Adaptive Submodeling

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Outline and Contributors

Outline:

- Submodeling
- A posteriori Error Estimation for Submodeling
- Mesh refinement respecting geometry.

Contributors:

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- August Johansson
- Klas Samuelsson

Funding: EU-ViSiCADE project, the Swedish Science Foundation, and the Foundation for Strategic Research.

Main goal

- Develop efficient adaptive finite element techniques for complex models arising in industrial applications.
- Particular focus on applications in solid mechanics.
- Adaptivity should be done in an interactive fashion in real time.

Typical problem

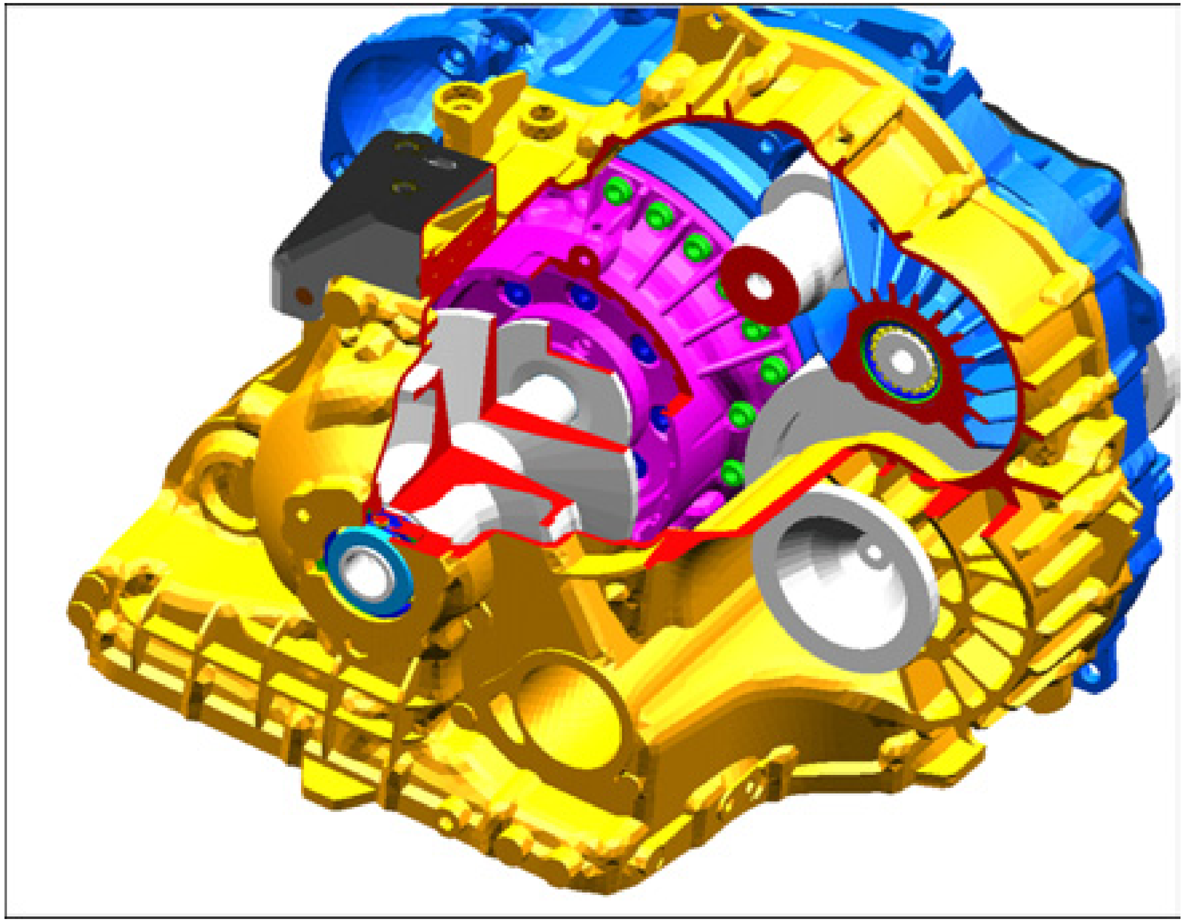


Figure 1: Gearbox model 2.3 Mdofs

Typical problem

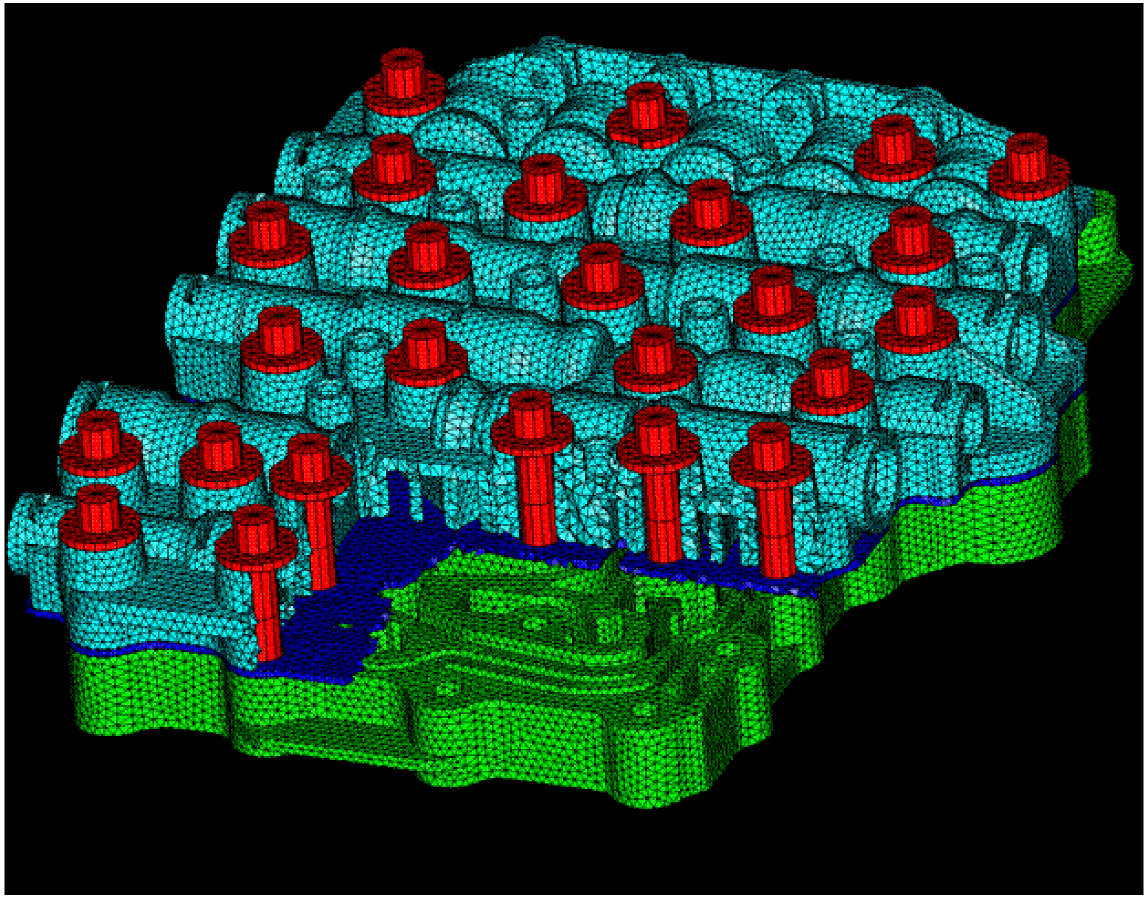


Figure 2: Control housing 2.9 Mdofs

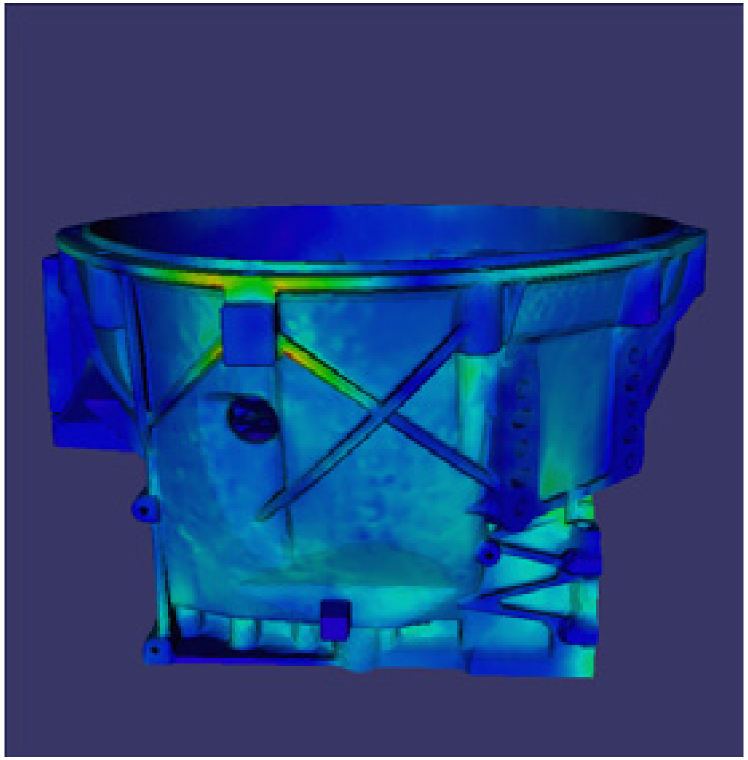
Typical problem: features

- Problems have **several million dofs** in the initial coarse grid!
- Coarse grid model based on **simplified geometric model** with small details removed.
- Small details may be critical for stress levels.
- Enhanced local resolution necessary to compute accurate stresses.
- Automatic residual based refinement may manufacture models which are too large.
- Seek to minimize/avoid global solves on refined grid.

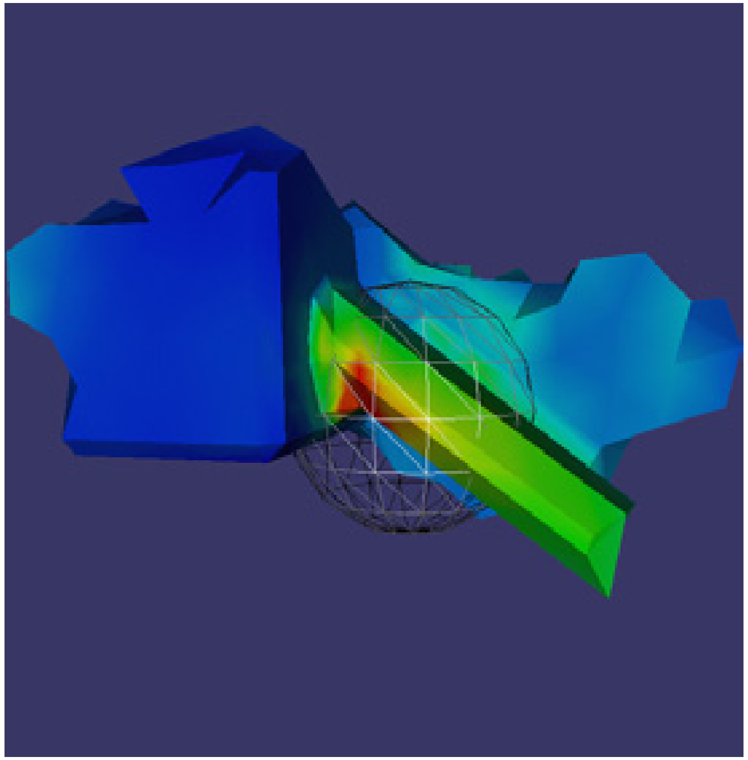
Submodeling

- Solve global coarse grid problem and store solution.
- Identify area of interest interactively.
- Cut out suitable local model containing area of interest.
- Compute boundary conditions from coarse grid solution.
- Refine the mesh in the area of interest.
- Compute enhanced local solution by solving local problem.

Submodeling: Example 1

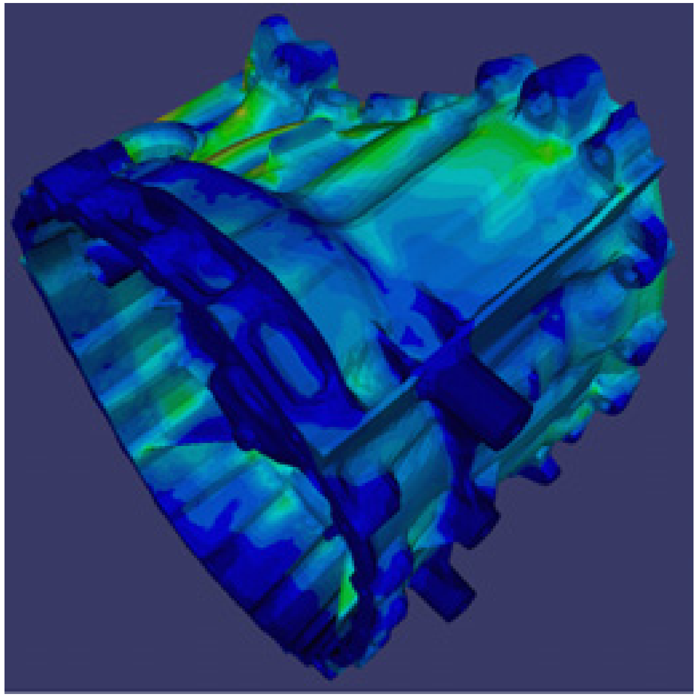


(a) Global problem

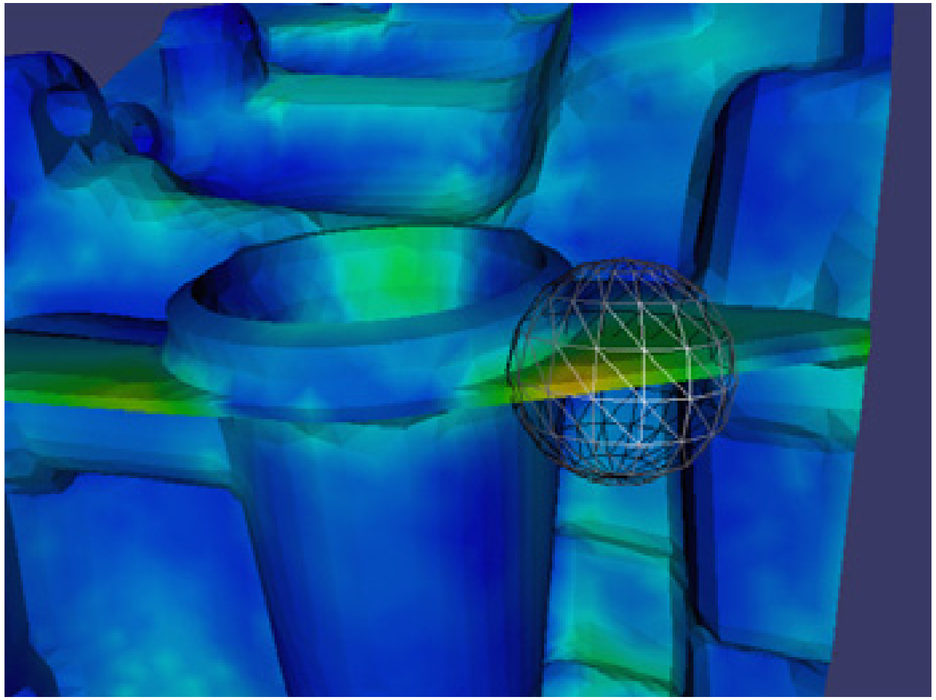


(b) Submodel

Submodeling: Example 2

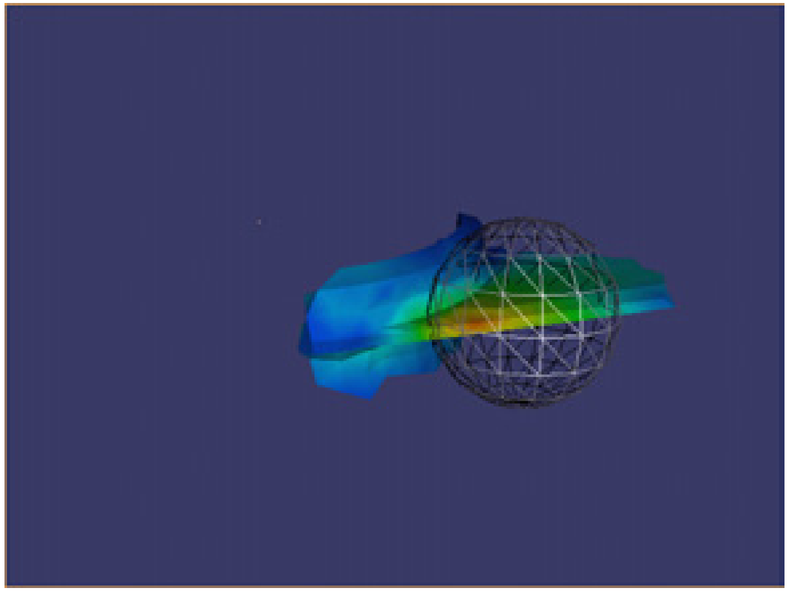


(c) Global problem

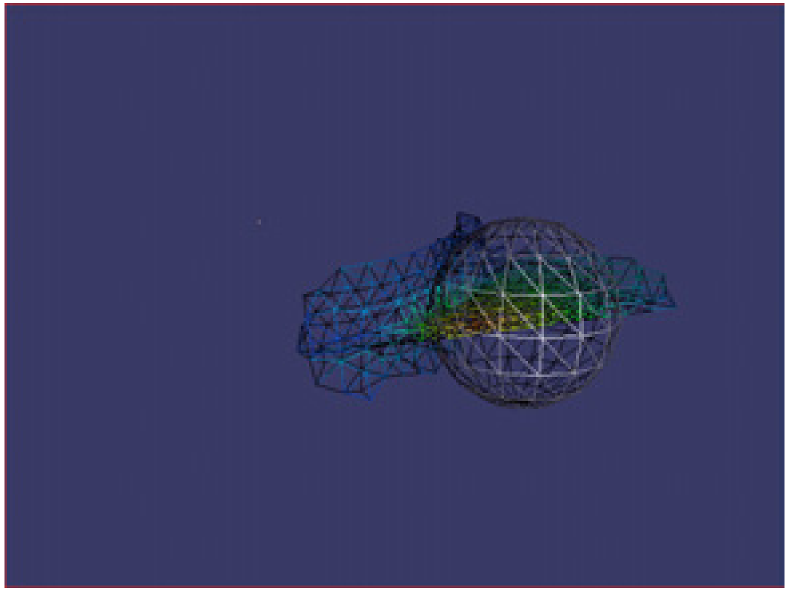


(d) Definition of area of interest

Submodeling: Example 2



(e) Submodel



(f) Submodel mesh

Submodeling: Interface

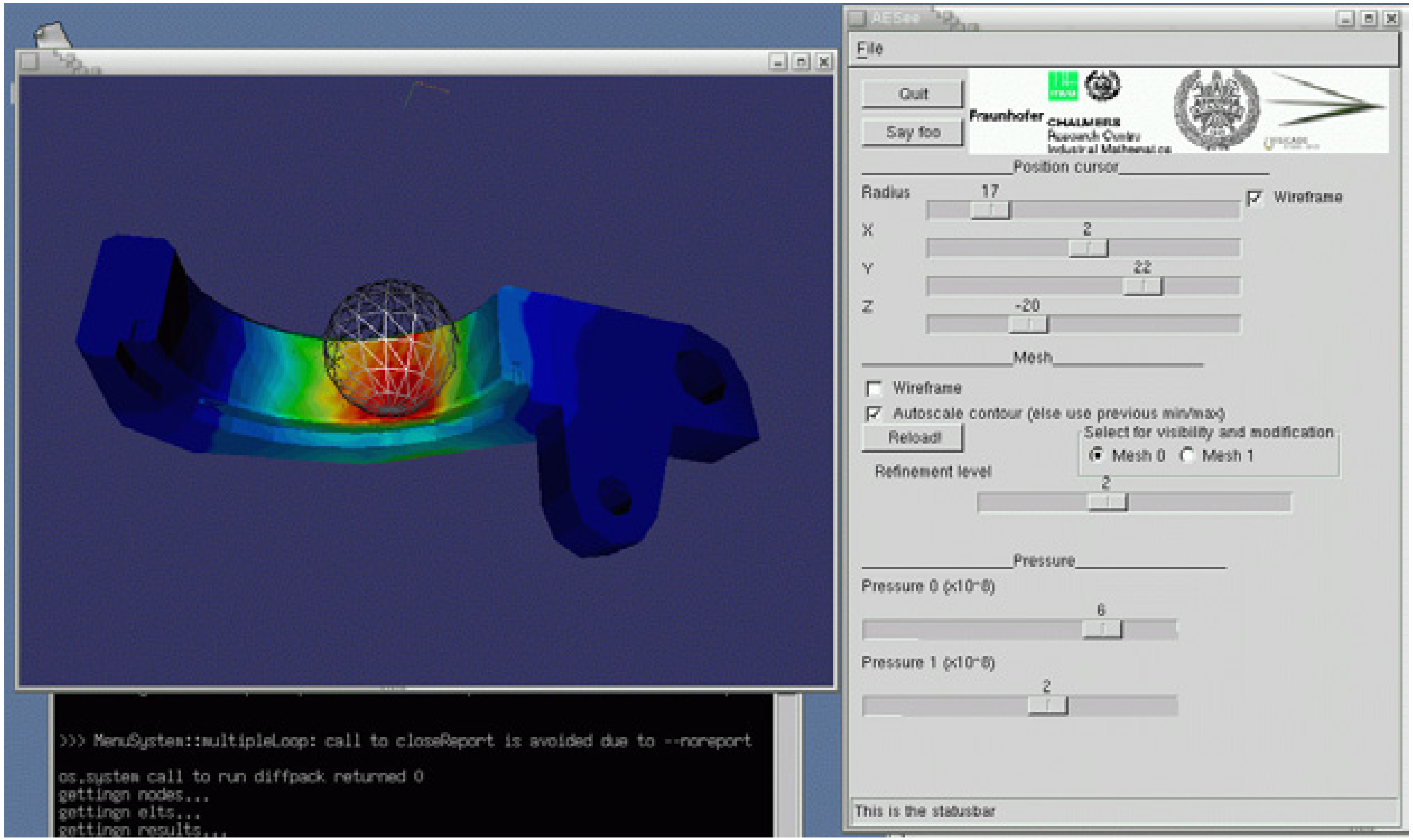


Figure 3: GUI for interactive submodeling

Submodeling: Notation

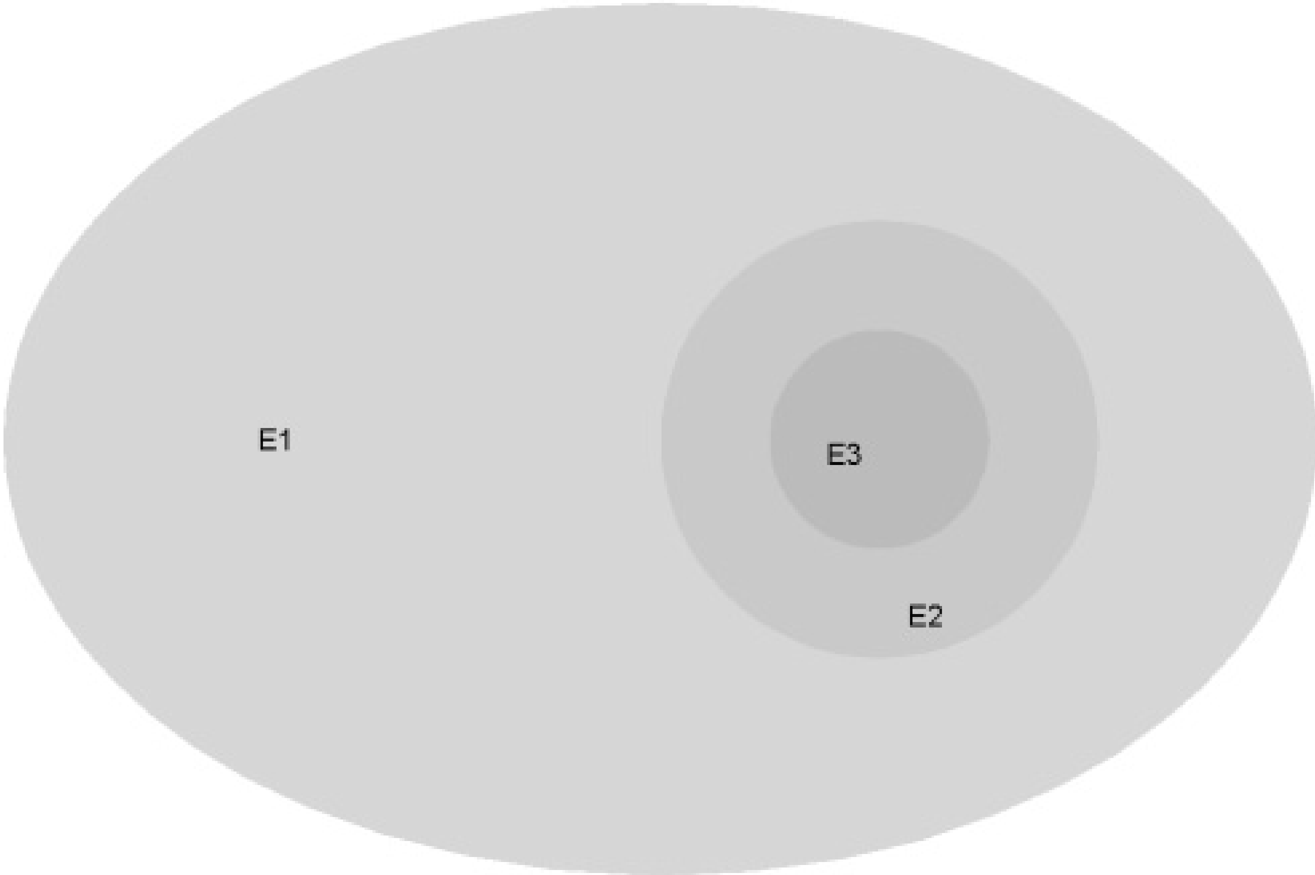


Figure 4: The domain Ω, the submodel ω, domain of interest ω_0

Submodeling: Elasticity

Find $u : \Omega \to \mathbf{R}^3$ such that

$$
\begin{aligned}
-\nabla \cdot \sigma = f \quad &\text{in } \Omega, \\
\sigma = \lambda \, \nabla \cdot u I + 2\mu\epsilon(u) \quad &\text{in } \Omega, \\
u = g_D \quad &\text{on } \Gamma_{\mathrm{D}}, \\
n \cdot \sigma = g_N \quad &\text{on } \Gamma_{\mathrm{N}}.
\end{aligned}
$$

where $\epsilon(u) = (\nabla u + \nabla u^T)/2$ is the strain and λ and μ are the Lame parameters.

Submodeling: FEM

Global FEM: Find $\boxed{u_H \in V_H(\Omega)}$ such that

$$a(u_H, v) = l(v) \quad \text{for all } v \in V_H(\Omega)$$

Submodel FEM: Find $\boxed{u^h \in V^h_{u_H}(\omega)}$ such that

$$a(u_h, v) = l(v) \quad \text{for all } v \in V_{h,0}(\omega)$$

where $V^h_g \subset V$ is a finite element space of piecewise polynomials with $v = g$ on $\partial\omega$.

$$\boxed{u^h_H = \begin{cases} u_H & \text{in } \Omega \setminus \omega \\ u^h & \text{in } \omega \end{cases}}$$

A posteriori error estimation

Contributions to error in submodel:

- Coarse grid error gives error in submodel boundary conditions.
- Resolution in the submodel.
- Resolution of the geometry.

Seek to construct algorithms which balance these three contributions.
A posteriori error estimates can be derived by duality based methods.

Neglect geometry resolution for simplicity.

Goal oriented error estimates

Objective: Let $m(\cdot)$ be a linear functional on V. We seek to estimate the error

$$m(u) - m(u_H^h)$$

in the functional in terms of the computed solution.

Examples of functionals

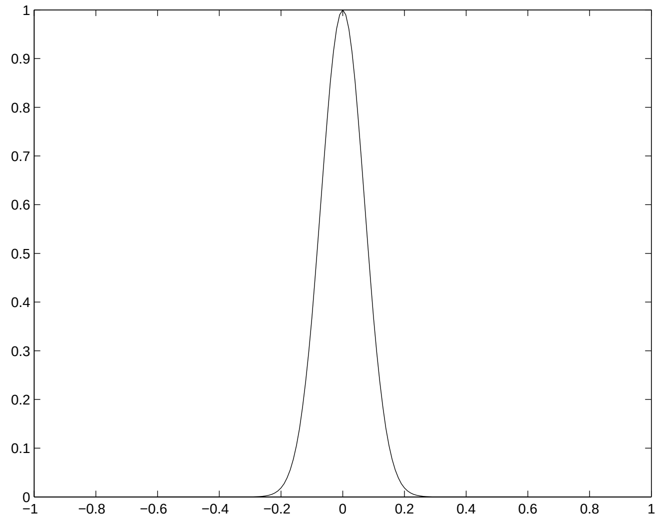
- Average of error in subdomain

$$m(e) = \int_\omega e\psi dx$$

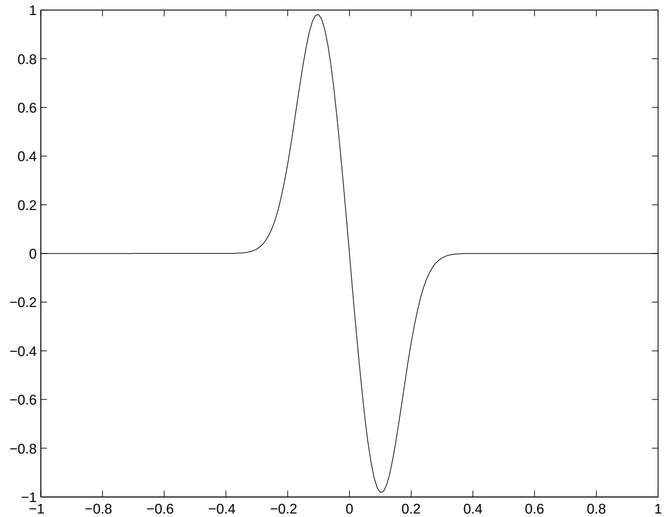
- Average of error in derivative in subdomain

$$m(e) = -\int_\omega e\partial_x\psi dx$$

Typical weight functions



(a) Point value



(b) Derivative point value

Dual problem

To represent the error we introduce the dual problem:
Find $\phi \in V$ such that

$$m(v) = a(v, \phi) \quad \text{all } v \in V.$$

Error representation

Setting $v = e = u - u_H^h$ in the dual problem we get

$$\begin{aligned} m(u) - m(u_H^h) &= m(e) \\ &= a(e, \phi) \\ &= l(\phi) - a(u_H^h, \phi) \\ &= l(\phi - \pi\phi) - a(u_H^h, \phi - \pi\phi) \\ &\qquad + l(\pi\phi) - a(u_H^h, \pi\phi). \end{aligned}$$

Here we used the linearity of $a(\cdot, \cdot)$ and $m(\cdot)$ and subtracted and added an interpolant $\pi\phi \in V_h$ of ϕ.

Note that the last term is **not zero** due to variational crime.

Last term

The term

$$l(\pi\phi) - a(u_H^h, \pi\phi)$$

only depends on the elements neighboring $\partial\omega$.

In fact:

$$l(\pi\phi) - a(u_H^h, \pi\phi) = l(\pi\phi - w) - a(u_H^h, \pi\phi - w)$$

for all $w \in V_H$ such that $w = 0$ on $\partial\omega$.

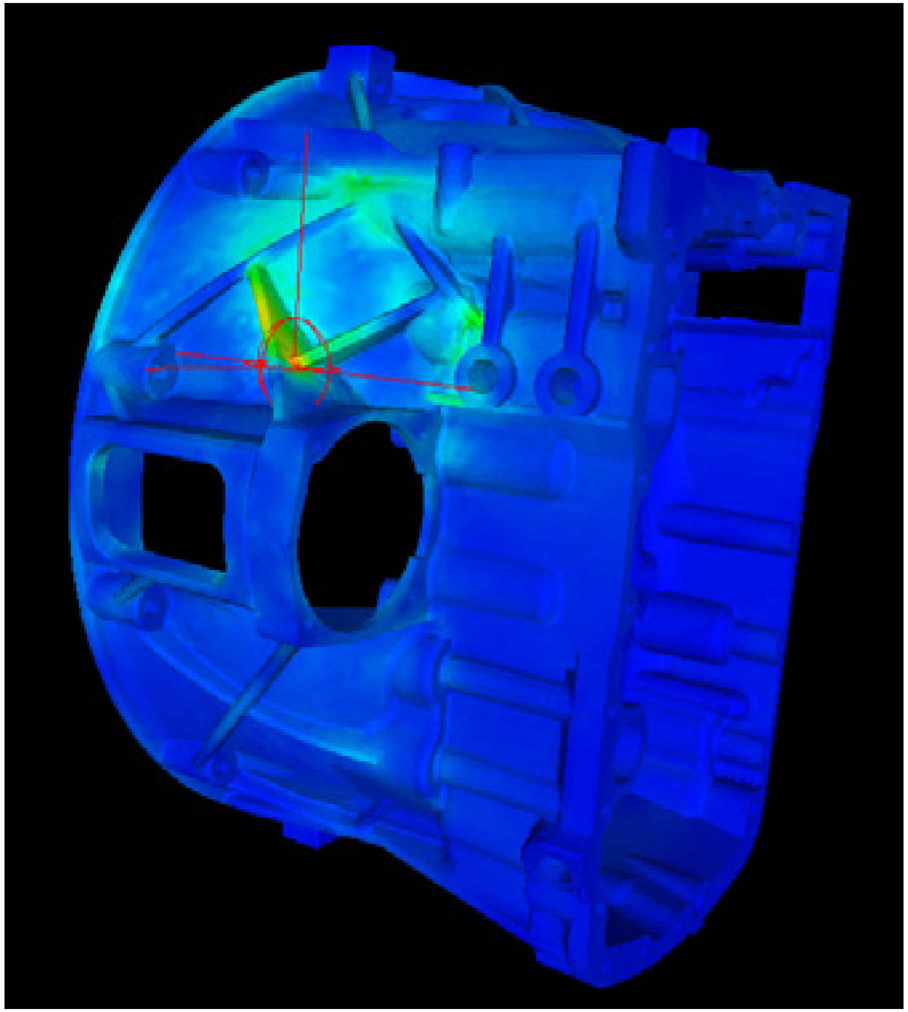
Elasticity: dual problem

Find $\phi : \Omega \to \mathbf{R}^3$ such that

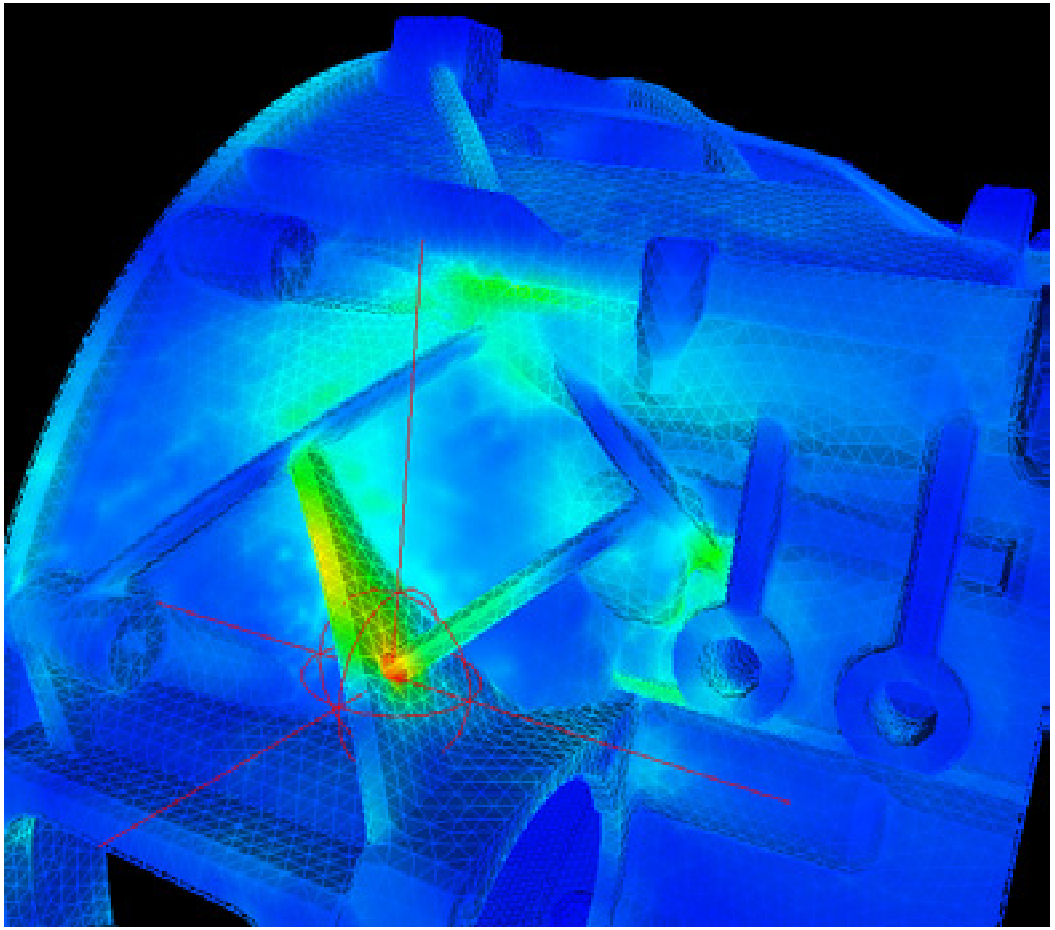
$$
\begin{aligned}
-\nabla \cdot \sigma = \psi \quad & \text{in } \Omega, \\
\sigma = \lambda \, \nabla \cdot \phi I + 2\mu\epsilon(\phi) \quad & \text{in } \Omega, \\
\phi = 0 \quad & \text{on } \Gamma_{\mathrm{D}}, \\
n \cdot \sigma = 0 \quad & \text{on } \Gamma_{\mathrm{N}}.
\end{aligned}
$$

Taking $\psi = \delta_{x_0} m$ controls the displacement error $(u - U)(x_0) \cdot m$.

Example: solution to dual



Example: solution to dual



Local dual problem

Idea: Derive an estimate based on

- Dual argument on the subdomain.
- A posteriori error estimate of error in boundary condition.

Local dual problem: find $\phi : \Omega \to \mathbf{R}^3$ such that

$$
\begin{aligned}
-\nabla \cdot \sigma = \psi \quad & \text{in } \omega, \\
\sigma = \lambda\, \nabla \cdot \phi I + 2\mu\epsilon(\phi) \quad & \text{in } \omega, \\
\phi = 0 \quad & \text{on } \partial\omega.
\end{aligned}
$$

Error representation

We have

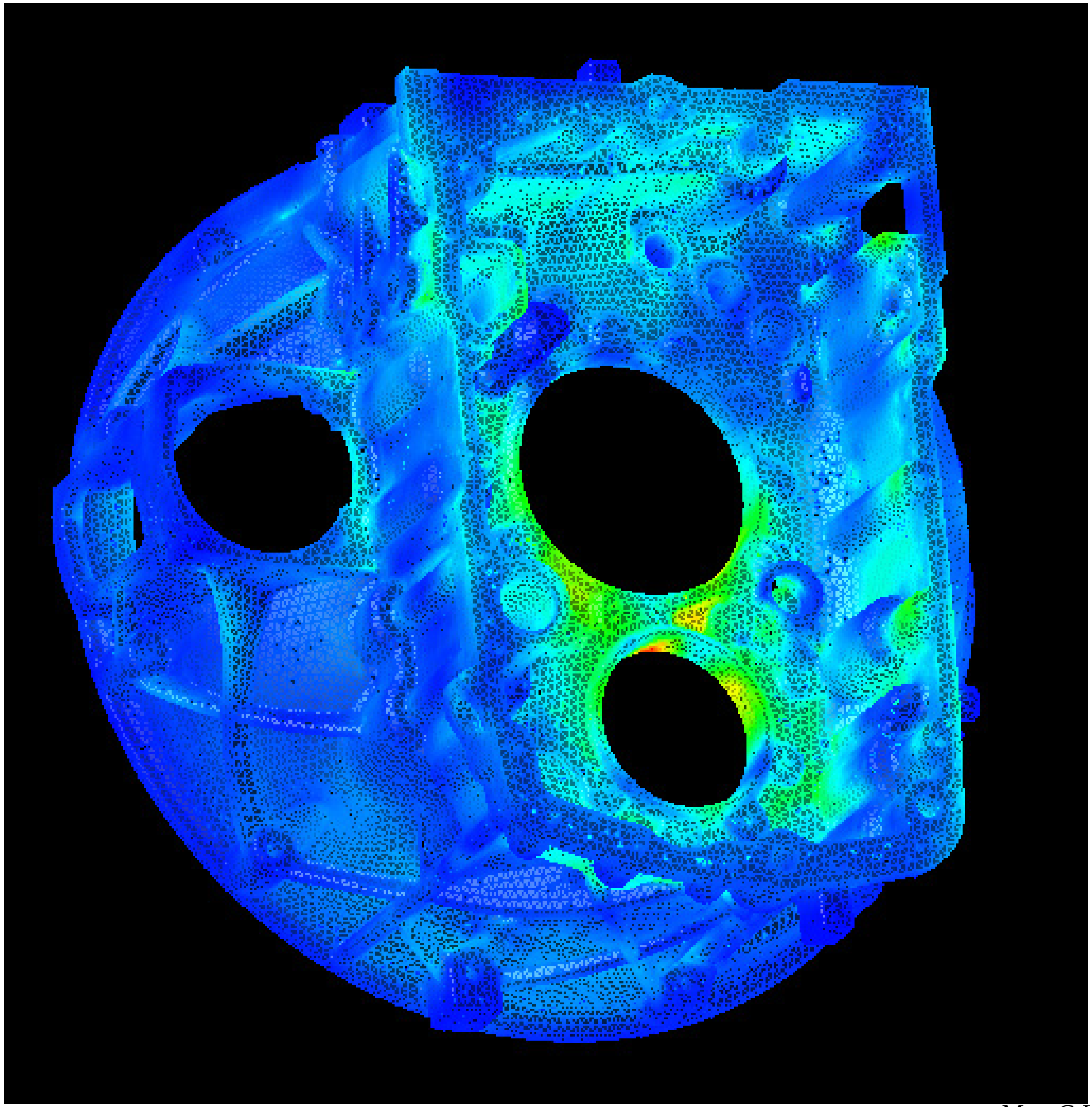
$$\int_\omega e\psi = \int_\omega e \cdot (-\nabla \cdot \sigma(\phi))$$
$$= \int_\omega \nabla \cdot \sigma(e) : \epsilon(\phi - \pi\phi) + \int_{\partial\omega} e \cdot (n \cdot \sigma(\phi))$$

First term is standard and second term can be estimated using a global duality argument and standard estimates

$$\left| \int_{\partial\omega} e \cdot (n \cdot \sigma(\phi)) \right| \leq C \| H^\alpha R(u_H) \|$$

with $\alpha \geq 3/2$ (or a more detailed approach)

Elasticity: solution to primal



Elasticity: solution to local dual

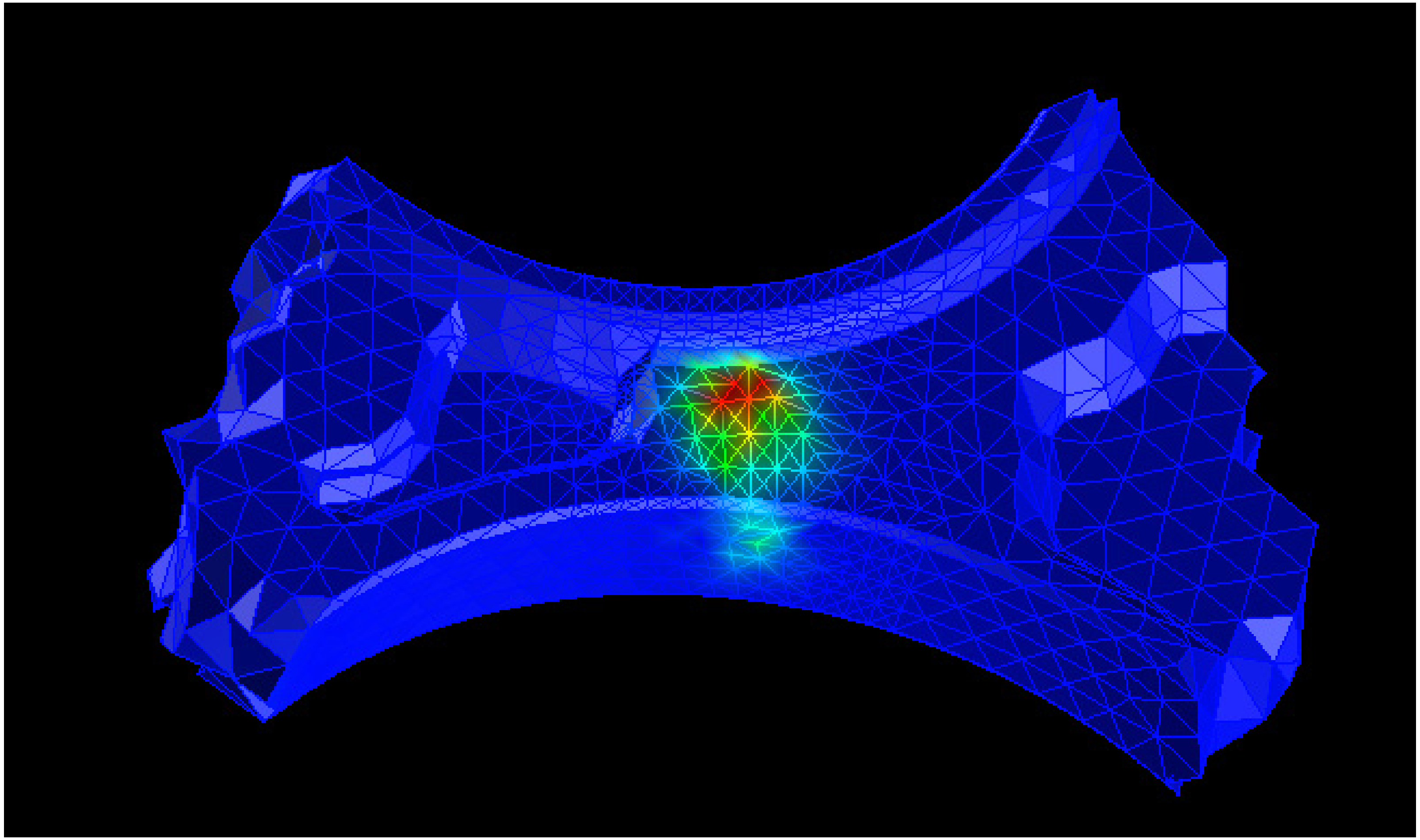


Figure 5: data = dipole$\times e_x$

Elasticity: solution to local dual

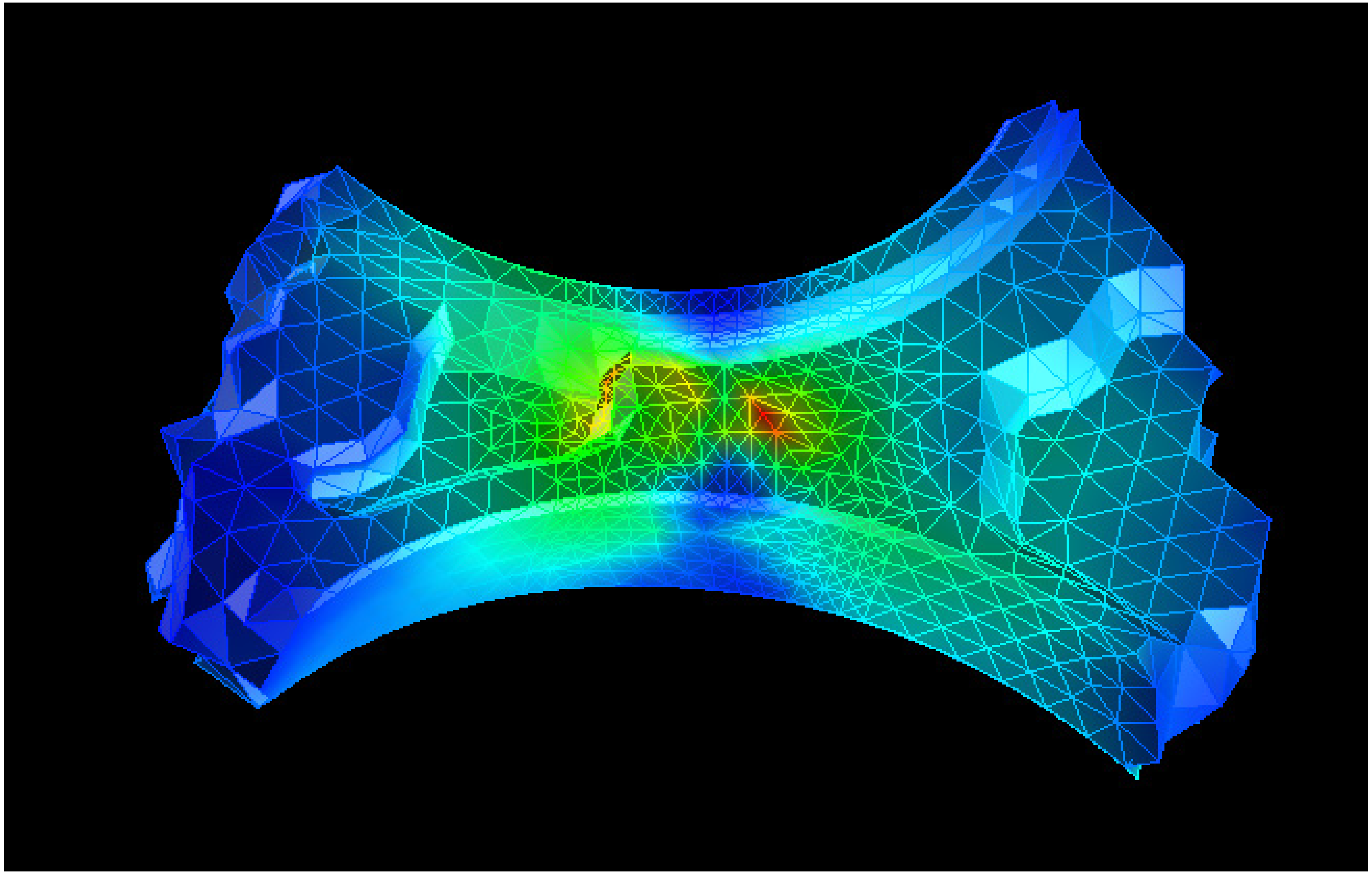


Figure 6: data = dipole $\times e_y$

Elasticity: solution to local dual

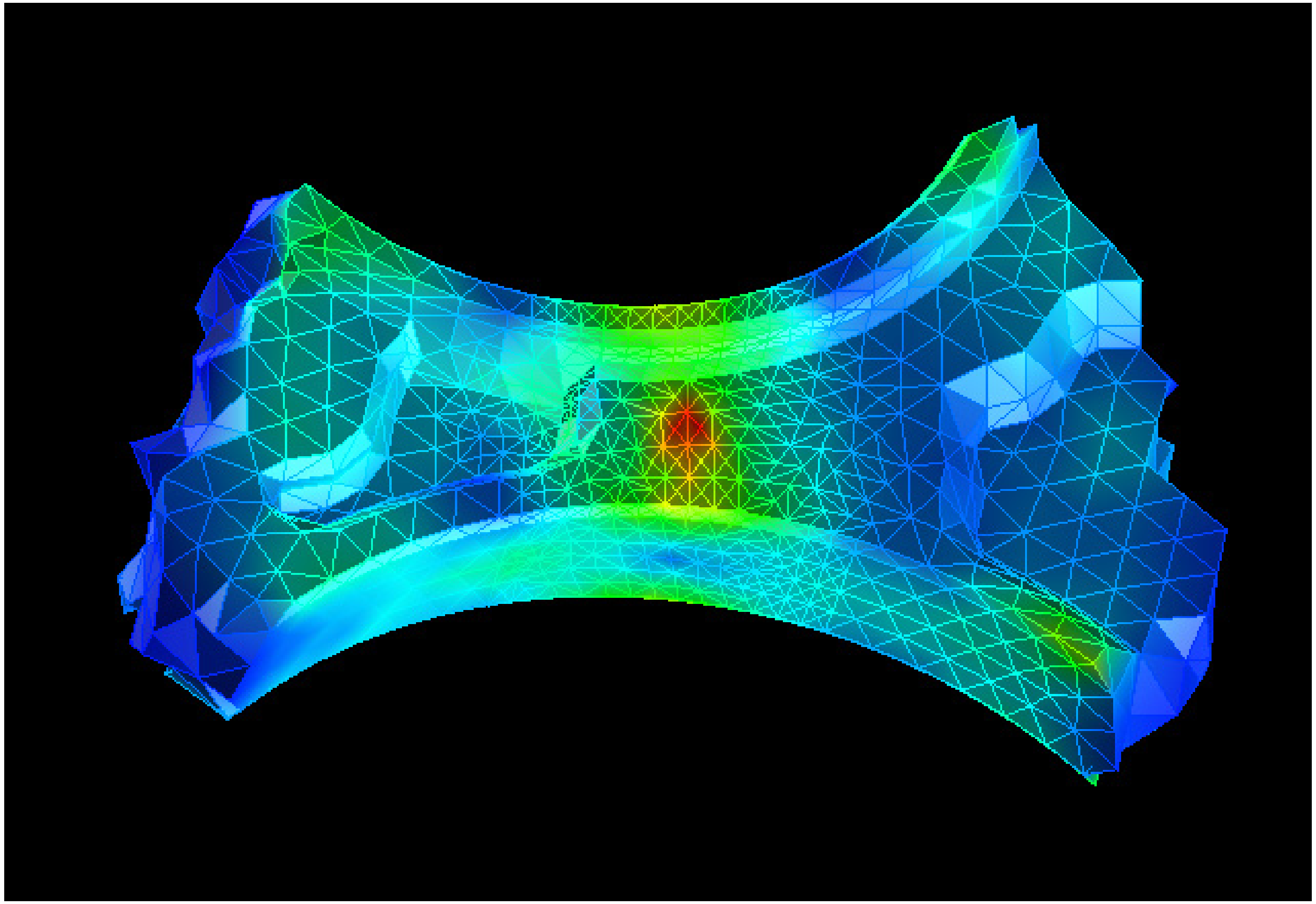


Figure 7: data = dipole $\times e_y$

Mesh refinement

- Mesh refinement **must respect CAD** geometry
- Otherwise we **do not get convergence** to the true solution and artificial stress concentrations may occur.

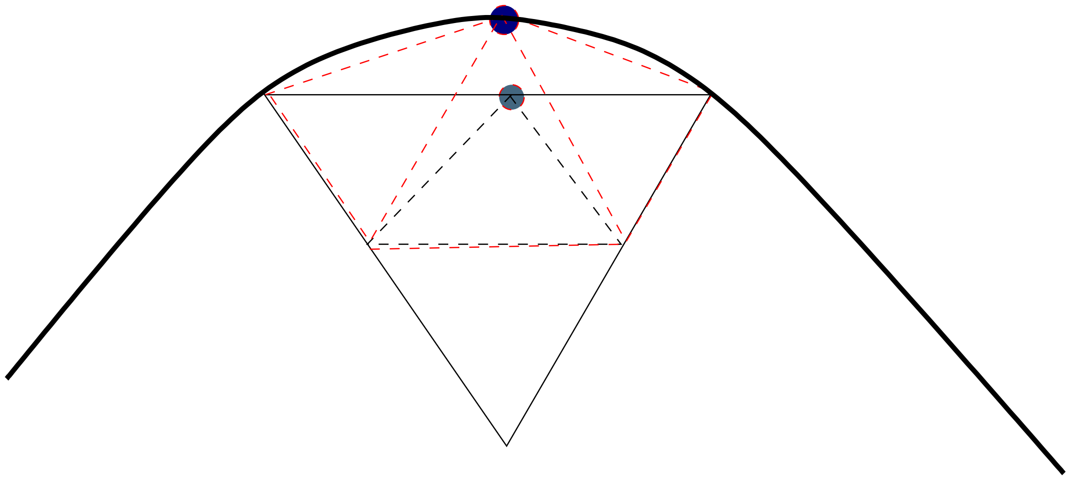


Figure 8: Projection of new node to CAD geometry

Mesh refinement

Basic principle:

- Find NURBS patches corresponding to the triangles under refinement.
- When the coarse mesh is refined the new nodes are projected to the true geometry using the surface descriptions.

Requirements:

- Sufficiently good quality of initial grid.
- Not too coarse initial grid.

Alternative:

- Remesh the submodel. Useful for instance when geometry changes locally.

Mesh refinement: example

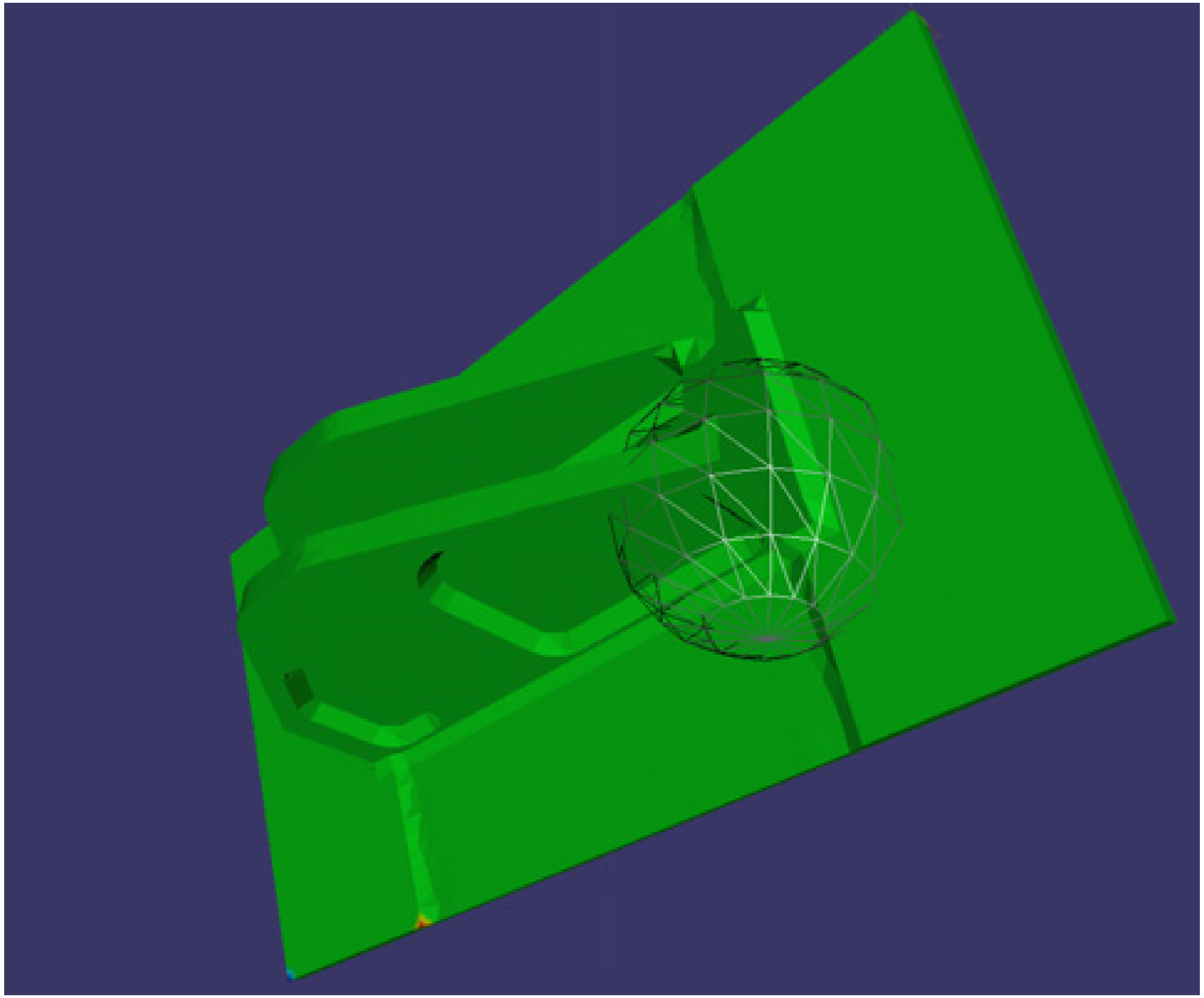


Figure 9: Sphere defining area of interest.

Mesh refinement: example

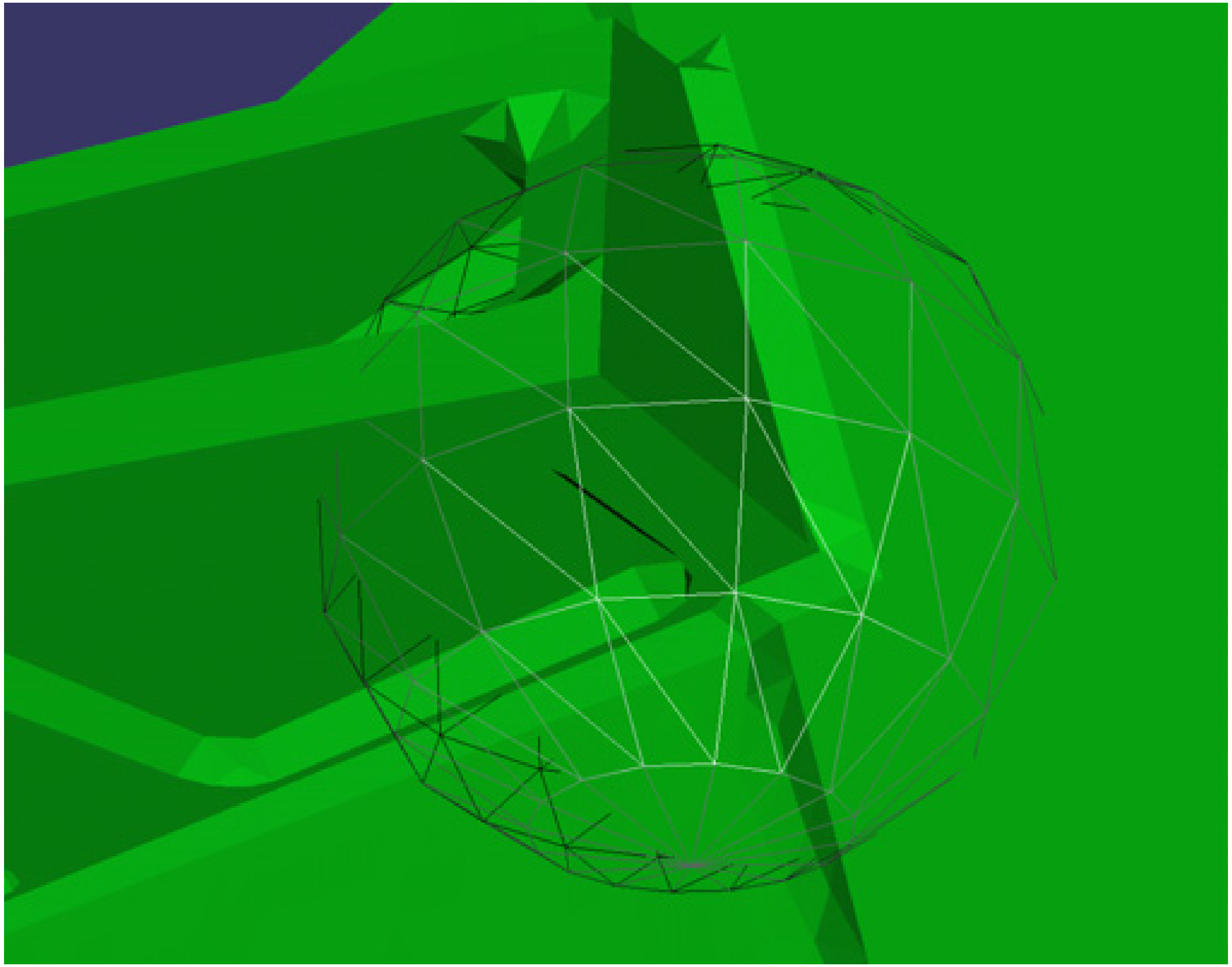


Figure 10: Close up of sphere defining area of interest.

Mesh refinement: example

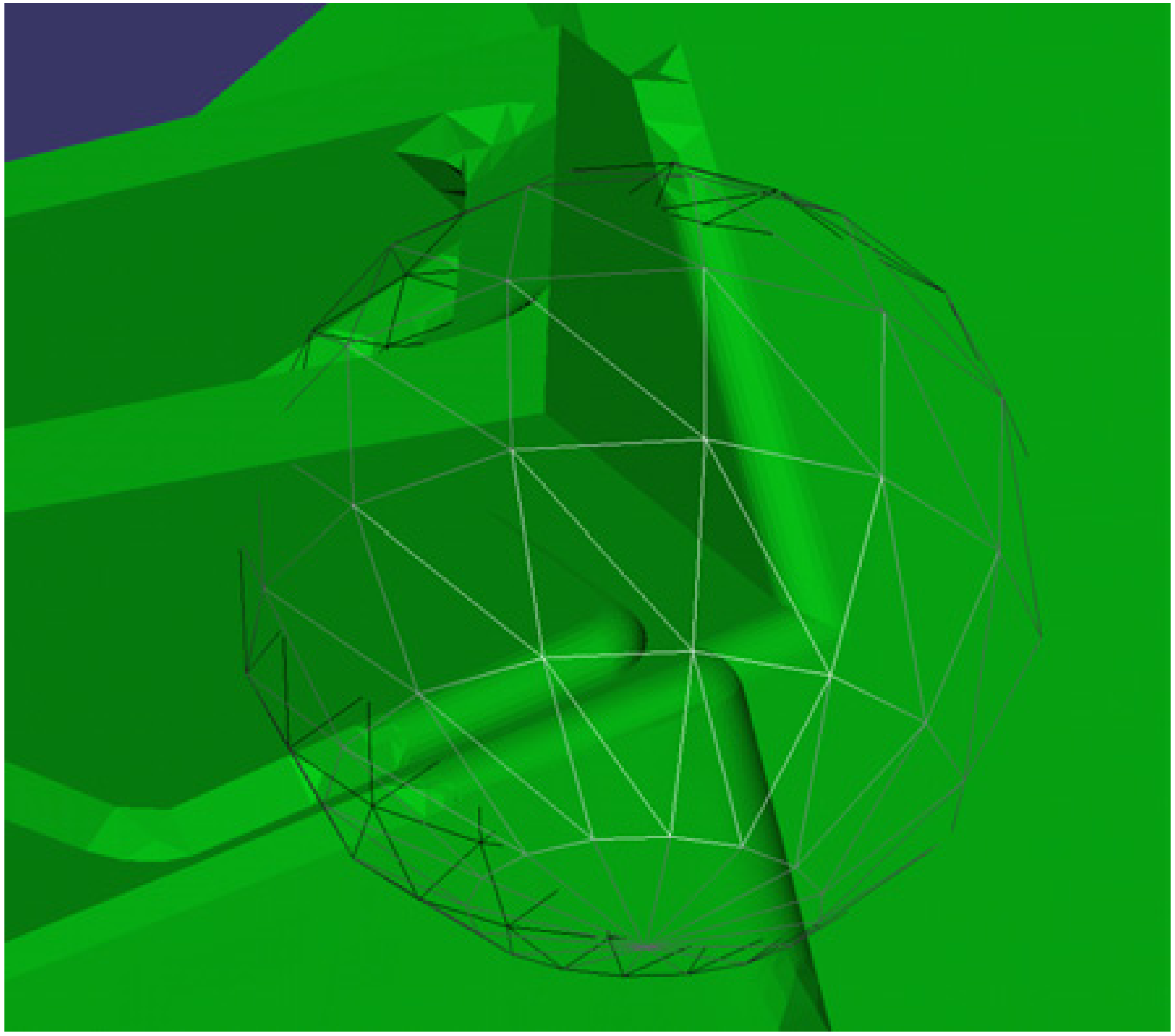


Figure 11: Solid after three refinements

Mesh refinement: example

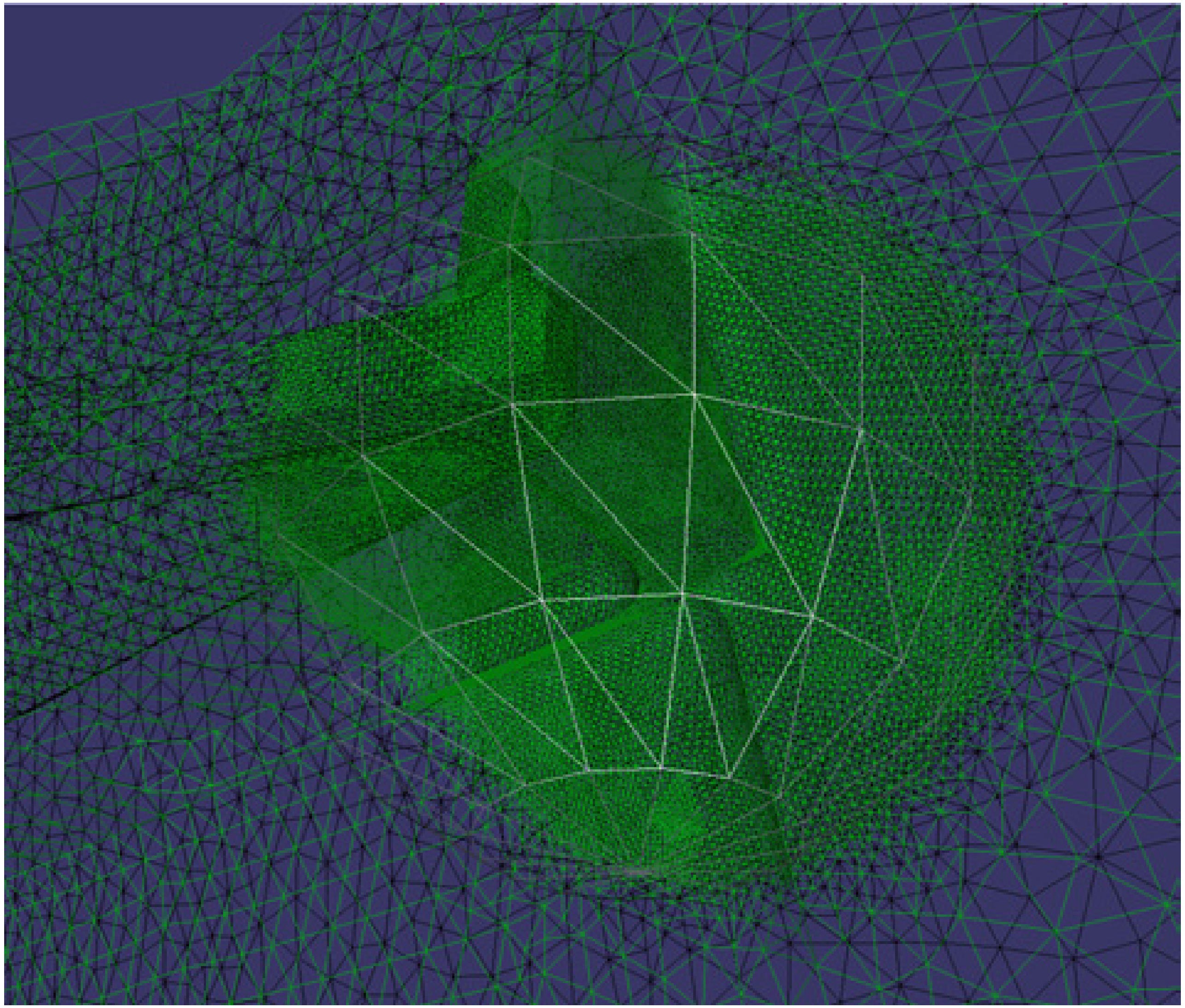


Figure 12: Mesh after three refinements

Conclusions and current work

Conclusions:

- Submodeling appears to be an attractive technique for practical use.
- Initial a posteriori error estimates have been derived.
- Mesh refinement techniques respecting CAD geometry have been developed.
- Interactive environment developed.

Current work:

- Construct suitable adaptive algorithm choosing, mesh size and size of subdomain.
- Couple the submodeling with local shape optimization.