Differentiation of the cost-functional

The complete form of this minimization problem can be written as

$$\min_{p_1,p_2,\ldots,p_M} J(p_1,p_2,\ldots,p_M) =$$
$$\min_{p_1,p_2,\ldots,p_M} \left(\frac{1}{2} \int_0^{t^*} \int_{\partial\Omega} \big[d(x,t) - v(x,t;p_1,p_2\ldots,p_M) \big]^2 dx\, dt \right) \quad (1)$$

subject to the constraint that v solves

$$v_t + g_\alpha(\phi) I(v) = \nabla \cdot [k_\alpha(\phi)\nabla v] \qquad \text{in } \Omega \quad (2)$$
$$k_\alpha(\phi)\nabla v \cdot n = 0 \qquad \text{along } \partial\Omega \quad (3)$$
$$v(x,0) = v_0(x) \qquad \text{in } \Omega \quad (4)$$

where

$$\phi = \phi(x;\ p_1,p_2\ldots,p_M) = \sum_{i=1}^{M} p_i N_i(x), \quad (5)$$

$$g_\alpha(\phi) = H_\alpha(\phi) \approx \begin{cases} 0 & \text{if } x \text{ in } D \\ 1 & \text{if } x \text{ in } \Omega \setminus D \end{cases} \quad (6)$$

and

$$k_\alpha(\phi) = k_1(1 - H_\alpha(\phi)) + k_2 H_\alpha(\phi) \approx \begin{cases} k_1 & \text{if } x \text{ in } D \\ k_2 & \text{if } x \text{ in } \Omega \setminus D. \end{cases} \quad (7)$$

To obtain $\frac{\partial J}{\partial p_i}$ we must differentiate the cost-functional with respect to p_i. Let $i \in \{1,2,\ldots,M\}$ be arbitrary and define $p = p_i$. Now we see that

$$\frac{\partial J}{\partial p} = -\int_0^{t^*} \int_{\partial\Omega} \big[d(x,t) - v(x,t;p_1,\ldots,p_M) \big] v_p(x,t;p_1,\ldots,p_M)\, dx\, dt. \quad (8)$$

Next, let us look at the weak form of the problem (2)-(3): Find v such that

$$\int_0^{t^*} \int_\Omega v_t \psi\, dx\, dt + \int_0^{t^*} \int_\Omega g_\alpha(\phi) I(v) \psi\, dx\, dt + \int_0^{t^*} \int_\Omega k_\alpha(\phi) \nabla v \cdot \nabla\psi\, dx\, dt = 0, \quad (9)$$

for all test functions $\psi \in V(\Omega)$.

Now we differentiate (9) with respect to p and find that

$$\int_0^{t^*}\int_\Omega v_{pt}\psi \, dx \, dt + \int_0^{t^*}\int_\Omega g'_\alpha(\phi)\phi_p I(v)\psi \, dx \, dt + \int_0^{t^*}\int_\Omega g_\alpha(\phi)I'(v)v_p\psi \, dx \, dt$$
$$+ \int_0^{t^*}\int_\Omega k'_\alpha(\phi)\phi_p \nabla v \cdot \nabla\psi \, dx \, dt + \int_0^{t^*}\int_\Omega k_\alpha(\phi)\nabla v_p \cdot \nabla\psi \, dx \, dt = 0, \tag{10}$$

for all $\psi \in V(\Omega)$, and we use (6)-(7) to get

$$\int_0^{t^*}\int_\Omega v_{pt}\psi \, dx \, dt + \int_0^{t^*}\int_\Omega g_\alpha(\phi)I'(v)v_p\psi \, dx \, dt + \int_0^{t^*}\int_\Omega k_\alpha(\phi)\nabla v_p \cdot \nabla\psi \, dx \, dt$$
$$= -\int_0^{t^*}\int_\Omega \delta_\alpha(\phi)\phi_p I(v)\psi \, dx \, dt - \int_0^{t^*}\int_\Omega (k_2 - k_1)\delta_\alpha(\phi)\phi_p \nabla v \cdot \nabla\psi \, dx \, dt. \tag{11}$$

Introducing the operator

$$a(\xi, \psi) = \int_0^{t^*}\int_\Omega \xi_t\psi \, dx \, dt + \int_0^{t^*}\int_\Omega g_\alpha(\phi)I'(v)\xi\psi \, dx \, dt + \int_0^{t^*}\int_\Omega k_\alpha(\phi)\nabla\xi \cdot \nabla\psi \, dx \, dt$$

for $\xi, \psi \in V(\Omega)$, equation (11) may be written on the form

$$a(v_p, \psi) = -\int_0^{t^*}\int_\Omega \Big(I(v)\psi + (k_2 - k_1)\nabla v \cdot \nabla\psi\Big)\delta_\alpha(\phi)\phi_p \, dx \, dt. \tag{12}$$

Next, let w denote the solution of the following auxiliary problem; find $w \in V(\Omega)$ such that

$$a(\psi, w) = -\int_0^{t^*}\int_{\partial\Omega} \big[d(x,t) - v(x,t;p_1,\ldots,p_M)\big]\psi(x,t) \, dx \, dt. \tag{13}$$

By choosing $\psi = v_p$ in (13) we find from (8) that

$$\frac{\partial J}{\partial p} = a(v_p, w). \tag{14}$$

Further (12) and (14) imply that

$$\frac{\partial J}{\partial p} = -\int_0^{t^*}\int_\Omega \Big(I(v)w + (k_2 - k_1)\nabla v \cdot \nabla w\Big)\delta_\alpha(\phi)\phi_p \, dx \, dt. \tag{15}$$

We thus conclude that the partial derivatives $\partial J/\partial p_1, \partial J/\partial p_2, \ldots, \partial J/\partial p_M$ of J can be computed by the following procedure;

a) Solve (9) for v

b) Solve (13) for w

c) For $i = 1, 2, \ldots, M$, compute

$$\frac{\partial J}{\partial p_i} = -\int_0^{t^*}\int_\Omega \Big(I(v)w + (k_2 - k_1)\nabla v \cdot \nabla w\Big)\delta_\alpha(\phi)\phi_{p_i}\, dx\, dt. \qquad (16)$$

The complexity of this method is (almost) independent of the number M of infarction parameters!

Note that the classical form of the adjoint problem (13) reads

$$-w_t + g_\alpha(\phi)I'(v)w = \nabla \cdot [k_\alpha \nabla w] \qquad \text{in } \Omega \qquad (17)$$

$$k_\alpha(\phi)\nabla w \cdot n = -\,(d - v) \qquad \text{along } \partial\Omega \qquad (18)$$

$$w(x, t^*) = 0 \qquad \text{in } \Omega. \qquad (19)$$

We find the classical form of the adjoint problem

$$a(\psi, w) = -\int_0^{t^*}\int_{\partial\Omega} \big[d(x,t) - v(x,t;p_1,\ldots,p_M)\big]\psi\, dx\, dt, \quad \text{for all } \psi \in V(\Omega)$$

by using the operator

$$a(\psi, w) = \int_0^{t^*}\int_\Omega \psi_t w\, dx\, dt + \int_0^{t^*}\int_\Omega g_\alpha(\phi)I'(v)\psi w\, dx\, dt + \int_0^{t^*}\int_\Omega k_\alpha(\phi)\nabla\psi \cdot \nabla w\, dx\, dt.$$

Now, integrating by parts (the first term) and using Green's lemma (the third term) we get

$$\Big[\int_\Omega \psi w\, dx\Big]_{t=0}^{t=t^*} - \int_0^{t^*}\int_\Omega \psi w_t\, dx\, dt + \int_0^{t^*}\int_\Omega g_\alpha(\phi)I'(v)\psi w\, dx\, dt$$
$$+ \int_0^{t^*}\int_{\partial\Omega} k_\alpha(\phi)\psi\nabla w \cdot n\, dx\, dt + \int_0^{t^*}\int_\Omega \nabla \cdot \big[k_\alpha(\phi)\nabla w\big]\psi\, dx\, dt = -\int_0^{t^*}\int_{\partial\Omega} \big[d - v\big]\psi\, dx\, dt.$$

By choosing appropriate test functions ψ, the we will end up with the classical form (17)-(19).