Domain Decomposition Methods.

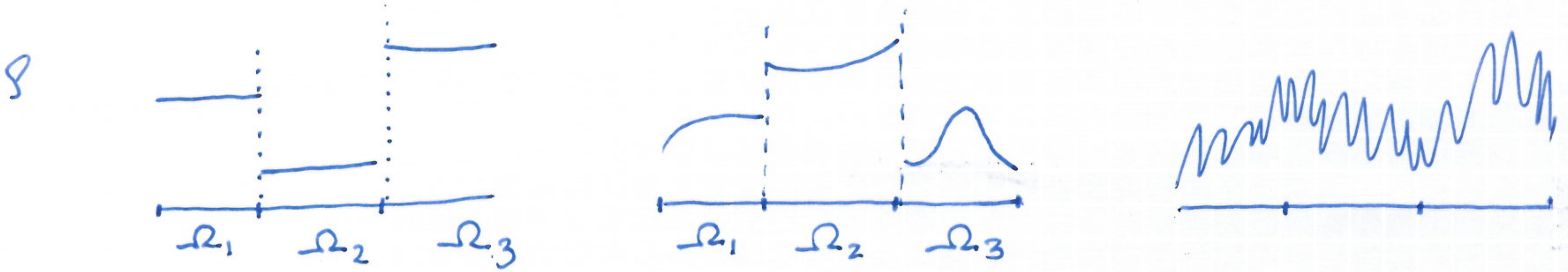
Problems of interest

$$-\nabla \cdot \rho(x) \nabla u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

$\rho(x)$: piecewise constant in each subdomain Ω_i, ρ_i,
or smoothly varying in each Ω_i,
with jump across Γ.

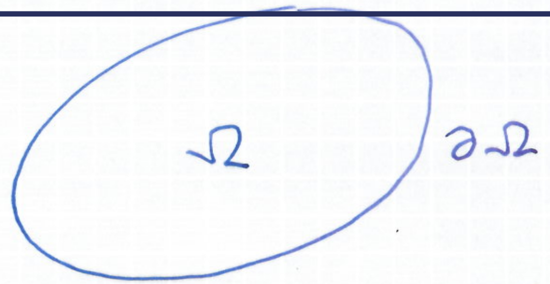
$\rho(x)$: highly varying, heterogeneous.



Basic idea

Consider the problem:

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$



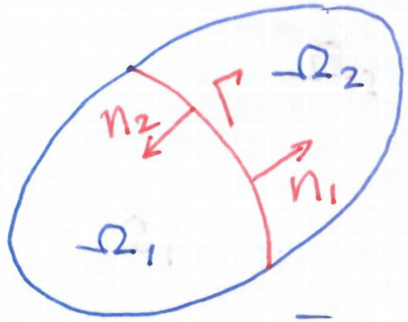
An equivalent coupled problem*:

$$-\Delta u_1 = f \quad \text{in } \Omega_1$$
$$u_1 = 0 \quad \text{on } \partial\Omega_1 \setminus \Gamma$$

Transmission condition
$$\begin{cases} u_1 = u_2 & \text{on } \Gamma \\ \dfrac{\partial u_1}{\partial n_1} = -\dfrac{\partial u_2}{\partial n_2} & \text{on } \Gamma \end{cases}$$

$$-\Delta u_2 = f \quad \text{in } \Omega_2$$
$$u_2 = 0 \quad \text{on } \partial\Omega_2 \setminus \Gamma$$

$\Gamma = \Omega_1 \cap \Omega_2$



$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$

$\Omega_1 \cap \Omega_2 = \emptyset$

Non-overlapping

* regularity assumption on f and $\partial\Omega$

Matrix representation

Linear system:

$$Au = f$$

A: spd

mesh size: h

$\kappa(A) \sim h^{-2}$

Partitioned:

$$A = \left(\begin{array}{c:c:c} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ \hdashline 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ \hdashline A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{array}\right) \qquad u = \left(\begin{array}{c} u_I^{(1)} \\ \hdashline u_I^{(2)} \\ \hdashline u_\Gamma \end{array}\right) \qquad f = \left(\begin{array}{c} f_I^{(1)} \\ \hdashline f_I^{(2)} \\ \hdashline f_\Gamma \end{array}\right)$$

Locally

$$A^{(i)} = \left(\begin{array}{c:c} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ \hdashline A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{array}\right) \qquad f = \left(\begin{array}{c} f_I^{(i)} \\ \hdashline f_\Gamma^{(i)} \end{array}\right)$$

I: interior

Γ: interface

$i = 1, 2$

$$A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)}$$

$$f_\Gamma = f_\Gamma^{(1)} + f_\Gamma^{(2)}$$

Approximation of normal derivatives, $\frac{\partial u_i}{\partial n_i}$

Using Green's formula

φ_j : nodal basis function on Γ.

$$\int_\Gamma \frac{\partial u_i}{\partial n_i}\varphi_j = \int_{\Omega_i} \nabla u_i \cdot \nabla \varphi_j + \underbrace{\int_\Omega \Delta u_i \varphi_j}_{-\int_\Omega f\varphi_j}$$

flux

Let $\lambda^{(i)}$: approximation of functional representing $\frac{\partial u_i}{\partial n_i}$.

$$\lambda_\Gamma^{(i)} = \underbrace{A_{\Gamma I} u_I^{(i)} + A_{\Gamma\Gamma} u_\Gamma^{(i)} - f_\Gamma^{(i)}}_{\text{Local residual on } \Gamma}$$

Letting j run over the nodes on Γ.

$$\begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{pmatrix} \quad \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{pmatrix}$$

Approximation of the coupled problem

a) $A_{II}^{(1)} u_I^{(1)} + A_{I\Gamma}^{(1)} u_\Gamma^{(1)} = f^{(1)}$

Dirichlet data vanashing on $\partial\Omega_1 \setminus \Gamma$ and $\partial\Omega_2 \setminus \Gamma$

b) $u_\Gamma^{(1)} = u_\Gamma^{(2)} = u_\Gamma$

c) $\underbrace{A_{\Gamma I}^{(1)} u_I^{(1)} + A_{\Gamma\Gamma}^{(1)} u_\Gamma - f_\Gamma^{(1)}}_{\lambda_\Gamma^{(1)}} = -\big(\underbrace{A_{\Gamma I}^{(2)} u_I^{(2)} + A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(2)} - f_\Gamma^{(2)}}_{\lambda_\Gamma^{(2)}}\big) = \lambda_\Gamma$

d) $A_{II}^{(2)} u_I^{(2)} + A_{I\Gamma}^{(2)} u_\Gamma^{(2)} = f^{(2)}$

a $\wedge$ d $\wedge$ b : Dirichlet problems ('0' on $\partial\Omega_i$ and common val. on Γ)

a $\wedge$ c : Mixed problem $u^{(1)}$ (Neumann λ_Γ on Γ and '0' on $\partial\Omega_1 \setminus \Gamma$)

d $\wedge$ c : —"— $u^{(2)}$

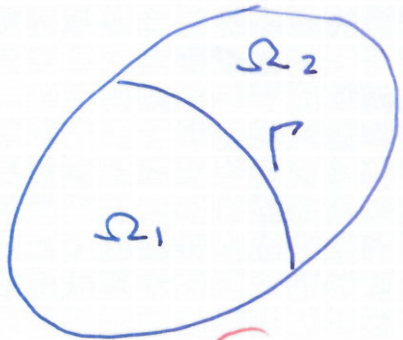
Non Overlapping Methods

2- subdomain case

(Simple substructuring algorithms)

$$A u = f$$

$$\underbrace{\begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} f_I^{(1)} \\ f_I^{(2)} \\ f_\Gamma \end{pmatrix}}_{f}$$



u_Γ

λ_Γ

An equation for u_Γ

$$u_\Gamma^{(1)} = u_\Gamma^{(2)} = u_\Gamma \quad \text{(unknown)}$$

$$A = LR = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A_{\Gamma I}^{(1)} A_{II}^{(1)^{-1}} & A_{\Gamma I}^{(2)} A_{II}^{(2)^{-1}} & I \end{pmatrix} \begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ 0 & 0 & S \end{pmatrix}$$

$$S = A_{\Gamma\Gamma} - A_{\Gamma I}^{(1)} A_{II}^{(1)^{-1}} A_{I\Gamma}^{(1)} - A_{\Gamma I}^{(2)} A_{II}^{(2)^{-1}} A_{I\Gamma}^{(2)}$$

Schur Complement

Resulting system:

$$\begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} f_I^{(1)} \\ f_I^{(2)} \\ g_\Gamma \end{pmatrix}$$

$$g_\Gamma = f_\Gamma - A_{\Gamma I}^{(1)} A_{II}^{(1)^{-1}} f_I^{(1)} - A_{\Gamma I}^{(2)} A_{II}^{(2)^{-1}} f_I^{(2)}$$

To get u_Γ: Solve $\boxed{S u_\Gamma = g_\Gamma}$

$u_I^{(i)}$: Local Dirichlet prob.

$$u_I^{(i)} = A_{II}^{(i)^{-1}} \left(f_I^{(i)} - A_{I\Gamma}^{(i)} u_\Gamma \right)$$

Schur complement system for u_Γ

$$(S^{(1)} + S^{(2)})\, u_\Gamma = \boxed{S\, u_\Gamma = g_\Gamma} = (g_\Gamma^{(1)} + g_\Gamma^{(2)})$$

where

$$S^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{I\Gamma}^{(i)} \qquad g_\Gamma^{(i)} = f_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} f_\Gamma^{(i)}$$

Local Schur complement

Can we derive the same system from the coupled problem ?

- Solve Local Dirichlet problem:

$$\boxed{u_I^{(i)} = A_{II}^{(i)^{-1}} \left(f_I^{(i)} - A_{I\Gamma} u_\Gamma^{(i)} \right) \qquad i = 1, 2}$$

- Use These expressions for $u_I^{(1)}$ and $u_I^{(2)}$ in the transmission condition to get:

$$(S^{(1)} + S^{(2)})\, u_\Gamma = g_\Gamma$$

An equation for the flux λ_Γ

$$\lambda_\Gamma^{(1)} = -\lambda_\Gamma^{(2)} = \lambda_\Gamma \qquad \text{(unknown)}$$

- Local Neumann problem for $u^{(i)}$, $i = 1, 2$

$$\boxed{\begin{aligned} A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} &= f_I^{(i)} \\ A_{\Gamma I}^{(i)} u_I^{(i)} + A_{\Gamma\Gamma}^{(i)} u_\Gamma^{(i)} &= f_\Gamma^{(i)} + \lambda_\Gamma^{(i)} \end{aligned}}$$

or

$$\begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{pmatrix} = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} + \lambda_\Gamma^{(i)} \end{pmatrix}$$

- Using block factorization

$$u_\Gamma^{(1)} = S^{(1)^{-1}} \left(g_\Gamma^{(1)} + \lambda_\Gamma^{(1)} \right) \quad \text{and} \quad u_\Gamma^{(2)} = S^{(2)^{-1}} \left(g_\Gamma^{(2)} + \lambda_\Gamma^{(2)} \right)$$

- With $u_\Gamma^{(1)} = u_\Gamma^{(2)}$

$$\underbrace{\left(S^{(1)^{-1}} + S^{(2)^{-1}} \right)}_{F} \lambda_\Gamma = \underbrace{- S^{(1)^{-1}} g_\Gamma^{(1)} + S^{(2)^{-1}} g_\Gamma^{(2)}}_{d_\Gamma}$$

$$\boxed{F \lambda_\Gamma = d_\Gamma}$$

- Algorithms to be described, involve preconditioners to solve

<u>either</u>

$$\boxed{S u_\Gamma = g_\Gamma}$$

$$S = S^{(1)} + S^{(2)}$$

- Application of $S^{(i)}$ corresponds to solving a Dirichlet problem.

<u>or</u>

$$\boxed{F \lambda_\Gamma = d_\Gamma}$$

$$F = {S^{(1)}}^{-1} + {S^{(2)}}^{-1}$$

- Application of ${S^{(i)}}^{-1}$ $\rightarrow$ solving a Neumann prob.

- Several of the Substructuring algorithms are, in fact, preconditioned Richardson methods to solve the two systems.

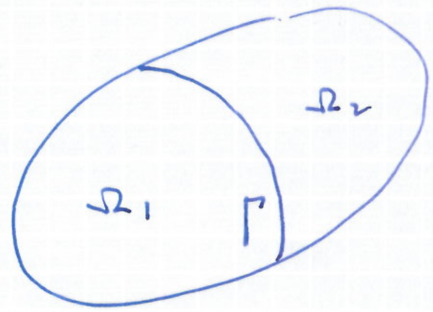
Dirichlet - Neumann

Bjørstad - Widlund (1986)
Bramble - Pasciak - Schatz (1986)

Consists of two fractional steps

Given u_Γ^n on Γ

$$\text{(D)} \begin{cases} -\Delta u_1^{n+1} = f & \text{in } \Omega_1 \\ u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \\ u_1^{n+1} = u_\Gamma^n & \Gamma \end{cases}$$



$$\text{(N)} \begin{cases} -\Delta u_2^{n+1} = f & \text{in } \Omega_2 \\ u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \\ \dfrac{\partial u_2^{n+1}}{\partial n_2} = -\dfrac{\partial u_1^{n+1}}{\partial n_1} & \text{on } \Gamma \end{cases}$$

Update $\quad u_\Gamma^{n+1} = u_2^{n+1} \quad$ on Γ

$\underline{D-N}$

given u_Γ^n

$$\left.\begin{aligned} -\Delta u_1^{n+1} &= f && \Omega_1 \\ u_1^{n+1} &= 0 && \partial\Omega_1 \backslash \Gamma \\ u_1^{n+1} &= u_\Gamma^n && \Gamma \end{aligned}\right\} \text{D}$$

$$A_{11}^{(1)} u_1^{(1)\,n+1} + A_{1\Gamma}^{(1)} u_\Gamma^n = f_1^{(1)}$$

$$\left.\begin{aligned} -\Delta u_2^{n+1} &= f && \Omega_2 \\ u_2^{n+1} &= 0 && \partial\Omega_2 \backslash \Gamma \\ \frac{\partial u_2^{n+1}}{\partial n_2} &= -\frac{\partial u_1^{n+1}}{\partial n_1} && \Gamma \end{aligned}\right\} \text{N}$$

$$\begin{pmatrix} A_{11}^{(2)} & A_{1\Gamma}^{(2)} \\ A_{\Gamma 1}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{pmatrix} \begin{pmatrix} u_1^{(2)\,n+1} \\ \omega \end{pmatrix} = \begin{pmatrix} f_1^{(2)} \\ f_\Gamma^{(2)} - \lambda_\Gamma^{n+1} \end{pmatrix}$$

$$\lambda_\Gamma^{n+1} = A_{1\Gamma}^{(1)} u_1^{(1)\,n+1} + A_{\Gamma\Gamma}^{(1)} u_\Gamma^n - f_\Gamma^{(1)}$$

$$u_\Gamma^{n+1} = \omega$$

D-N

Eliminate $u_I^{(1)^{n+1}}$ from (D) to get:

$$\lambda_\Gamma^{n+1} = -\left(g_\Gamma^{(1)} - S^{(1)} u_\Gamma^{n} \right)$$

i.e. -ve of the local residual.

Eliminate $u_I^{(2)^{n+1}}$ from (N) to get:

$$\boxed{u_\Gamma^{n+1} = u_\Gamma^{n} + S^{(2)^{-1}} \left(g_\Gamma - S u_\Gamma^{n} \right)}$$

residual of Schur Compl.

D-N is preconditioned Richardson iteration for the Schur complement system: $S u_\Gamma = g_\Gamma$

Preconditioner : $S^{(2)^{-1}}$

Preconditioned matrix : $S^{(2)^{-1}} S = I + S^{(2)^{-1}} S^{(1)}$

$$\varkappa\left(S^{(2)^{-1}} S \right) \le C$$

Optimal: independent of mesh size h