Basics of Multigrid Methods

Harald van Brummelen

TU/e, Dept. of Mechanical Engineering, Multiscale Engineering Fluid Dynamics section

TU/e, Dept. of Mathematics and Computer Science, Centre for Analysis, Scientific computing and Applications (CASA)

Mathematical and Numerical Methods for Multiscale Problems
Multigrid Methods

Outline

Direct solution methods

Direct solution methods (Gauss elimination / LU factorization) for solving linear systems are robust, but:

1. have very inefficient computational complexity: a system with N unknowns requires $O(N^3)$ operations
2. have poor properties in terms of memory usage: pivot matrices in GE and L and U factors are generally full
3. require a complete representation (linearization) of the problem in the matrix, which is often not available or undesirable:
   - coupled problems: fluid-structure interaction, electro-thermo-mechanics, etc. (coupling terms)
   - problems involving nonlocal operators (full matrices)

Direct solution methods

Nonlinear problems: Given $R : \mathbb{R}^N \to \mathbb{R}^N$, find $u \in \mathbb{R}^N$ such that

$$R(u) = 0$$

Newton's method: given initial approximation $u^0 \in \mathbb{R}^N$, repeat for $n = 0, 1, 2, \ldots$:

$$\begin{aligned} Ad &= -R(u^n) \\ u^{n+1} &= u^n + d \end{aligned} \qquad \text{(T)}$$

with $A := R'(u^n)$. If the tangent problem (T) is solved with a direct method, then the nonlinearity is treated globally $\Rightarrow$ non-robust

Iterative solution methods

Iterative solution methods can generally be formulated as defect-correction processes*. Consider

$$Au = b \tag{L}$$

with $A \in \mathbb{R}^{N\times N}$. Let $\tilde{A}$ be a suitable approximation to A. The defect-correction process reads:

$$\tilde{A}u^0 = b \qquad \text{(init. approx.)}$$

Repeat for $n = 1, 2, \ldots$

$$\tilde{A}u^n = \tilde{A}u^{n-1} + (b - Au^{n-1}) \tag{DeC}$$

(Note: also possible for nonlinear problems)

*K. Bøhmer, P.W. Hemker, and H.J. Stetter, *The defect correction approach*, Computing **5** (1984), 1-32.

Relaxation methods

Relaxation methods

a class of iterative solution procedures in which (blocks of) equations are solved consecutively $\Rightarrow$ only small problems need to be inverted

Relaxation methods: example

Gauss-Seidel relaxation for Poisson's equation in 1D

Set $\Omega = (0, \ell)$. Consider

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{at } \partial\Omega = \{0, \ell\} \end{aligned}$$

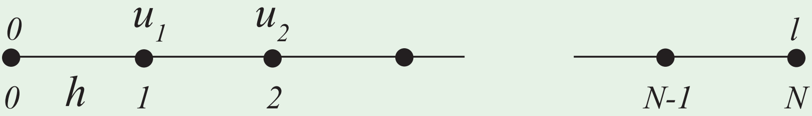
Relaxation methods: example

Gauss-Seidel relaxation for Poisson's equation in 1D

Standard discretization: partition Ω by point $\{0, h, 2h, \ldots, Nh(:= \ell)\}$. Let u_i denote approximation to $u(ih)$, defined by

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(ih) \qquad i \in \{1, 2, \ldots, N-1\}$$

$$u_i = 0 \qquad i \in \{0, N\}$$



Relaxation methods: example

Gauss-Seidel relaxation for Poisson's equation in 1D

$\Leftrightarrow$

$$-\frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & & \\ 1 & -2 & 1 & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & 0 & 1 & -2 & 1 \\ & & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f(h) \\ f(2h) \\ \vdots \\ f((N-2)h) \\ f((N-1)h) \end{pmatrix}$$

$\Leftrightarrow$

$$Au = b$$

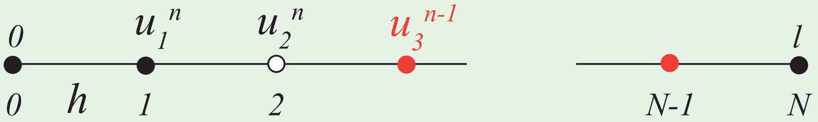
Relaxation methods: example

Gauss-Seidel relaxation for Poisson's equation in 1D

Gauss-Seidel relaxation: given initial approximation $u^0 = (u_1^0, \ldots, u_{N-1}^0)$, successively solve equations pointwise:

$$-\frac{u_{i+1}^{n-1} - 2u_i^n + u_{i-1}^n}{h^2} = f(ih) \qquad i = 1, 2, \ldots, N-1$$

for $n = 1, 2, \ldots$.



Relaxation methods: example

Gauss-Seidel relaxation for Poisson's equation in 1D

$\Leftrightarrow$

$$\tilde{A}u^n = \tilde{A}u^{n-1} + (b - Au^{n-1}) \qquad \text{(DeC)}$$

with

$$\tilde{A} = -\frac{1}{h^2}\begin{pmatrix} -2 & \color{red}{0} & 0 & & \\ 1 & -2 & \color{red}{0} & 0 & \\ 0 & \ddots & \ddots & \color{red}{\ddots} & 0 \\ & 0 & 1 & -2 & \color{red}{0} \\ & & 0 & 1 & -2 \end{pmatrix}$$

($\tilde{A}$ is lower triangular $\Rightarrow$ solve by forward substitution)

Iterative methods: Example

Partitioned methods for fluid-structure interaction

Structure of FSI problems (within each time step):

$$\begin{pmatrix} A_{ss} & A_{sf} \\ A_{fs} & A_{ff} \end{pmatrix} \begin{pmatrix} u_s \\ u_f \end{pmatrix} = \begin{pmatrix} b_s \\ b_f \end{pmatrix}$$

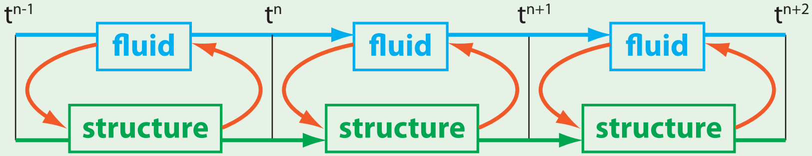
Properties:

- coupling matrices A_{sf} and A_{fs} respectively correspond to continuity of tractions and displacements
- generally, coupling matrices have complicated structure (nonlocal, shape derivatives) and/or are unavailable explicitly (code modularity)
- XXL systems

Iterative methods: Example

Partitioned methods for fluid-structure interaction

Partitioned iterative solution methods:



$$\tilde{A}u^n = \tilde{A}u^{n-1} + (b - Au^{n-1}) \qquad \text{(DeC)}$$

with

$$\tilde{A} = \begin{pmatrix} A_{ss} & 0 \\ A_{fs} & A_{ff} \end{pmatrix}$$

($\tilde{A}$ is block-lower triangular matrix $\Rightarrow$ solve by forward substitution)

Iterative methods

Convergence

Let $\bar{u}$ denote solution of $A\bar{u} = b$. Define error $e^n = u^n - \bar{u}$. Then

$$\tilde{A}u^n = \tilde{A}u^{n-1} + (b - Au^{n-1}) \qquad \text{(DeC)}$$

$\Leftrightarrow$ (add partition of zero)

$$\tilde{A}(u^n - \bar{u}) = \tilde{A}(u^{n-1} - \bar{u}) + (A\bar{u} - Au^{n-1})$$

$\Leftrightarrow$

$$\tilde{A}e^n = \tilde{A}e^{n-1} - Ae^{n-1}$$

$\Leftrightarrow$

$$e^n = \left(I - \tilde{A}^{-1}A\right)e^{n-1}$$

Iterative methods

Convergence

$$\boxed{\frac{\|e^n\|}{\|e^{n+1}\|} \leq \left\|I - \tilde{A}^{-1}A\right\|}$$

- Fast convergence if $\tilde{A}$ is "close to" A
- Monotonous convergence if $\left\|I - \tilde{A}^{-1}A\right\| < 1$
- Often, $I - \tilde{A}^{-1}A$ has many eigenvalues close to zero and relatively few large eigenvalues $\Rightarrow$ the iterative method efficiently confines the error to a small subspace (but still gives slow convergence)

Iterative methods

Convergence of combined iterative methods

Sequential application of two distinct iterative methods with approximate operators $\tilde{A}_1$ and $\tilde{A}_2$ yields:

$$e^n = \big(I - \tilde{A}_2^{-1}A\big)\big(I - \tilde{A}_1^{-1}A\big)e^{n-1}$$

Effective convergence is obtained by combining different iterative methods that reduce different parts of the error spectrum.

Multigrid

Concept

Iterative methods (in particular relaxation methods) generally* effectively reduce oscillatory (*high frequency*) components of the error. The remaining smooth (*low-frequency*) part of the error can be effectively represented on a coarse mesh, and can be reduced by coarse-grid correction.

- the common terminology is *frequency*, although *wave number* is more appropriate
- *generally does not mean always: problems near boundaries (re-entrant corners), non-elliptic problems, . . .
- for difficult problems, the development of a good smoother (and a good coarse-grid correction) is a daunting challenge

Multigrid: historical references

- N.S. Bahvalov, *Convergence of a relaxation method with natural constraints on an elliptic operator*, Z. Vyčisl. Mat. i Mat. Fiz. **6** (1966), 861–885, (in Russian).
- R.P. Fedorenko, *The speed of convergence of one iterative process*, USSR Comput. Math. Math. Phys. **4** (1964), p. 227.
- A. Brandt, *Multi-level adaptive techniques (MLAT) for fast numerical solution to boundary value problem*s, Proc. 3rd Internat. Conf. on Numer. Meth. in Fluid Mech. (Paris, 1972), Lecture Notes in Physics, vol. 18, Springer-Verlag, 1973, pp. 82–89.
- A. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp. **31** (1977), 333–390.
- P. Wesseling, *Numerical solution of stationary Navier–Stokes equations by means of multiple grid method and Newton iteration*, Tech. Report Report NA-18, Dept. of Math., Delft University of Technology, 1977.
- A. Jameson, *Acceleration of transonic potential flow calculations on arbitrary meshes by the multiple grid method*, AIAA (1979), Paper 79-1458.
- W. Hackbusch and U. Trottenberg (eds.), *Multigrid methods*, Lecture Notes in Math. 960, Springer-Verlag, 1982.
- D. Braess and W. Hackbusch, *A new convergence proof for the multigrid method including the v-cycle*, SIAM Journal on Numerical Analysis **20** (1983), 967–975.
- W. Hackbusch, *Multi-grid methods and applications*, Springer, Berlin, 1985.

Outline

Fourier analysis

Historcal note

Fourier analysis, similar to *Von-Neumann stability analysis*, was developed as an analysis tool for MG methods by Brandt in the late 1970s.

Fourier analysis: Hilbert-space setting

Consider an interval on the real line, $\Omega = (0, \ell)$. Denote by $L^2(\Omega, \mathbb{C})$ the space of square-integrable complex functions, equipped with inner product

$$(u, v)_{L^2(\Omega,\mathbb{C})} = \int_\Omega u(x) v^*(x)\,\mathrm{d}x$$

$L^2(\Omega, \mathbb{C})$ is a *separable Hilbert space*: there exists a countable basis $\{a_1, a_2, \ldots\}$ of $L^2(\Omega, \mathbb{C})$. In particular, the *Fourier modes*

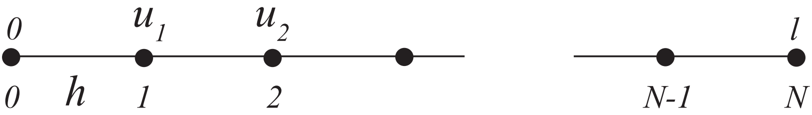
$$a_k(x) = \ell^{-1/2} \mathrm{e}^{\iota 2\pi\, kx/\ell}$$

form a basis of $L^2(\Omega, \mathbb{C})$. Conversely, any function $u \in L^2(\Omega, \mathbb{C})$ can be represented as

$$\boxed{u(x) = \sum_{k=-\infty}^{\infty} \hat{u}_k a_k(x)} \quad \text{with} \quad \hat{u}_k = (u, a_k)_{L^2(\Omega,\mathbb{C})}$$

Fourier analysis: Grid functions

Let $\{0, h, 2h, \ldots, Nh(:= \ell)\}$ denote a uniform partition of Ω. We call $u_{(\cdot)} : \{0, 1, \ldots, N\} \to \mathbb{R}$ a grid function.



Fourier analysis: Grid functions

Let $\{0, h, 2h, \ldots, Nh(:= \ell)\}$ denote a uniform partition of Ω. We call $u_{(\cdot)} : \{0, 1, \ldots, N\} \to \mathbb{R}$ a grid function.

Nyquist-Shannon theorem

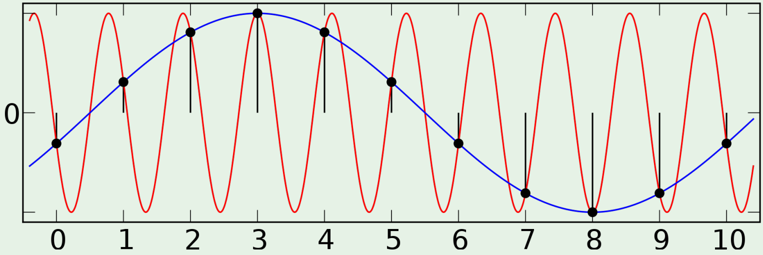
If and only if a function $u : (0, \ell) \to \mathbb{R}$ contains no wave numbers higher than $\ell/2h$ (wave length $\geq 2h$), then it is uniquely determined by its values on a grid with mesh size h.

Fourier analysis: Grid functions

Nyquist-Shannon theorem

If and only if a function $u : (0, \ell) \to \mathbb{R}$ contains no wave numbers higher than $\ell/2h$ (wave length $\geq 2h$), then it is uniquely determined by its values on a grid with mesh size h.

Aliasing



Fourier analysis: Grid functions

Corollary

In examining grid functions on a uniform mesh with mesh parameter h, we can restrict our attention to

$$u(x) = \sum_{k=-\ell/2h}^{\ell/2h} \hat{u}_k \ell^{-1/2} e^{\iota 2\pi\, kx/\ell}$$

$\Rightarrow$

$$u_j = u(jh) = \sum_{\theta \in \Theta} \hat{\hat{u}}_\theta\, e^{\iota\, \theta j}$$

with

$$\Theta = \left\{ -\pi, -\pi + \frac{2\pi}{N}, -\pi + \frac{4\pi}{N}, \ldots, \pi - \frac{2\pi}{N}, \pi \right\} \subset [-\pi, \pi]$$

Convergence analysis: example

Recall discretization of Poisson problem:

$$-\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h^2} = f(ih) \qquad i \in \{1, 2, \ldots, N-1\} \qquad \text{(D)}$$

$$\bar{u}_i = 0 \qquad i \in \{0, N\}$$

and the Gauss-Seidel relaxation

$$-\frac{u_{i+1}^{n-1} - 2u_i^n + u_{i-1}^n}{h^2} = f(ih) \qquad i = 1, 2, \ldots, N-1 \qquad \text{(GS)}$$

Define *relaxation error* by $e_{(\cdot)}^n = u_{(\cdot)}^n - \bar{u}_{(\cdot)}$. Subtract (D) from (GS):

$$\frac{e_{i+1}^{n-1} - 2e_i^n + e_{i-1}^n}{h^2} = 0 \qquad i = 1, 2, \ldots, N-1 \qquad \text{(E)}$$

Convergence analysis: example

$$\frac{e_{i+1}^{n-1} - 2e_i^n + e_{i-1}^n}{h^2} = 0 \qquad i = 1, 2, \ldots, N-1 \qquad \text{(E)}$$

Insert the Fourier expansion:

$$e_i^n = \sum_{|\theta| \le \pi} \hat{e}_\theta^n \mathrm{e}^{\iota\theta i}$$

$\Rightarrow$

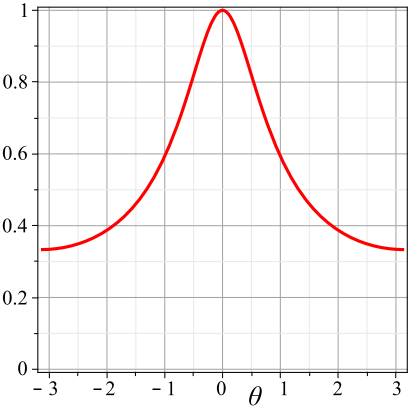
$$\hat{e}_\theta^{n-1} \mathrm{e}^{\iota\theta} - 2\hat{e}_\theta^n + \hat{e}_\theta^n \, \mathrm{e}^{-\iota\theta} = 0$$

$\Rightarrow$

$$\boxed{\frac{|\hat{e}_\theta^n|}{|\hat{e}_\theta^{n-1}|} \le \left| \frac{\mathrm{e}^{\iota\theta}}{2 - \mathrm{e}^{-\iota\theta}} \right|} \qquad \text{(error amplification)}$$

Convergence analysis: example

$$\frac{\left|\hat{e}_\theta^n\right|}{\left|\hat{e}_\theta^{n-1}\right|} \leq \left|\frac{\mathrm{e}^{\iota\theta}}{2-\mathrm{e}^{-\iota\theta}}\right|$$



Observations

1. The iteration is *stable*: the amplification factor ≤ 1 for all $\theta \in [-\pi, \pi]$.
2. It holds that

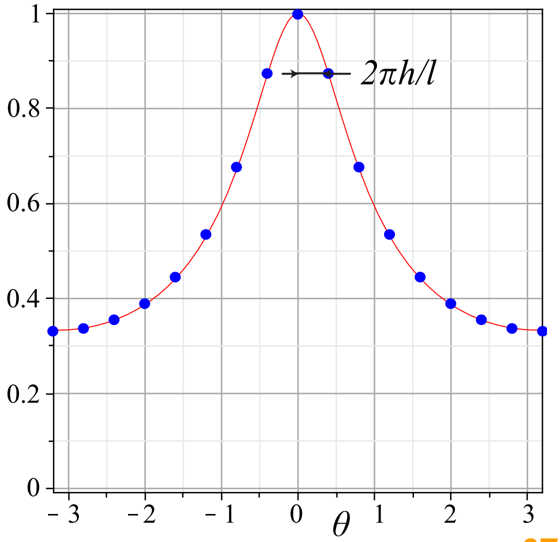
$$\theta \in \Theta := \left\{ -\pi, -\pi + 2\pi\frac{h}{\ell}, \ldots, -2\pi\frac{h}{\ell}, 2\pi\frac{h}{\ell}, \ldots, \pi - \frac{2\pi}{N}, \pi \right\}$$

the constant component, $\hat{e}_0$, is zero on account of boundary conditions. Therefore,

$$\sup_{\theta \in \Theta} \left| \frac{e^{\iota\theta}}{2 - e^{-\iota\theta}} \right| = 1 - O(h^2) \quad \text{as } h \to 0$$

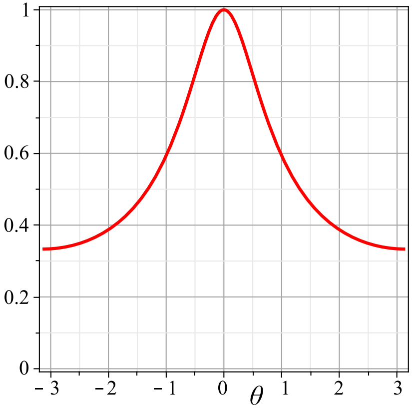
This implies that the convergence rate deteriorates as the mesh is refined!

Observations



Observations (cont'd)

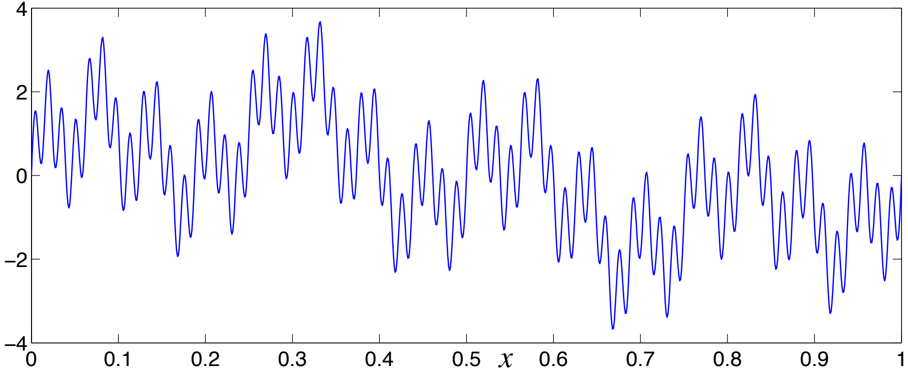
3. The Gauss-Seidel relaxation procedure yields fast convergence for high-frequency components in the error, and very slow convergence for low frequency components



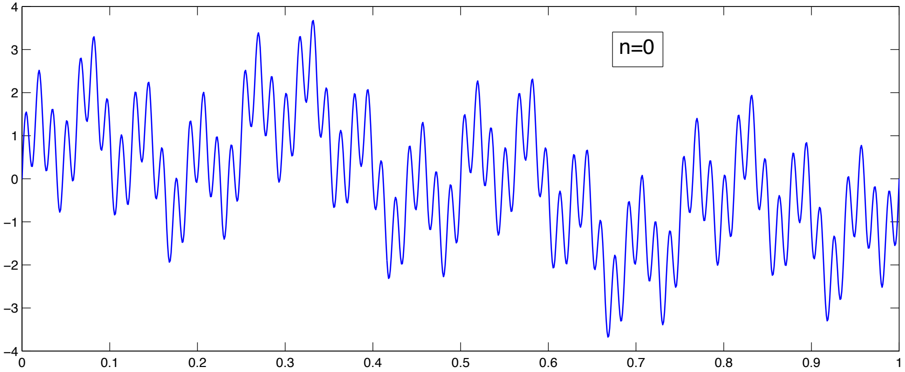
Illustration

Set $\Omega = (0, 1)$. Consider the Poisson problem with $f = 0$ and $u(0) = u(\ell) = 0$ ($\Rightarrow \bar{u} = 0$). Set $N = 2^{10}$ and $u^0_{(\cdot)}$ according to:

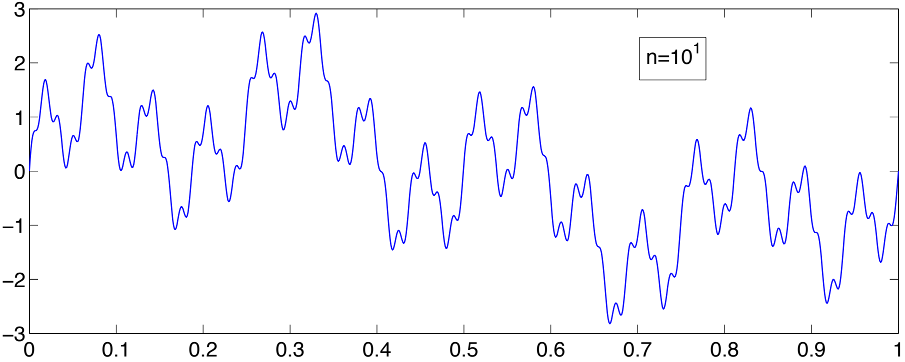
$$u^0_i = \sin(2\pi x_i) + \sin(8\pi x_i) + \sin(32\pi x_i) + \sin(128\pi x_i)$$



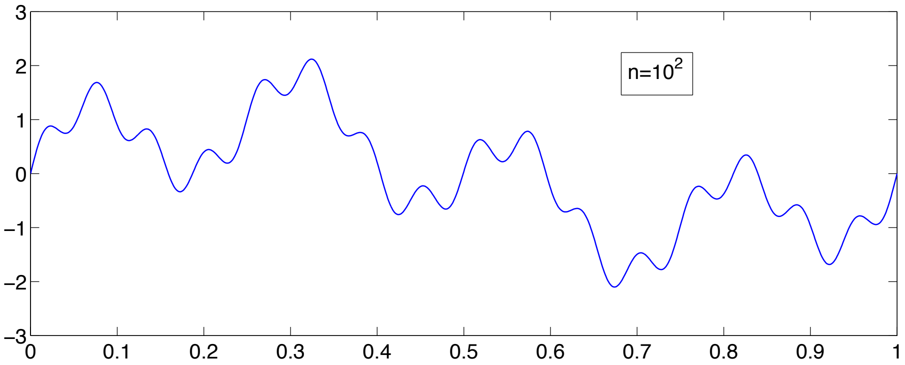
Illustration



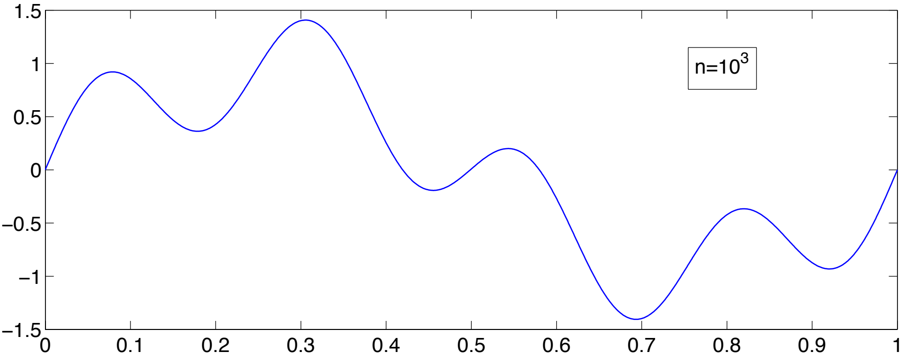
Illustration



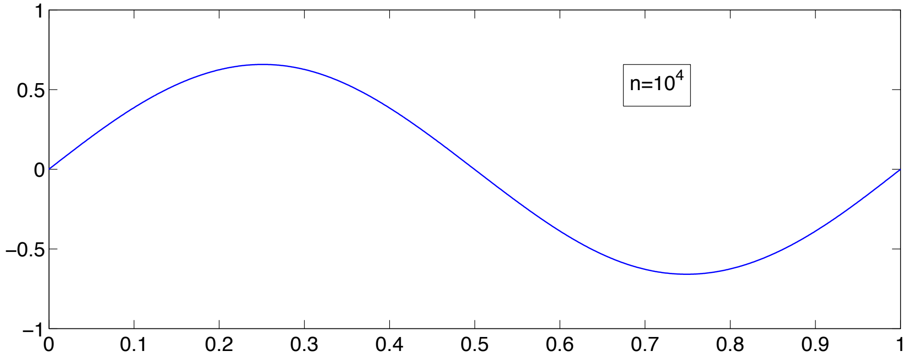
Illustration



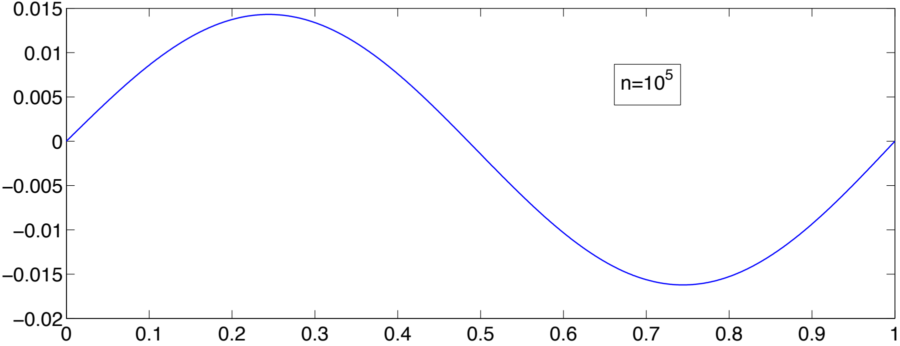
Illustration



Illustration



Illustration



Outline

Coarse-grid function

Consider a coarse grid with mesh size $H = 2h$ (standard coarsening). On such a grid, we can represent all functions with wave numbers up to $\ell/2H = \ell/4h$ (Nyquist-Shannon Thm.):

$$u^H(x) = \sum_{k=-\ell/2H}^{\ell/2H} \hat{u}_k^H \ell^{-1/2} e^{\iota 2\pi kx/\ell} = \sum_{k=-\ell/4h}^{\ell/4h} \hat{u}_k^H \ell^{-1/2} e^{\iota 2\pi kx/\ell}$$

If we evaluate u^H as on the fine grid, by comparison, there exist coefficients $\hat{u}_\theta^H$ such that

$$u_j^H = u^H(jh) = \sum_{\theta\in\Theta} \hat{u}_\theta^H e^{\iota \theta j}$$

with

$$\Theta = \left\{ -\frac{\pi}{2}, -\frac{\pi}{2} + \frac{2\pi}{N}, \ldots, \frac{\pi}{2} - \frac{2\pi}{N}, \frac{\pi}{2} \right\} \subset \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

Coarse-grid correction

Coarse-grid correction (concept)

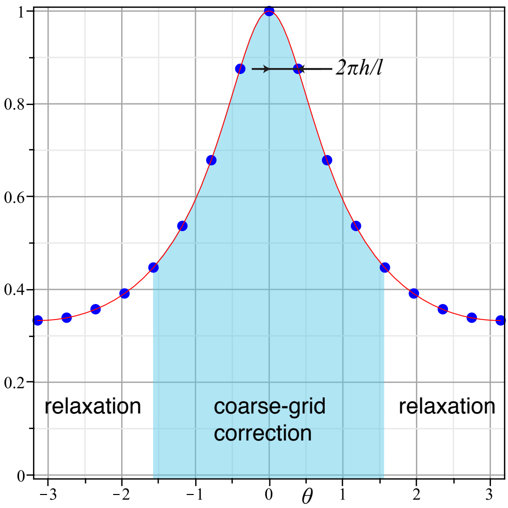
If we can construct the coarse-grid function such that $\hat{u}_\theta^H = \hat{u}_\theta^h$ for $|\theta| \leq \pi/2$, then on the fine grid (h), only the error components with $\pi/2 \leq |\theta| \leq \pi$ have to be resolved by relaxation!

1. relaxation gives very effective error reduction for $\pi/2 \leq |\theta| \leq \pi$:

$$\sup_{\pi/2 \leq |\theta| \leq \pi} \left| \frac{\mathrm{e}^{\iota\theta}}{2 - \mathrm{e}^{-\iota\theta}} \right| = \left| \frac{\mathrm{e}^{\iota\pi/2}}{2 - \mathrm{e}^{-\iota\pi/2}} \right| = 0.447\ldots$$

2. the mesh dependence of the convergence behavior of Gauss-Seidel relaxation appears in the limit $\theta \to 0 \Rightarrow$ relaxation + coarse-grid correction gives mesh-independent convergence behavior

Coarse-grid correction



Restriction and Prolongation

Denoting by V^h and V^H the spaces of fine- and coarse-grid functions, we require transfer operators between V^h and V^H.

Prolongation & restriction

$$P : V^H \to V^h \qquad \text{(prolongation)}$$

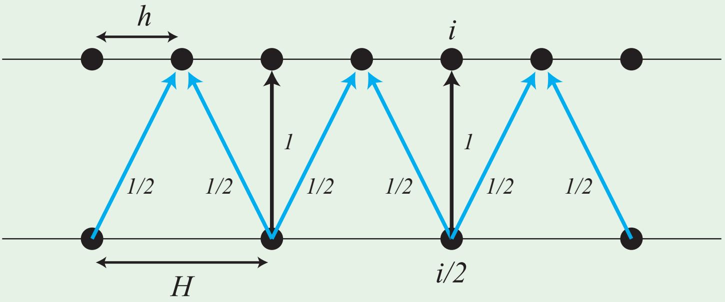
$$R : V^h \to V^H \qquad \text{(restriction)}$$

- P and R are also often denoted by I_H^h and I_h^H, respectively
- Generally, $R = P^*$ (restriction is adjoint/transpose of prolongation)
- General requirement: $R \circ P = \mathrm{Id}$ in V^H

Restriction and Prolongation

Prolongation: example

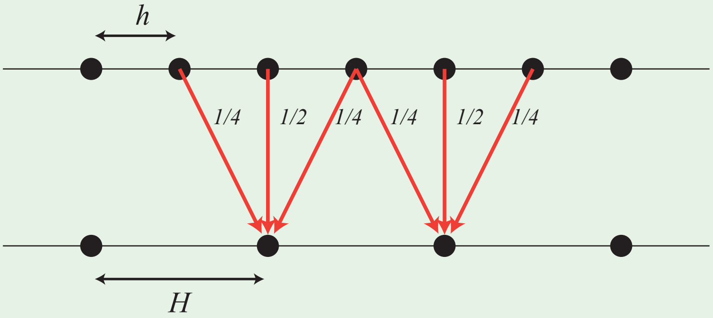
$$u_i^h = \begin{cases} u_{i/2}^H & i \text{ even} \\ \frac{1}{2}u_{(i-1)/2}^H + \frac{1}{2}u_{(i+1)/2}^H & i \text{ odd} \end{cases}$$



Restriction and Prolongation

Restriction: example

$$u_i^H = \frac{1}{4}u_{2i+1}^h + \frac{1}{2}u_{2i}^h + \frac{1}{4}u_{2i-1}^h$$



Restriction and Prolongation

Remarks

1. Because much of the MG theory was historically developed in a finite-difference context, restriction and prolongation operators are often interpreted as point-wise operators. In a finite-element context, variational projection is more natural.
2. Note that $R = P^*$ does not mean that the matrix of R is the transpose of the matrix of P.
3. Depending on the PDE, P and R have to satisfy certain conditions*. These are rather obvious in the FEM context (Sobolev-space projections).

*P. Hemker, On the order of prolongation and restriction in multigrid procedures, J. Comm. Appl. Math. **32** (1990), 423-429.

Coarse-grid equations: linear problems

Given a (relaxed) approximation $\tilde{u}^h \in V^h$, define the residual as:

$$r^h(\tilde{u}^h) = f^h - A^h \tilde{u}^h$$

The error $e^h := \tilde{u}^h - \bar{u}^h$ satisfies

$$A^h e^h = r^h(\tilde{u}^h)$$

A coarse-grid approximation $e^H \in V^H$ to $e^h \in V^h$ can be computed from:

$$A^H e^H = R r^h(\tilde{u}^h)$$

This approximation can be used to correct the fine-grid approximation $\tilde{u}^h$ according to:

$$\tilde{u}^h + P e^H$$

Correction scheme: MG for linear problems

Correction scheme

Given initial approximation $u^{h,0} \in V^h$, repeat for $n = 1, 2, \ldots$

1. Perform ν GS iterations:

$$\tilde{u}^{h,n} = \mathrm{GS}^{\nu} u^{h,n-1}$$

2. Construct (and restrict) fine-grid residual: $r^h(\tilde{u}^{h,n}) = f^h - A^h \tilde{u}^{h,n}$
3. Solve coarse-grid-correction problem (by approximation):

$$A^H e^H = R r^h(\tilde{u}^{h,n})$$

4. Prolongate and apply correction:

$$u^{h,n} = \tilde{u}^{h,n} + P e^H$$

Correction scheme: MG for linear problems

Remarks

1. Multigrid by recursion: the coarse-grid problem can again be approximated by smoothing and coarse-grid correction
2. Post-smoothing (after the coarse-grid correction) is then applied to ensure smoothness of the correction before the prolongation
3. How should the coarse-grid operator be constructed? The natural choice is $A^H = RA^hP$ (Galerkin projection). This is automatic in Galerkin FEM.

Correction scheme: MG for linear problems

V-cycle

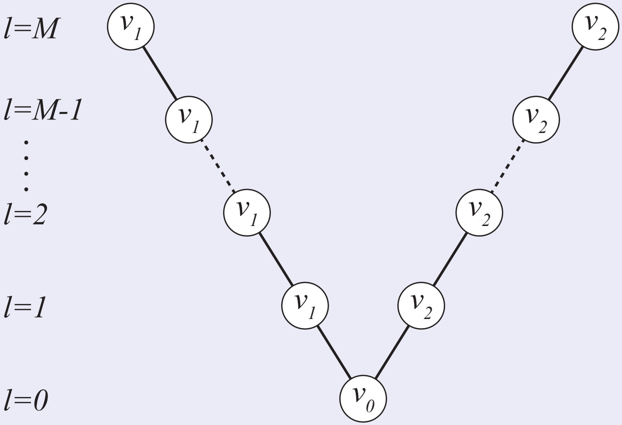


Illustration: two-grid convergence

Testcase

Set $\Omega = (0, 1)$. Consider:

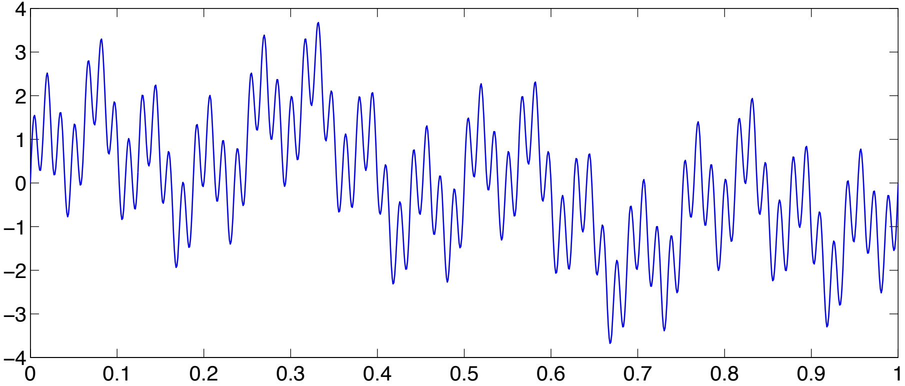
$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{at } \partial\Omega = \{0, 1\} \end{aligned}$$

Set $N = 2^{10}$ and $u^0_{(\cdot)}$ according to:

$$u_i^0 = \sin(2\pi x_i) + \sin(8\pi x_i) + \sin(32\pi x_i) + \sin(128\pi x_i)$$

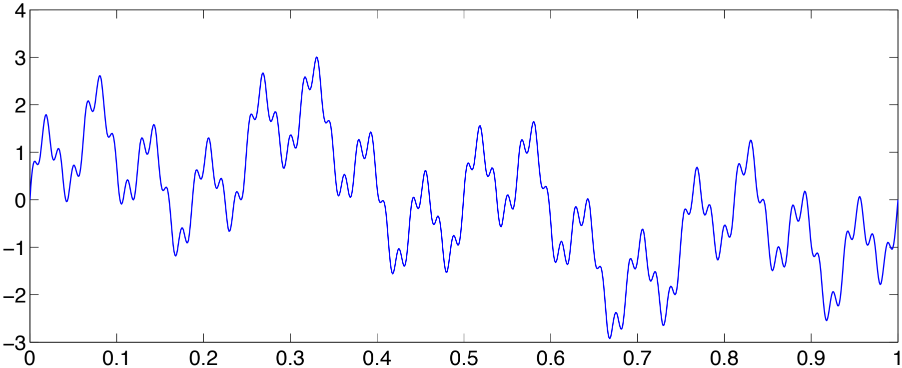
Perform $\nu = 8$ GS relaxations followed by a coarse-grid correction.

Illustration: two-grid convergence



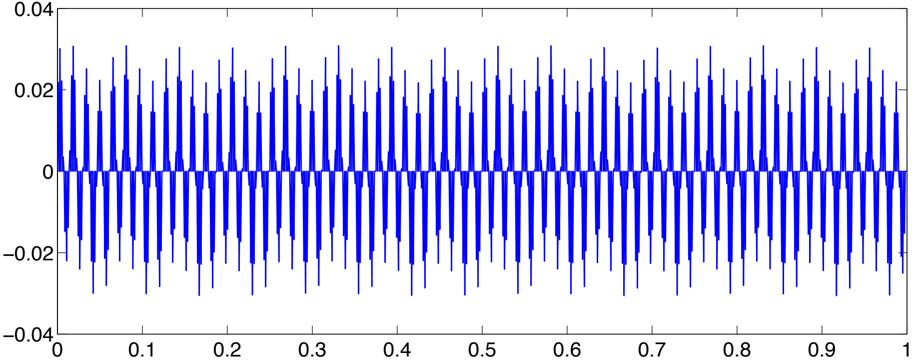
0 GS^8 relaxation 0 CG correction

Illustration: two-grid convergence



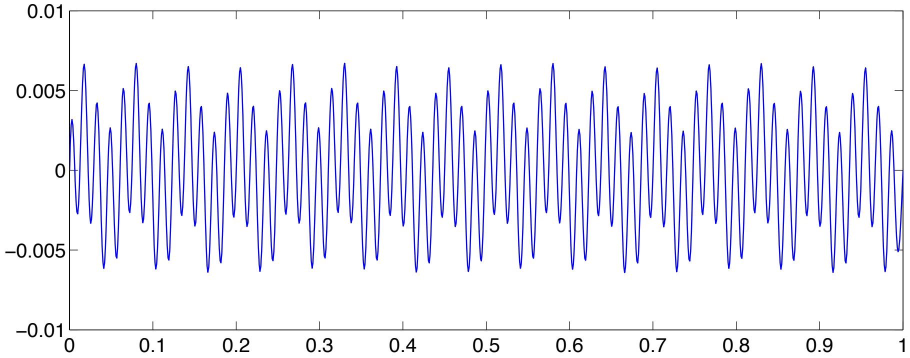
1 GS^8 relaxation 0 CG correction

Illustration: two-grid convergence



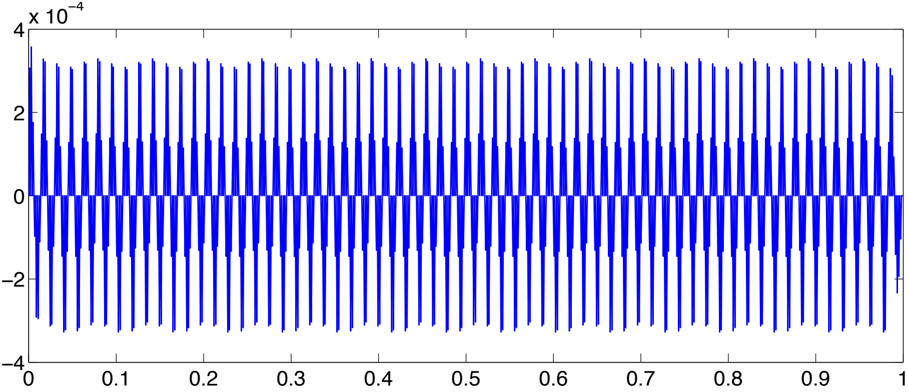
1 GS^8 relaxation   1 CG correction

Illustration: two-grid convergence



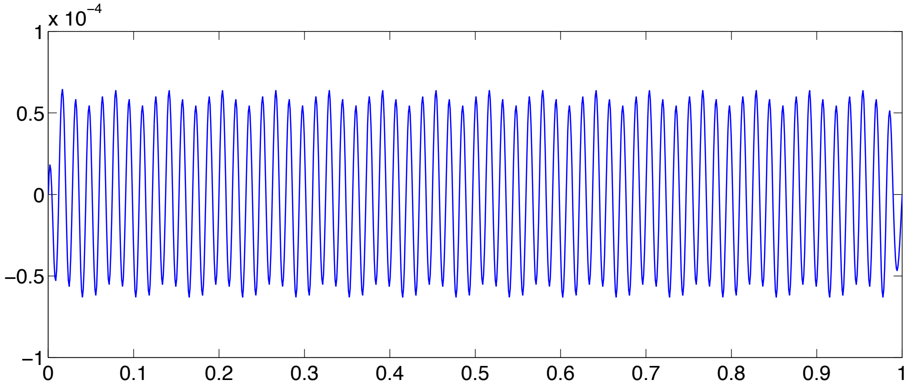
2 GS^8 relaxation 1 CG correction

Illustration: two-grid convergence



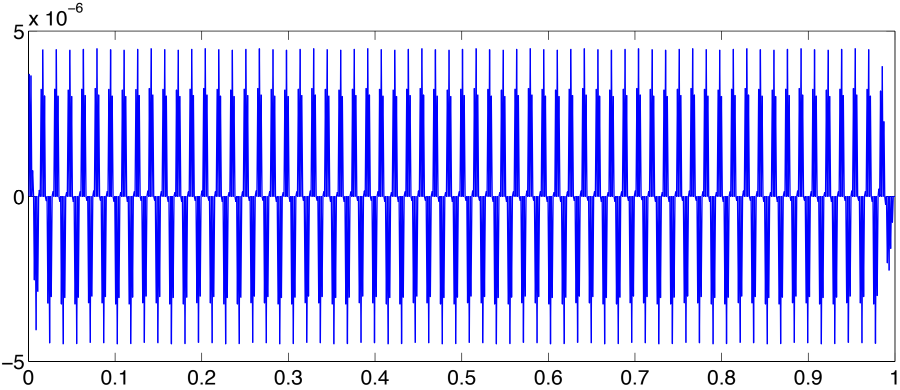
2 GS^8 relaxation   2 CG correction

Illustration: two-grid convergence



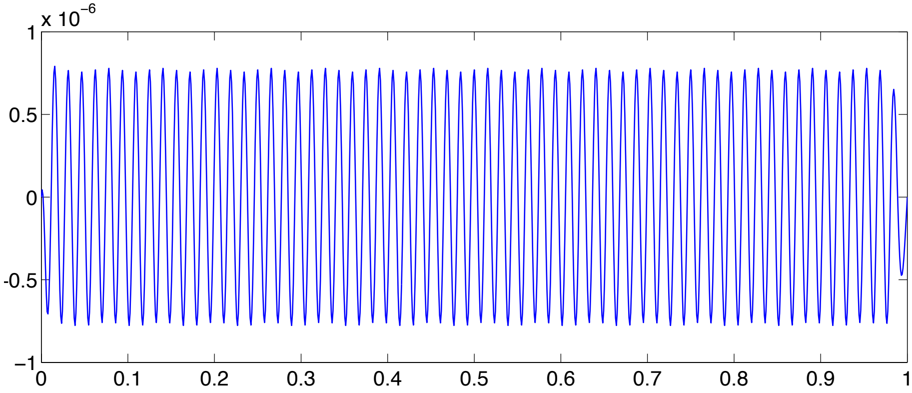
3 GS^8 relaxation   1 CG correction

Illustration: two-grid convergence



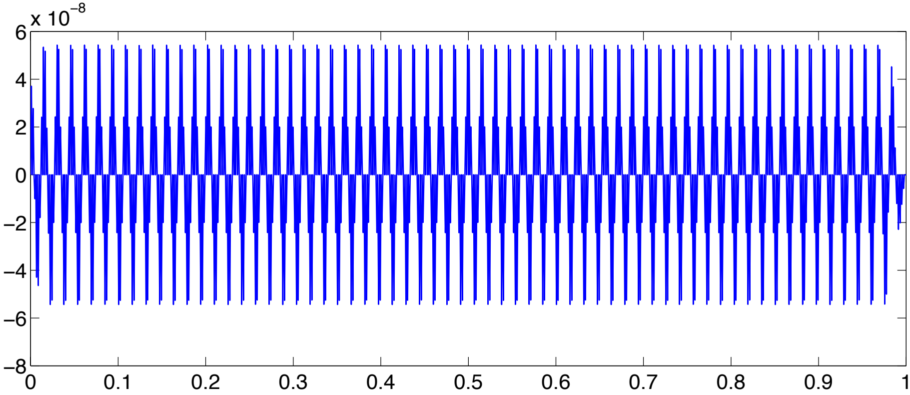
3 GS^8 relaxation   3 CG correction

Illustration: two-grid convergence



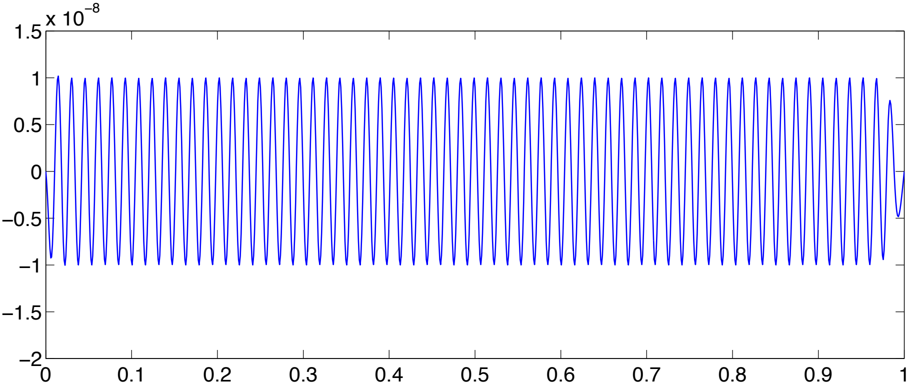
4 GS^8 relaxation 3 CG correction

Illustration: two-grid convergence



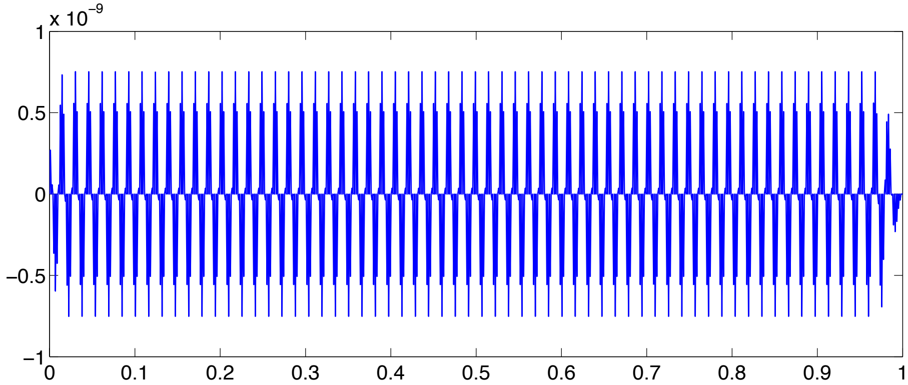
4 GS^8 relaxation   4 CG correction

Illustration: two-grid convergence



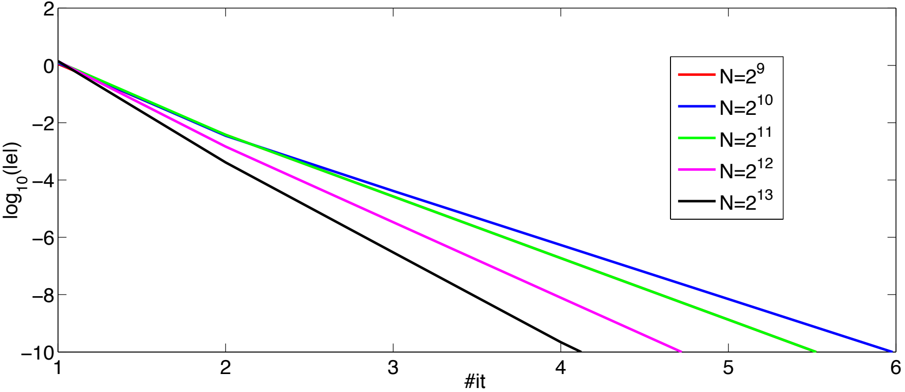
5 GS^8 relaxation   4 CG correction

Illustration: two-grid convergence



5 GS^8 relaxation 5 CG correction

Illustration: two-grid convergence



Coarse-grid equations: non-linear problems

The error relation applied in the correction scheme,

$$A^h e^h = f^h - A^h \tilde{u}^h$$

is invalid for nonlinear problems:

$$A^h(e^h + \bar{u}^h) \neq A^h e^h + A^h \bar{u}^h$$

Coarse-grid equations: non-linear problems

Consider the decomposition $V^h = V^H \oplus V^\perp$. Let $\tilde{u}^h \in V^h$ denote a *post-relaxation* approximation.

1. Since the error in $\tilde{u}^h$ is smooth:

$$\tilde{u}^h - R\tilde{u}^h \approx \bar{u}^h - R\bar{u}^h \in V^\perp$$

   (The projection of $\tilde{u}^h$ onto $V^\perp$ (oscillatory component) is close to the projection of the actual solution)

2. So, we require $u^H \in V^H$ such that

$$A^h\big(u^H + (\tilde{u}^h - R\tilde{u}^h)\big) = f^h \qquad (*)$$

   but this is not a useful equation for u^H (ill posed).

Coarse-grid equations: non-linear problems

3. To obtain a meaningful coarse-grid equation
   1. project $(*)$ onto V^H:

$$RA^h\big(u^H + (\tilde{u}^h - R\tilde{u}^h)\big) = Rf^h \qquad (*^H)$$

   2. perform a defect-correction step with approximate operator

$$RA^h\big(u^H + (\tilde{u}^h - R\tilde{u}^h)\big) \approx \widetilde{RA^h}\big(u^H + (\tilde{u}^h - R\tilde{u}^h)\big) := A^H u^H$$

   and initial estimate $\tilde{u}^H = R\tilde{u}^h$

$$\boxed{A^H u^H = A^H R\tilde{u}^h + R(f^h - A^h\tilde{u}^h)}$$

4. The approximation u^H can be used to correct the fine-grid approximation $\tilde{u}^h$ according to:

$$\tilde{u}^h + P(u^H - R\tilde{u}^h)$$

Coarse-grid equations: non-linear problems

Remarks

1. The equation

$$\boxed{A^H u^H = A^H R \tilde{u}^h + R(f^h - A^h \tilde{u}^h)} \qquad \text{(FAS)}$$

   is called the (coarse-grid equation of) the Full-Approximation Scheme (Note: not the error but the solution itself is approximated)
2. Multigrid recursion in same manner as for linear problems
3. The derivation is much more elegant in a variational setting (Galerkin FEM) $\Rightarrow$ FAS-MG directly related to *multiscale formulations*

Coarse-grid equations: non-linear problems

Full-Approximation Scheme: dual perspective

If $\tilde{u}^h = \bar{u}^h$ then

$$A^H u^H = A^H R \tilde{u}^h + R(f^h - A^h \tilde{u}^h) \qquad \text{(FAS)}$$

Because $f^h - A^h \bar{u}^h = 0$, the correction equation (FAS) implies

$$A^H u^H = A^H R \bar{u}^h \quad \Leftrightarrow \quad \boxed{u^H = R \bar{u}^h}$$

- $\Rightarrow$ in the multigrid process, the coarse-grid solution converges to the restriction (projection) of the fine-grid solution
- $\Rightarrow$ interpretation: the right-hand side in (FAS) represents the effect of fine scales (in $V^{\perp}$) on the coarse-grid solution

3TU. TU/e Technische Universiteit Eindhoven University of Technology

Outline

Full Multi-Grid (FMG)

Coarse-grid prediction

The coarse grid can also be used to construct an initial approximation for the fine grid.

1. Solve the H-grid problem:

$$A^H u^H = f^H$$

2. Construct an initial approximation for h-grid by prolongation:

$$u^{h,0} = Pu^H$$

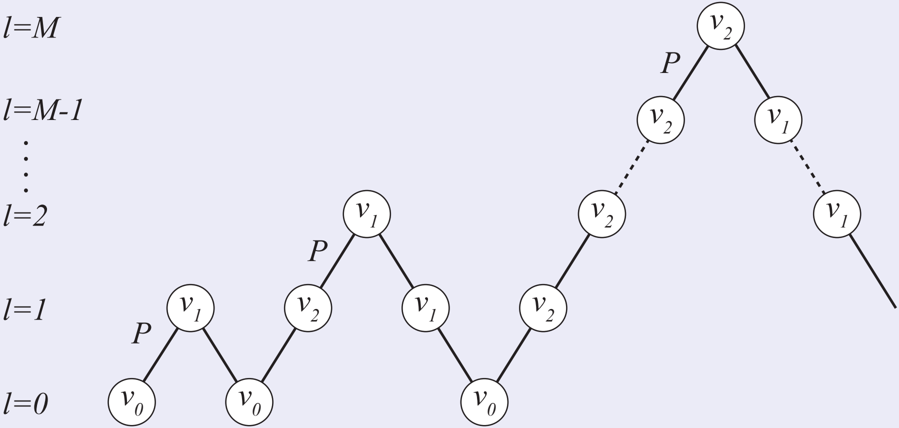
Full Multi-Grid (FMG)

Remarks

1. Of course, the coarse-grid prediction can again be applied recursively
2. Since Pu^H is only an initial approximation, it is not necessary to fully resolve u^H: a suitable approximation will do

Full Multi-Grid (FMG)

FMG 1V-cycle



Outline

Challenges

'Theory and practice are the same in theory;
in practice, they are not'

$\neq$

Challenges

When MG does not (trivially) work ...

1. Non-elliptic problems: hyperbolic PDEs, integro-differential equations, ...
2. Anisotropic problems: direction dependent coefficients (and smoothing)
3. Near boundaries: corner singularities, non-linear BCs, ...
4. Indefinite problems: Helmholtz-type equations, standing waves, ...

3TU. TU/e Technische Universiteit Eindhoven University of Technology

Current developments

Research directions

1. p-multigrid: applying MG to high-order discretizations and using low-order corrections
2. Algebraic Multi Grid (AMG): black-box multigrid, providing MG as standard preconditioner/solver, similar to LU factorization
3. Applications
4. Combination with optimization
5. ...

Further reading

- A. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp. **31** (1977), 333–390.
- A. Brandt, *Multigrid Techniques: 1984 Guide with Applications to Fluid Dynamics*, Tech. Report GMD **85**, 1984
- W. Hackbusch,*Multi-grid methods and applications*, Springer, Berlin, 1985.
- P. Wesseling, *An introduction to multigrid methods*, Pure and Applied Mathematics, Wiley, Chichester, 1992.
- W.L. Briggs, V.A. Henson, and S.F. McCormick, *A multigrid tutorial*, 2nd ed., SIAM, 2000.
- U. Trottenberg, C.W. Oosterlee, and A. Schuller, *Multigrid*, Academic Press, 2001.