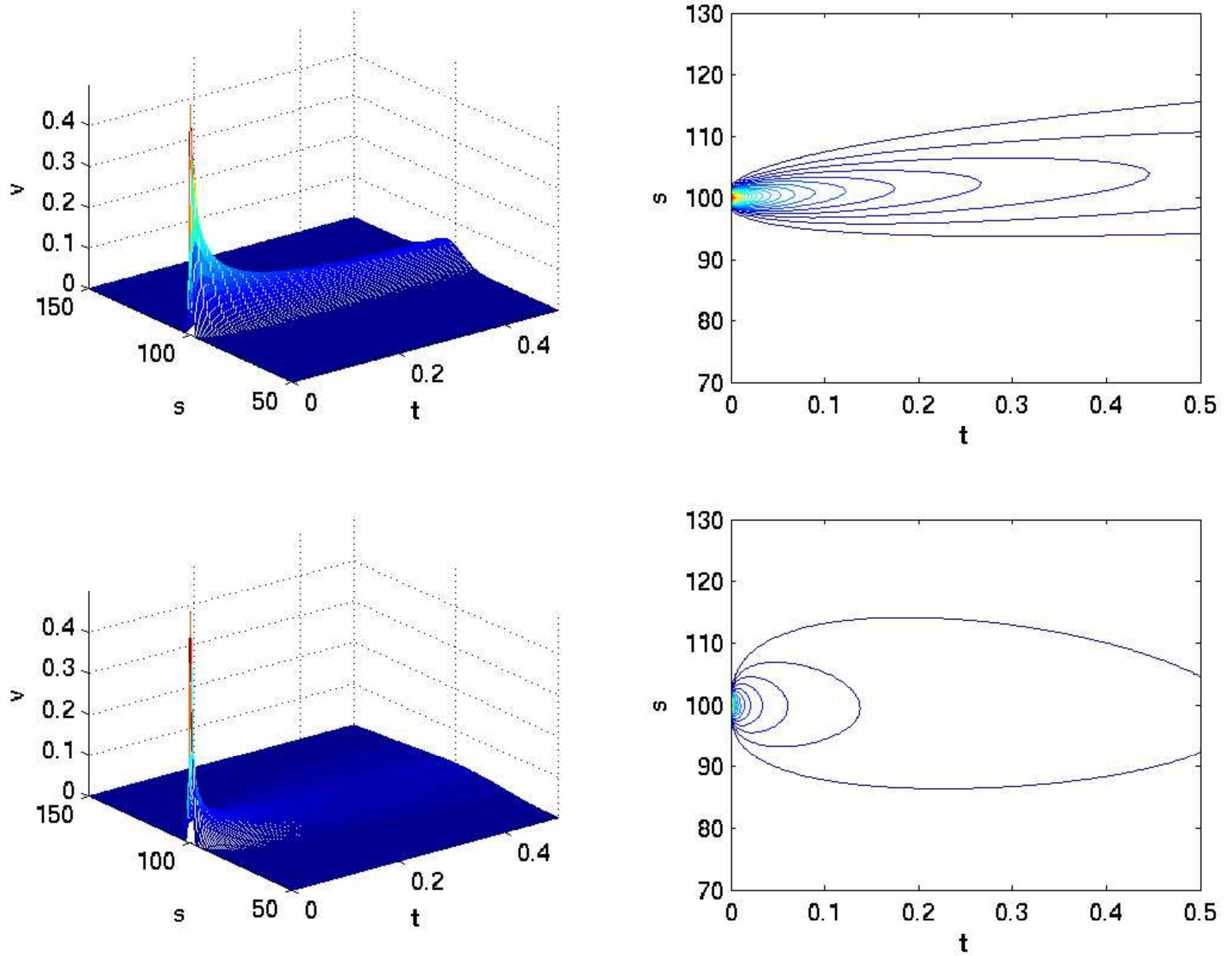
Examples of dual solutions



Fixed strike lookback put. To the left, ϕ. To the right φ. Below, contour plots using 30 levels. In both cases $\sigma = 0.2$, $r = 0.05$, $q = 0.0$, and weekly sampling was used.

Example 1

To estimate the error at $s = s_\alpha$ we let $\psi = \delta_{s_\alpha}(s)$. Figure shows ϕ, for $\sigma = 0.1$ and $\sigma = 0.3$ when $r = 0.1$.



Example 2

We approximate the derivative of u by $\frac{\partial u}{\partial s} \approx \frac{u(s+\mu)-u(s-\mu)}{2\mu}$. To estimate the error of the derivative of the solution we thus choose $\psi(s) = (\delta_{s_\alpha}(s-\mu) - \delta_{s_\alpha}(s+\mu))/2\mu$.

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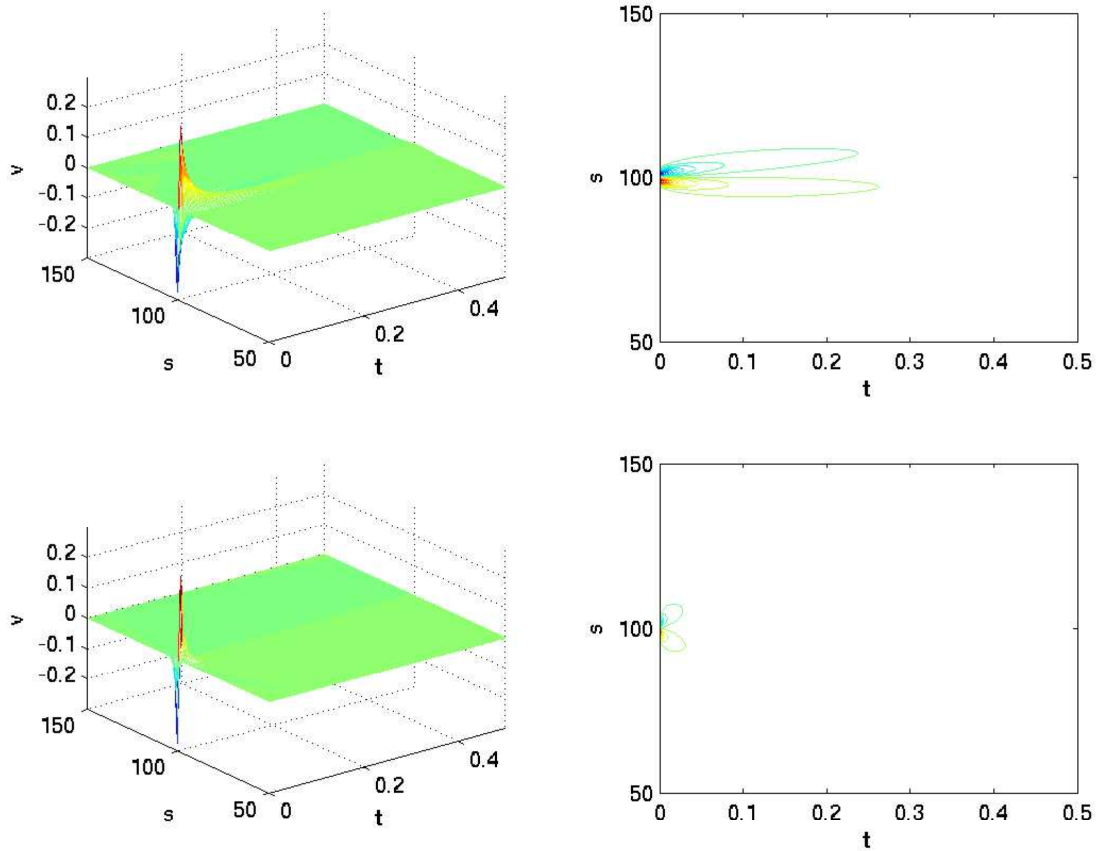

Barrier options

- Up-and-out call option with payoff

$$\max(S(T) - K, 0)1_{\{max_{t\in[0,T]}S(t)<H\}},$$

at maturity T.

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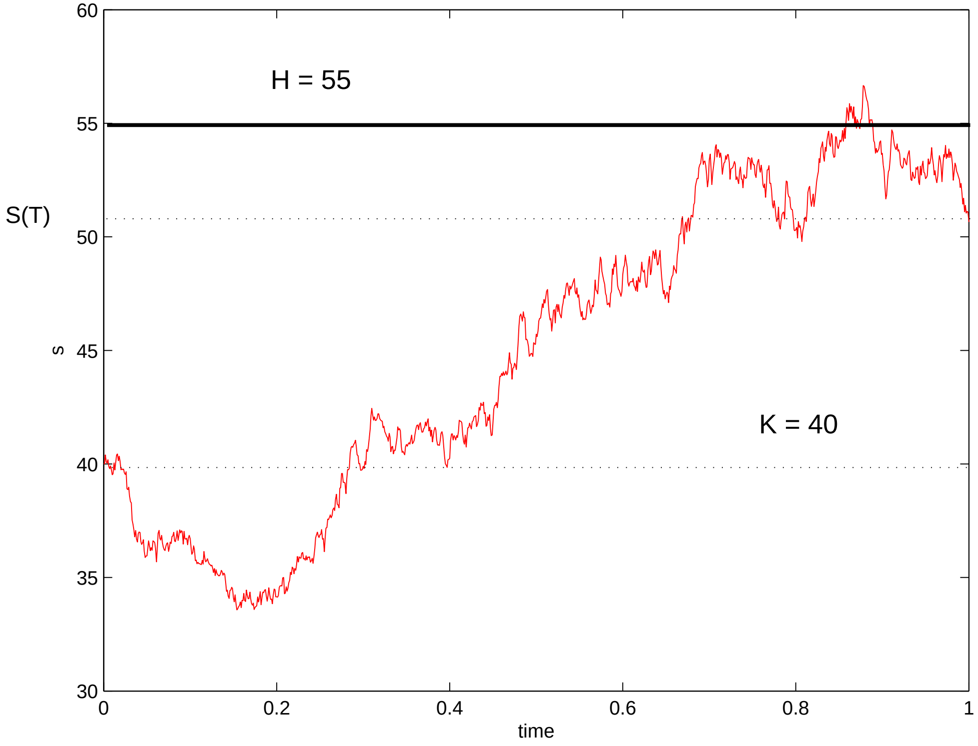

Barrier options

- Up-and-out call option with payoff

$$\max(S(T) - K, 0)1_{\{max_{t\in[0,T]} S(t) < H\}},$$

at maturity T.

- Down-and-out call option with payoff

$$\max(S(T) - K, 0)1_{\{min_{t\in[0,T]} S(t) > H\}}.$$

at maturity T.

Barrier options - overview

- The same pricing PDE (BS) still holds between monitoring dates

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- The finite element method is the same (except for the barrier constraints)
- The error representation formula is still valid with a suitable choice of the dual problem

Barrier constraint

For the value of the discretely monitored up-and-out call, $u(t,s)$, with monitoring dates t_k^*, we have the barrier constraint

$$u^-(t_k^*, s_j) = \begin{cases} 0 & \text{if } s_j \geq H, \quad j = 0, 1, \dots, J, \\ u^+(t_k^*, s_j) & \text{if } s_j < H, \quad j = 0, 1, \dots, J, \end{cases}$$

where H is the barrier.

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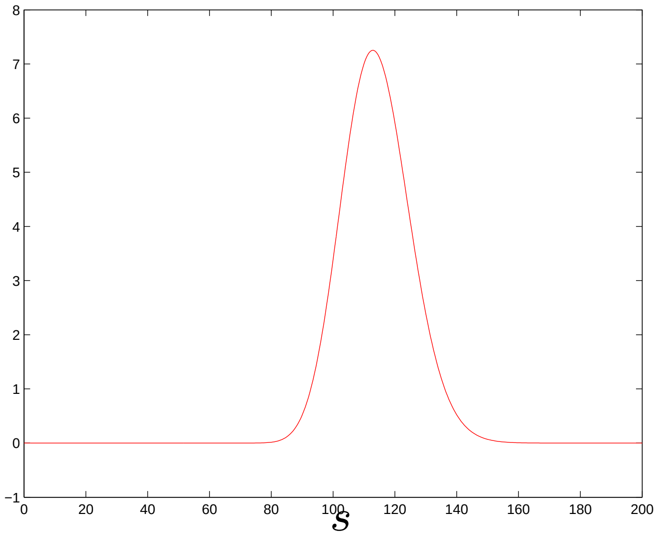

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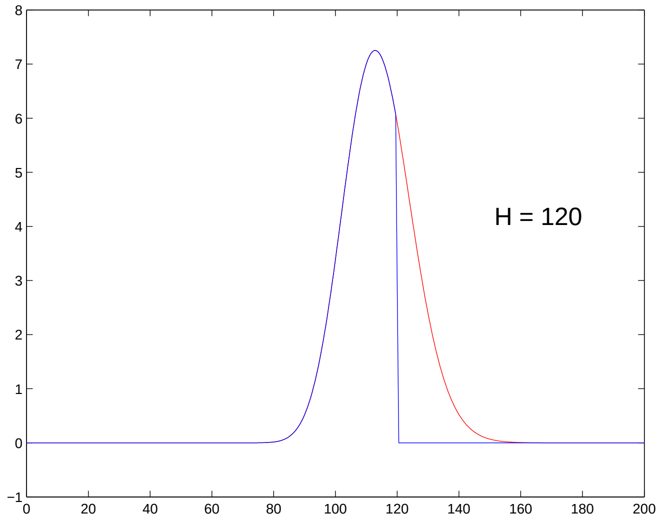
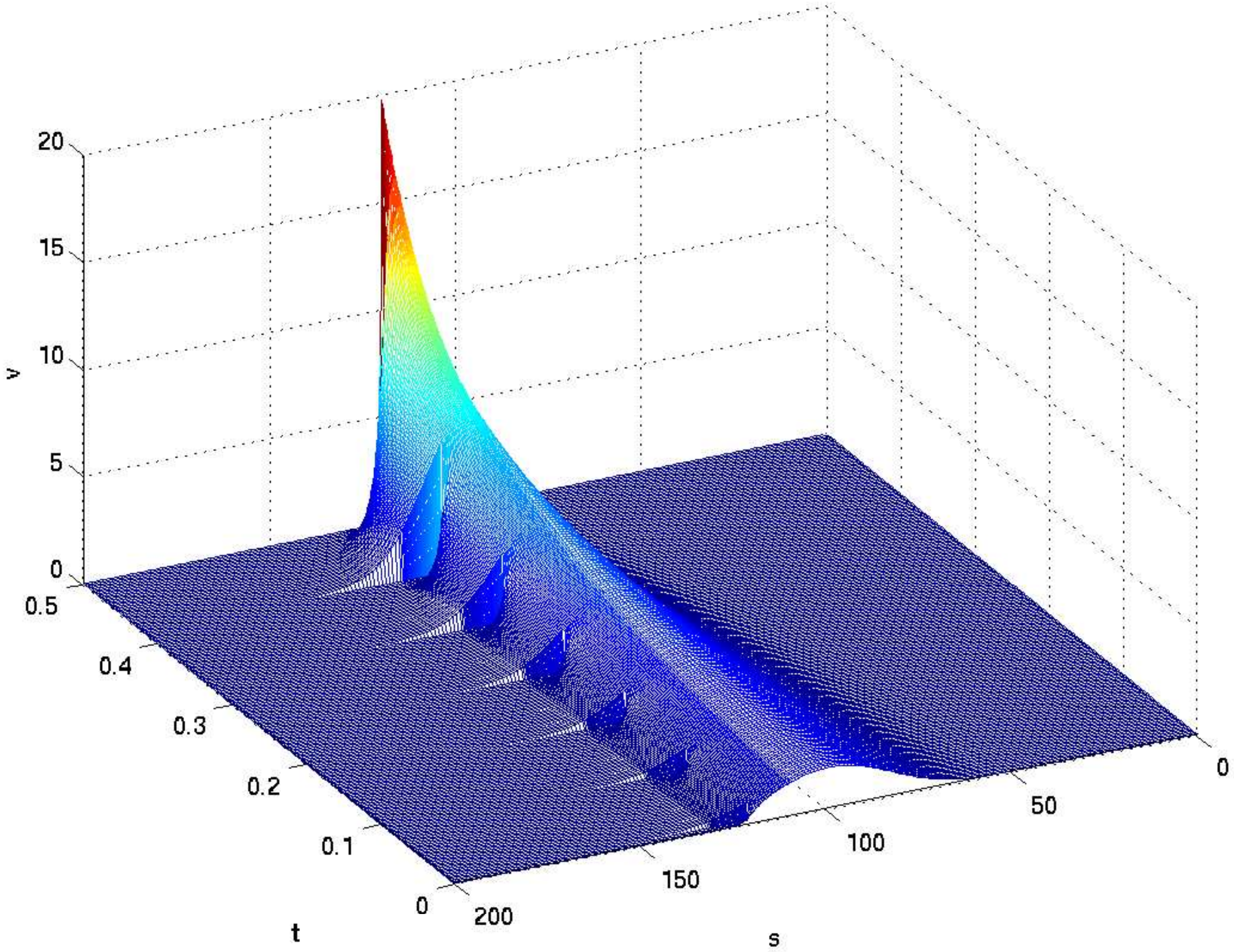


The up-and-out barrier call

Parameter values are $\sigma = 0.3$, $r = 0.1$, $q = 0.0$, $T = 0.5$, $t = 0.0$, $K = 100$, and $H = 120$. Monthly sampling.



A posteriori error estimation

Dual problem: Find $\phi \in \mathcal{W}$

$$\begin{cases} -\phi_t + (\sigma^2 + \nu - 2r)\phi - (r - \nu - 2\sigma^2)s\phi_s + \frac{\sigma^2}{2}s^2\phi_{ss} = 0, \\ \phi(0,s) = \delta_{s_\alpha}, \\ \phi^+(t_k^*) = \begin{cases} 0 & \text{if } s_j \geq H, \quad j = 0,1,\ldots,J, \\ \phi^-(t_k^*, s_j) & \text{if } s_j < H, \quad j = 0,1,\ldots,J, \end{cases} \qquad t_k^* \in D. \end{cases}$$

Error representation formula

Multiplying with the error $e = u - U \in \mathcal{W}$ and integrating in space and time we get

$$\sum_k \int_{t_{k-1}^*}^{t_k^*} \Big(-(\phi_t, e) + (\sigma^2 + \nu - 2r)(\phi, e)$$

$$- (r - \nu - 2\sigma^2)\big(s\phi_s, e\big) + \frac{\sigma^2}{2}\big(s^2\phi_{ss}, e\big)\Big) dt = 0.$$

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$$- (r - \nu - 2\sigma^2)\big(s\phi_s, e\big) + \frac{\sigma^2}{2}\big(s^2\phi_{ss}, e\big)\Big) dt = 0.$$

We now have jumps in ϕ at the monitoring dates, affecting only the first term. Studying this term in detail we see that

$$-\sum_k \int_{t_{k-1}^*}^{t_k^*} (\phi_t, e)\, dt$$

$$= \sum_k \left(\int_{t_{k-1}^*}^{t_k^*} (\phi, e_t)\, dt - (\phi^-(t_k^*), e^-(t_k^*)) + (\phi^+(t_{k-1}^*), e^+(t_{k-1}^*)) \right)$$

Error representation formula

Next we note that

$$
\begin{aligned}
&(\phi^-(t_k^*), e^-(t_k^*)) - (\phi^+(t_k^*), e^+(t_k^*)) \\
&= \int_{s<h(t_k^*)H} \big(\phi^-(t_k^*)e^-(t_k^*) - \phi^+(t_k^*)e^+(t_k^*)\big)\; ds \\
&\quad + \int_{s\geq h(t_k^*)H} \big(\phi^-(t_k^*)e^-(t_k^*) - \phi^+(t_k^*)e^+(t_k^*)\big)\; ds = 0,
\end{aligned}
$$

were the first term on the right is zero since $\phi^+(t_k^*) = \phi^-(t_k^*)$, and $e^-(t_k^*) = e^+(t_k^*)$ for $s < h(t_k^*)H$, and second term is zero since $\phi^+(t_k^*) = e^-(t_k^*) = 0$, for $s \geq h(t_k^*)H$.

Error representation formula

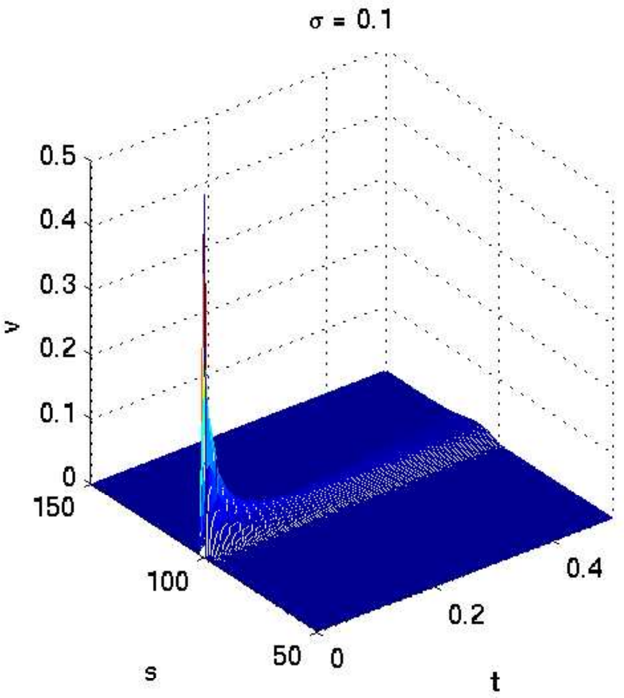
Integrating the other terms as well just as for the European option, we get the same error representation formula

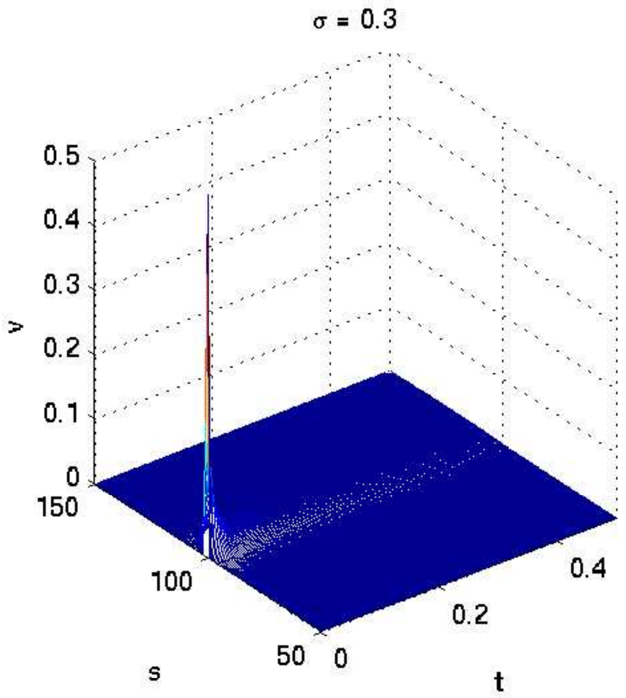
$$\boxed{e(0, s_\alpha) = \sum_k \int_{t_{k-1}^*}^{t_k^*} (m(U_t, \phi) + a(U, \phi))}$$

where the bilinear forms $m(u_t, v)$ and $a(u, v)$ are defined exactly as before.

Examples of dual solutions

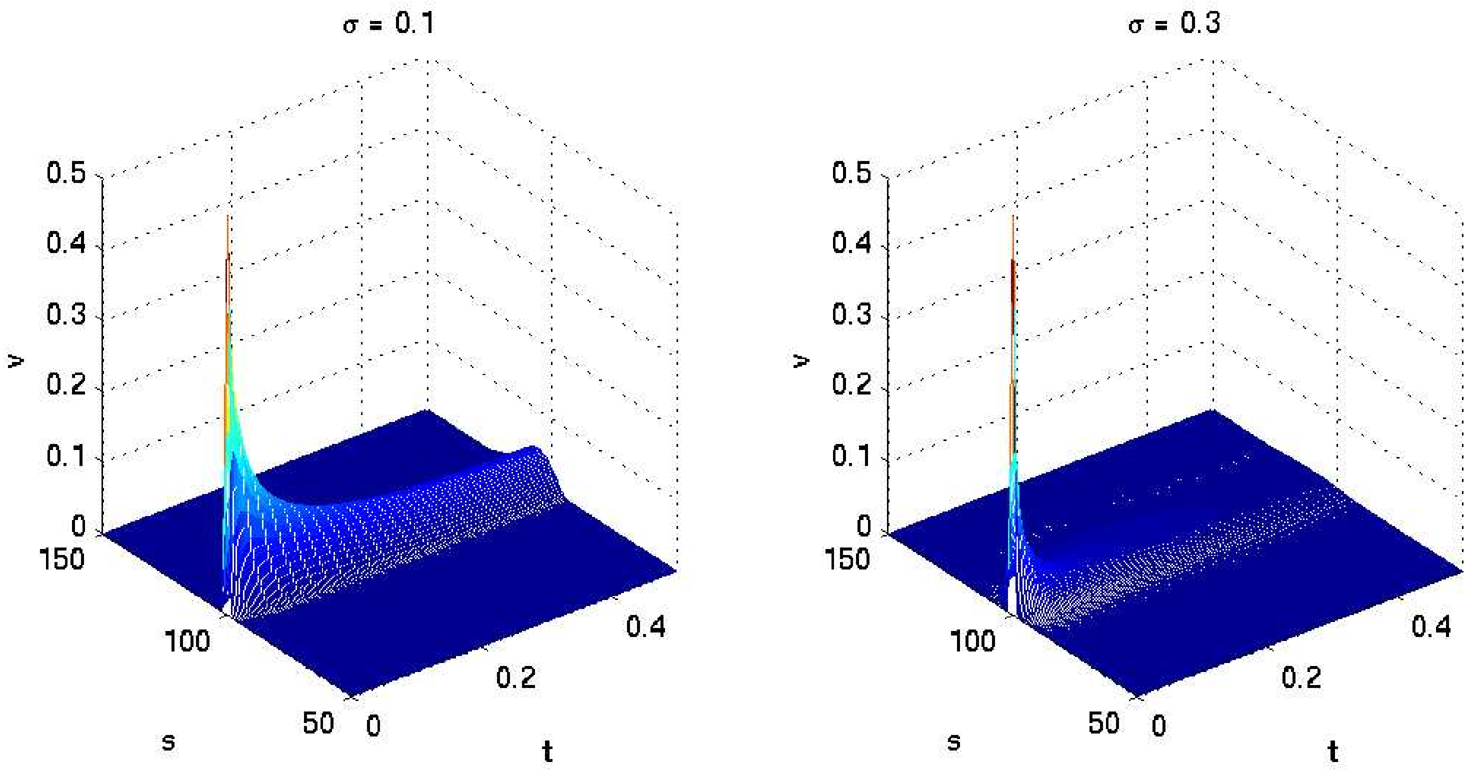
ϕ for two different values of σ when $r = 0.10$ and $q = 0.0$ for the weekly sampled down-and-out barrier call option with $S_0 = 100$ and barrier $H = 99.9$.





Examples of dual solutions

ϕ for the weekly sampled double barrier call option with $S_0 = 100$ and barriers $H_{low} = 95$ and $H_{high} = 125$, when $r = 0.10$ and $q = 0.0$.



Examples of dual solutions

ϕ for the monthly sampled double barrier call option with $S_0 = 100$ and barriers $H_{low} = 95$ and $H_{high} = 125$ when $\sigma = 0.2$, $r = 0.10$ and $q = 0.0$.