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ADAPTIVE METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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Open Problems and Outlook

Current developments:

- *3-D applications:* The full power of the DWR method is expected to be seen in applications to 3-D problems. Those are currently developed in the context of the nonstationary Navier-Stokes equations.
- *Multi-physics and multi-scale problems:* The DWR method allows the systematic treatment of rather complex systems involving various physical mechanisms and spatial or temporal scales. A typical example is seen in chemically reacting flow where the chemical reaction usually induces much faster time scales and smaller spatial scales compared to those of the flow.
- *Adaptivity in optimal control:* Most numerical simulation eventually is optimization. Complex multidimensional optimal control and parameter identification problems constitute highly demanding computational tasks. Goal-oriented model reduction by adaptive discretization has the potential of facilitating large-scale optimization problems in structural and fluid mechanics, such as for example, minimization of drag or control of flow-induced structural vibrations, as well as in the estimation of distributed parameters in PDE.
- *Model adaptivity:* The concept of *a posteriori* error control for single quantities of interest via duality may also be applicable to other situations when a full model, such as a differential equation, is reduced by projection to a subproblem, such as a finite element model. Model reduction within scales of hierarchical sub-models is a recent development in structural as well as fluid mechanics.
- *Open areas for applications:* There are several very promising and almost untouched areas for the application of the DWR method such as, e.g., electro-magnetics, semi-conductor theory, porous media flow, and fluid-structure interaction.
- *'Non-standard' finite element methods:* The DWR method can also be used in the context of 'non-standard' finite element methods such as *non-conforming* and *mixed* methods as well as the 'least-squares' stabilized cG-FEM for transport-dominated problems.

Open problems:

- *How to use the DWR method for multidimensional time-dependent problems?* Rigorous error control in the space-time frame requires us to solve a space-time dual problem. Especially for nonlinear problems, this may be prohibitive with respect to storage space and computing time. The question is how to exploit the option of solving on coarser meshes only and that of data compression.

- *How to organize anisotropic mesh refinement?* The rigorous extension of the DWR concept for generating solution-adapted *anisotropic* meshes, either by simple cell stretching or by more sophisticated mesh reorientation, is still to be developed. In the context of 'global' error estimation with respect to energy norm and L^2 norm a posteriori error estimates on anisotropic meshes have been derived.

- *How to effectively control the error caused by 'variational crimes'?* These are deviations from the pure Galerkin method such as numerical integration, boundary approximation, cutting off unbounded domains, transport stabilization.

- *How to control the 'algebraic' solution errors?* These are unavoidable using iterative solvers such as Newton's method, Krylov-space methods, multigrid methods, etc.

- *How to apply the DWR method to non-variational problems?* Residual-based methods for *a posteriori* error estimation like the DWR method rely on the variational formulation and the Galerkin orthogonality property of the finite element scheme. This allows us to locally extract additional powers of the mesh size, leading to sensitivity factors of the form $h_K^2\|\nabla^2 z\|_K$. Other 'non-variational' discretizations such as the finite volume method usually have a different error behavior, governed by sensitivity factors like $h_K\|\nabla z\|_K$. This may be seen by reinterpreting these discretizations as perturbed finite element schemes obtained by evaluating local integrals by special low-order quadrature rules. An *a posteriori* error analysis for finite volume schemes has been developed by exploiting superapproximation properties.

- *How to solve the technical problems discussed in the course for providing theoretical support for the DWR method?*