# Approximate Mortar Conditions for the CR Finite Element on Non-matching Grids

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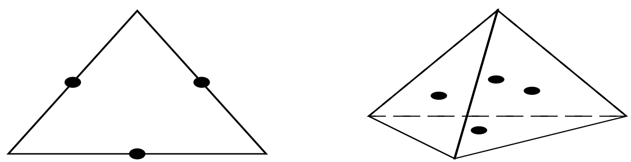
# The Crouzeix-Raviart Mortar FE

We consider the problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad \forall v \in H_0^1(\Omega),$$

where $\overline{\Omega} = \cup_i \overline{\Omega_i}$ (Non-overlapping), $a(u,v) = \sum_{i=1}^N \int_{\Omega_i} \nabla u \cdot \nabla v \, dx$ and $f(v) = \sum_{i=1}^N \int_{\Omega_i} fv \, dx$.

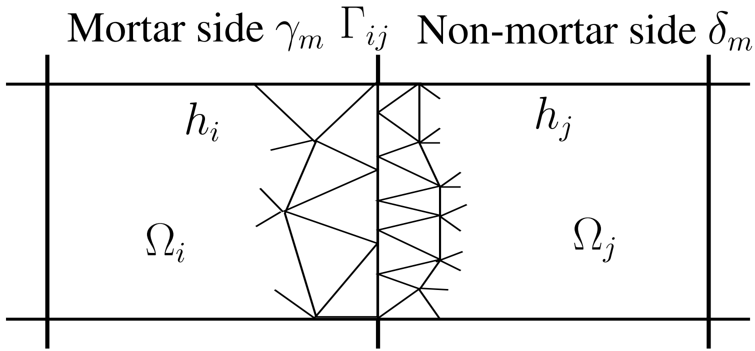


$X_h(\Omega_i)$: The CR (P1 Non-conforming) FE space on $\Omega_i$, vanishing at the edge-mid nodes on the boundary $\partial\Omega$.

**Mortar condition**: For $u_i \in X_h(\Omega_i)$ and $u_j \in X_h(\Omega_j)$,

$$Q_m u_i = Q_m u_j \quad \text{on } \delta_m.$$

$Q_m : L^2(\Gamma_{ij}) \rightarrow M(\delta_m)$ is the $L^2$ projection operator. $M(\delta_m)$ is the set of piecewise constant functions on the $\delta_m$-discretization.

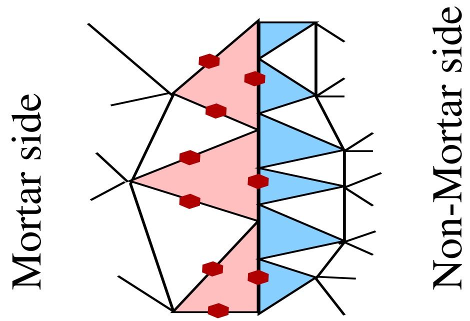



**CR mortar FE space**: [cf. Marcinkowski]

$$V_h = \{u \in \prod_i X_h(\Omega_i) : u \text{ satisfies the Mortar condition on } \delta_m\}$$

## Approximate Mortar Condition

Basis functions of $V_h$, associated with the subdomain interior, may have non-zero support on the non-mortar side (Not desirable, specially in 3D).




Aim is to avoid their direct use in applying the mortar condition.
Alternative: Approximate mortar condition ($I_m : X_h(\gamma_m) \rightarrow L^2(\gamma_m)$):

$$Q_m I_m u_i = Q_m u_j \quad \text{on } \delta_m$$

Approximate mortar condition in Wavelet context, cf. [Bertoluzza]

## Choice of $I_m$

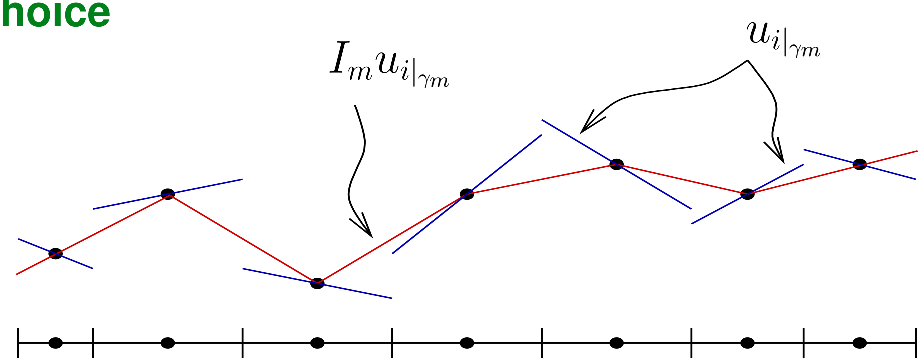
### First choice

Being inspired by the fact that the basis functions associated with the subdomain interior, have zero integrals on the subdomain boundary, an easy choice would be to define $I_m$ as

$$I_m u_{i|_e} = \frac{1}{|e|} \int_e u_i \, dx \quad \text{on } e \subset \gamma_m$$

The operator $I_m$ preserves only constants, and is therefore not enough for getting optimal error estimate. We need locally at least P1-preserving.

## Second choice




$$I_m u_i(x) = \begin{cases} u_i(x), & x \in \gamma_m^{CR}, \\ \frac{h_{\mathcal{E}_r}}{h_{\mathcal{E}_l}+h_{\mathcal{E}_r}} u_i(x_m^{\mathcal{E}_l}) + \frac{h_{\mathcal{E}_l}}{h_{\mathcal{E}_l}+h_{\mathcal{E}_r}} u_i(x_m^{\mathcal{E}_r}), & x \in \gamma_{mh}, \\ u_i(x_m^{\mathcal{E}_e}) + \frac{h_{\mathcal{E}_e}}{h_{\mathcal{E}_e}+h_{\mathcal{E}_e'}} \left( u_i(x_m^{\mathcal{E}_e}) - u_i(x_m^{\mathcal{E}_e'}) \right) & x \in \partial\gamma_{mh}. \end{cases}$$

The interpolation is done basically by first joining values at the neighboring edge-mid nodes by straight lines, and then simply extending the two end straight lines towards the end of the mortar $\gamma_m$.

## The discrete problem

Since $V_h$ does not belong to $H_0^1(\Omega)$, we use the broken bilinear form $a_h(\cdot,\cdot)$ defined as $a_h(u,v) = \sum_{i=1}^{N} \sum_{\tau \in \mathcal{T}_h(\Omega_i)} \int_\tau \nabla u \cdot \nabla v \; dx$. The discrete problem takes the following form: Find $u_h^* \in V_h$ such that

$$a_h(u_h^*, v_h) = f(v_h), \quad \forall v_h \in V_h.$$

$V_h$ is a Hilbert space with an inner product defined by $a_h(u_h, v_h)$.
An error estimate for the new mortar technique for CR FE (optimal):

$$|u^* - u_h^*|^2_{H_h^1(\Omega)} \le c \sum_{i=1}^{N} h_i^2 |u^*|^2_{H^2(\Omega_i)}$$

# An additive Schwarz method

## The splitting

$$V_h = \sum_{\gamma_m} V_\gamma + V_0 + \sum_{i}^{N} V_i$$

$V_i$ and $V_\gamma$ are the standard local subspaces of $V_h$ on subdomain $\Omega_i$ and mortar edge $\gamma_m$. $V_0 \subset V_h = span\{\Phi_i\}_{i=1,\cdots,N}$ is a coarse space similar to the one in [Dryja-Bjørstad-Rahman].

The function $\Phi_i$ is defined by its values at the CR nodes:

$$\Phi_i(x) = \begin{cases} 1, & x \in \Omega_i^{CR}, \\ \frac{\rho_i}{(\rho_i+\rho_j)}, & x \in \gamma_m^{CR}, \gamma_m \subset \partial\Omega_i \cap \partial\Omega_j, \\ 0, & \text{elsewhere.} \end{cases}$$

Values on the non-mortar sides are defined using the mortar condition.

## Projection-like

For $u \in V_h$, and $\alpha \in \{\{\gamma_m\}, 0, 1, \cdots, N\}$, $\quad T_\alpha : V_h \to V_\alpha$

$$a_h(T_\alpha u, v) = a_h(u, v), \quad v \in V_\alpha$$

## An equivalent discrete problem

Set $T = \sum_{\gamma_m} T_{\gamma_m} + T_0 + \sum_{i=1}^N T_i$.

$$a_h(Tu, v) = a_h(u, v), \quad v \in V_\alpha$$

## Convergence

$$\kappa_2(T) \leq c\frac{H}{h}$$

# Numerical Results

Unit square domain $\Omega$, initially divided into 9 square subdomains, each consisting of $2m_i^2$ triangles corresponding to the mesh size $h_i \approx 1/m_i$. For each neighboring subdomain pair $\{\Omega_i, \Omega_j\}$, $m_i \neq m_j$ in order to get non-matching grids. The function $f$ is so chosen that the exact solution is equal to $sin(\pi x) sin(\pi y)$.

| $\{h_i, h_j\}$ | Standard Mortar | | Approximate Mortar | |
|---|---|---|---|---|
| | $L^2 - error$ | $H^1 - error$ | $L^2 - error$ | $H1 - error$ |
| $\{1/6, 1/5\}$ | 0.00202040 | 0.065292 | 0.00248430 | 0.078409 |
| $\{1/12, 1/10\}$ | 0.00049658 | 0.032843 | 0.00066682 | 0.038768 |
| $\{1/24, 1/20\}$ | 0.00012307 | 0.016479 | 0.00017493 | 0.019321 |

$L^2 - norm$ and $H^1 - seminorm$ of the error of the two mortar methods, showing that the two methods are quite close.

| $\{h_i, h_j\}$ | Standard Mortar | Approximate Mortar |
|---|---|---|
| $\{1/6, 1/5\}$ | 28.85 (25) | 30.11 (23) |
| $\{1/12, 1/10\}$ | 63.44 (35) | 60.90 (31) |
| $\{1/24, 1/20\}$ | 134.18 (49) | 122.55 (45) |

Condition number estimates (PCG-iteration counts in parentheses), showing similar performance for the two mortar methods.