Domain Decomposition Methods.

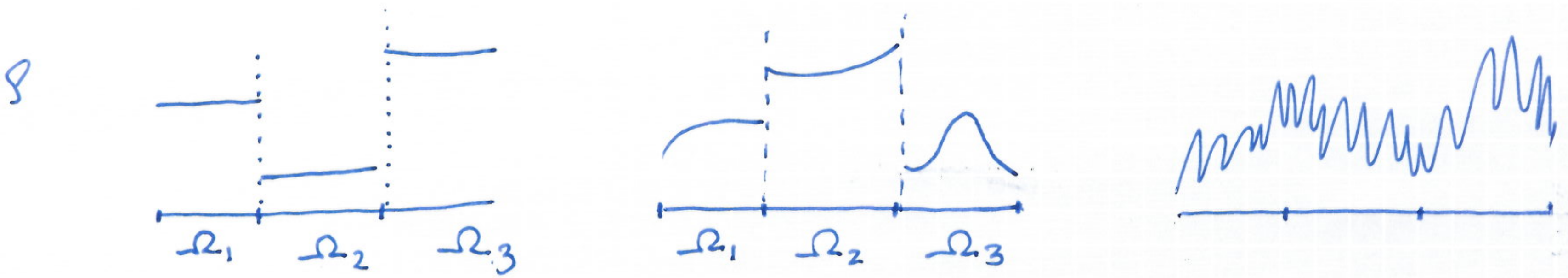
Problems of interest

$$-\nabla\cdot\rho(x)\nabla u = f \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega$$

$\rho(x)$: piecewise constant in each subdomain Ω_i, ρ_i,
or smoothly varying in each Ω_i,
with jump across Γ.

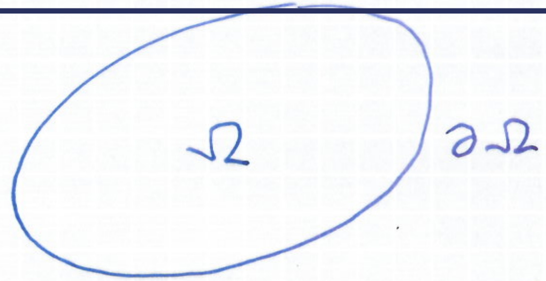
$\rho(x)$: highly varying, heterogeneous.



Basic idea

Consider the problem:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$



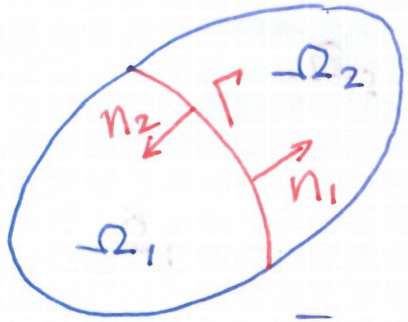
An equivalent coupled problem*:

$$\begin{aligned} -\Delta u_1 &= f && \text{in } \Omega_1 \\ u_1 &= 0 && \text{on } \partial\Omega_1 \setminus \Gamma \end{aligned}$$

$$\text{Transmission condition} \left\{ \begin{aligned} u_1 &= u_2 && \text{on } \Gamma \\ \frac{\partial u_1}{\partial n_1} &= -\frac{\partial u_2}{\partial n_2} && \text{on } \Gamma \end{aligned} \right.$$

$$\begin{aligned} -\Delta u_2 &= f && \text{in } \Omega_2 \\ u_2 &= 0 && \text{on } \partial\Omega_2 \setminus \Gamma \end{aligned}$$

$\Gamma = \Omega_1 \cap \Omega_2$



$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$

$\Omega_1 \cap \Omega_2 = \emptyset$

Non-overlapping

* regularity assumption on f and $\partial\Omega$

Matrix representation

Linear system:

$$Au = f$$

A: spd
mesh size: h
$\varkappa(A) \sim h^{-2}$

Partitioned:

$$A = \begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{pmatrix} \qquad u = \begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{pmatrix} \qquad f = \begin{pmatrix} f_I^{(1)} \\ f_I^{(2)} \\ f_\Gamma \end{pmatrix}$$

Locally:

$$A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \qquad f = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{pmatrix}$$

I: interior
Γ: interface

$i = 1, 2$

$$A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)}$$

$$f_\Gamma = f_\Gamma^{(1)} + f_\Gamma^{(2)}$$

Approximation of normal derivatives, $\frac{\partial u_i}{\partial n_i}$

Using Green's formula

φ_j : nodal basis function on Γ.

$$\int_\Gamma \frac{\partial u_i}{\partial n_i}\varphi_j = \int_{\Omega_i} \nabla u_i \cdot \nabla \varphi_j + \underbrace{\int_\Omega \Delta u_i \varphi_j}_{-\int_\Omega f\varphi_j}$$

flux

Let $\lambda^{(i)}$: approximation of functional representing $\frac{\partial u_i}{\partial n_i}$.

$$\lambda_\Gamma^{(i)} = \underbrace{A_{\Gamma I} u_I^{(i)} + A_{\Gamma\Gamma} u_\Gamma^{(i)} - f_\Gamma^{(i)}}_{\text{Local residual on } \Gamma}$$

Letting j run over the nodes on Γ.

$$\begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{pmatrix} \quad \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{pmatrix}$$

Approximation of the coupled problem

a) $A_{II}^{(1)} u_I + A_{I\Gamma}^{(1)} u_\Gamma^{(1)} = f^{(1)}$

Dirichlet data vanashing on $\partial\Omega_1 \setminus \Gamma$ and $\partial\Omega_2 \setminus \Gamma$

b) $u_\Gamma^{(1)} = u_\Gamma^{(2)} = u_\Gamma$

c) $$\underbrace{A_{\Gamma I}^{(1)} u_I + A_{\Gamma\Gamma}^{(1)} u_\Gamma - f_\Gamma^{(1)}}_{\lambda_\Gamma^{(1)}} = -\underbrace{\left(A_{\Gamma I}^{(2)} u_I^{(2)} + A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(2)} - f_\Gamma^{(2)} \right)}_{\lambda_\Gamma^{(2)}} = \lambda_\Gamma$$

d) $A_{II}^{(2)} u_I^{(2)} + A_{I\Gamma}^{(2)} u_\Gamma^{(2)} = f^{(2)}$

a $\wedge$ d $\wedge$ b : Dirichlet problems ('0' on $\partial\Omega_i$ and common val. on Γ)

a $\wedge$ c : Mixed problem $u^{(1)}$ (Neumann λ_Γ on Γ and '0' on $\partial\Omega_1 \setminus \Gamma$)

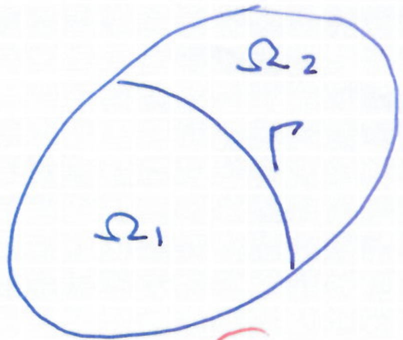
d $\wedge$ c : —— " —— $u^{(2)}$

Non Overlapping Methods

2- subdomain case

(Simple substructuring algorithms)

$$A u = f$$



u_Γ

λ_Γ

$$\underbrace{\begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{2\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma 2}^{(2)} & A_{\Gamma\Gamma} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} f_I^{(1)} \\ f_I^{(2)} \\ f_\Gamma \end{pmatrix}}_{f}$$

An equation for u_Γ

$$u_\Gamma^{(1)} = u_\Gamma^{(2)} = u_\Gamma \quad \text{(unknown)}$$

$$A = LR = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A_{\Gamma I}^{(1)} A_{II}^{(1)^{-1}} & A_{\Gamma I}^{(2)} A_{II}^{(2)^{-1}} & I \end{pmatrix} \begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ 0 & 0 & S \end{pmatrix}$$

$$S = A_{\Gamma\Gamma} - A_{\Gamma I}^{(1)} A_{II}^{(1)^{-1}} A_{I\Gamma}^{(1)} - A_{\Gamma I}^{(2)} A_{II}^{(2)^{-1}} A_{I\Gamma}^{(2)}$$

Schur Complement

Resulting system:

$$\begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} f_I^{(1)} \\ f_I^{(2)} \\ g_\Gamma \end{pmatrix}$$

$$g_\Gamma = f_\Gamma - A_{\Gamma I}^{(1)} A_{II}^{(1)^{-1}} f_I^{(1)} - A_{\Gamma I}^{(2)} A_{II}^{(2)^{-1}} f_I^{(2)}$$

To get u_Γ: Solve $\boxed{S u_\Gamma = g_\Gamma}$

$u_I^{(i)}$: Local Dirichlet prob.

$$u_I^{(i)} = A_{II}^{(i)^{-1}} \left(f_I^{(i)} - A_{I\Gamma}^{(i)} u_\Gamma \right)$$

Schur complement system for u_Γ

$$\left(S^{(1)} + S^{(2)}\right) u_\Gamma = \boxed{S\, u_\Gamma = g_\Gamma} = \left(g_\Gamma^{(1)} + g_\Gamma^{(2)}\right)$$

where

$$S^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{I\Gamma}^{(i)} \qquad g_\Gamma^{(i)} = f_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} f_\Gamma^{(i)}$$

Local Schur complement

Can we derive the same system from the coupled problem ?

- Solve Local Dirichlet problem:

$$\boxed{u_I^{(i)} = A_{II}^{(i)^{-1}} \left(f_I^{(i)} - A_{I\Gamma} u_\Gamma^{(i)} \right) \qquad i = 1, 2}$$

- Use these expressions for $u_I^{(1)}$ and $u_I^{(2)}$ in the transmission condition to get:

$$\left(S^{(1)} + S^{(2)}\right) u_\Gamma = g_\Gamma$$

An equation for the flux λ_Γ

$$\lambda_\Gamma^{(1)} = -\lambda_\Gamma^{(2)} = \lambda_\Gamma \qquad \text{(unknown)}$$

- Local Neumann problem for $u^{(i)}$, $i = 1, 2$

$$\boxed{\begin{aligned} A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} &= f_I^{(i)} \\ A_{\Gamma I}^{(i)} u_I^{(i)} + A_{\Gamma\Gamma}^{(i)} u_\Gamma^{(i)} &= f_\Gamma^{(i)} + \lambda_\Gamma^{(i)} \end{aligned}}$$

or

$$\begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{pmatrix} = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} + \lambda_\Gamma^{(i)} \end{pmatrix}$$

- Using block factorization

$$u_\Gamma^{(1)} = S^{(1)^{-1}} \left(g_\Gamma^{(1)} + \lambda_\Gamma^{(1)} \right) \quad \text{and} \quad u_\Gamma^{(2)} = S^{(2)^{-1}} \left(g_\Gamma^{(2)} + \lambda_\Gamma^{(2)} \right)$$

- With $u_\Gamma^{(1)} = u_\Gamma^{(2)}$

$$\underbrace{\left(S^{(1)^{-1}} + S^{(2)^{-1}} \right)}_{F} \lambda_\Gamma = \underbrace{- S^{(1)^{-1}} g_\Gamma^{(1)} + S^{(2)^{-1}} g_\Gamma^{(2)}}_{d_\Gamma}$$

$$\boxed{F \lambda_\Gamma = d_\Gamma}$$

- Algorithms to be described, involve preconditioners to solve

<u>either</u>

$$\boxed{S u_\Gamma = g_\Gamma}$$

$$S = S^{(1)} + S^{(2)}$$

- Application of $S^{(i)}$ corresponds to solving a Dirichlet problem.

<u>or</u>

$$\boxed{F \lambda_\Gamma = d_\Gamma}$$

$$F = S^{(1)^{-1}} + S^{(2)^{-1}}$$

- Application of $S^{(i)^{-1}}$ $\rightarrow$ solving a Neumann prob.

- Several of the Substructuring algorithms are, in fact, preconditioned Richardson methods to solve the two systems.

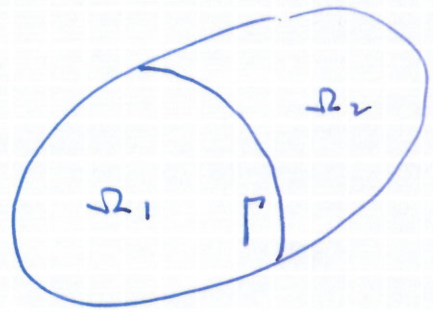
Dirichlet - Neumann

Bjørstad - Widlund (1986)
Bramble - Pasciak - Schatz (1986)

Consists of two fractional steps

Given u_Γ^n on Γ

$$(D) \begin{cases} -\Delta u_1^{n+1} = f & \text{in } \Omega_1 \\ u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \\ u_1^{n+1} = u_\Gamma^n & \Gamma \end{cases}$$



$$(N) \begin{cases} -\Delta u_2^{n+1} = f & \text{in } \Omega_2 \\ u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \\ \dfrac{\partial u_2}{\partial n_2}^{n+1} = -\dfrac{\partial u_1}{\partial n_1}^{n+1} & \text{on } \Gamma \end{cases}$$

Update $\quad u_\Gamma^{n+1} = u_2^{n+1} \quad$ on Γ

D-N

given u_Γ^n

$$\left.\begin{aligned} -\Delta u_1^{n+1} &= f && \Omega_1 \\ u_1^{n+1} &= 0 && \partial\Omega_1\setminus\Gamma \\ u_1^{n+1} &= u_\Gamma^n && \Gamma \end{aligned}\right\} \text{D}$$

$$A_{11}^{(1)} u_1^{(1)\,n+1} + A_{1\Gamma}^{(1)} u_\Gamma^n = f_1^{(1)}$$

$$\left.\begin{aligned} -\Delta u_2^{n+1} &= f && \Omega_2 \\ u_2^{n+1} &= 0 && \partial\Omega_2\setminus\Gamma \\ \frac{\partial u_2^{n+1}}{\partial n_2} &= -\frac{\partial u_1^{n+1}}{\partial n_1} && \Gamma \end{aligned}\right\} \text{N}$$

$$\begin{pmatrix} A_{11}^{(2)} & A_{1\Gamma}^{(2)} \\ A_{\Gamma 1}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{pmatrix} \begin{pmatrix} u_1^{(2)\,n+1} \\ \omega \end{pmatrix} = \begin{pmatrix} f_1^{(2)} \\ f_\Gamma^{(2)} - \lambda_\Gamma^{n+1} \end{pmatrix}$$

$$\lambda_\Gamma^{n+1} = A_{1\Gamma}^{(1)} u_1^{(1)\,n+1} + A_{\Gamma\Gamma}^{(1)} u_\Gamma^n - f_\Gamma^{(1)}$$

$$u_\Gamma^{n+1} = \omega$$

D-N

Eliminate $u_I^{(1)^{n+1}}$ from Ⓓ to get:

$$\lambda_\Gamma^{n+1} = -\left(g_\Gamma^{(1)} - S^{(1)} u_\Gamma^{n}\right)$$

i.e. -ve of the local residual.

Eliminate $u_I^{(2)^{n+1}}$ from Ⓝ to get:

$$\boxed{u_\Gamma^{n+1} = u_\Gamma^{n} + S^{(2)^{-1}}\left(g_\Gamma - S u_\Gamma^{n}\right)}$$

residual of Schur Compl.

D-N is preconditioned Richardson iteration for the Schur complement system: $S u_\Gamma = g_\Gamma$

Preconditioner : $S^{(2)^{-1}}$

Preconditioned matrix : $S^{(2)^{-1}} S = I + S^{(2)^{-1}} S^{(1)}$

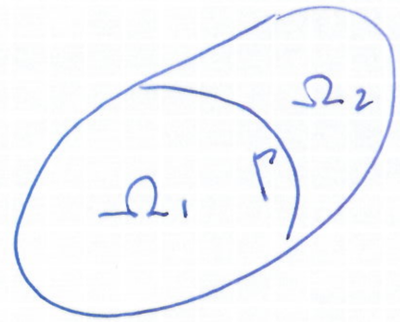
$$\varkappa\left(S^{(2)^{-1}} S\right) \leq C$$

Optimal: independent of mesh size h

Neumann – Neumann

Bourgat – Glowinski – Le Tallec – Vidrascu ('88)
Le-Tallec – De Roeck – Vidrasu ('91)

Given u_Γ^n on Γ



$$(D_i)\quad \begin{cases} -\Delta u_i^{n+1/2} = f & \text{in } \Omega_i \\ u_i^{n+1/2} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\ u_i^{n+1/2} = u_\Gamma^n & \text{on } \Gamma \end{cases} \qquad i = 1, 2$$

$$(N_i)\quad \begin{cases} -\Delta \psi_i^{n+1} = 0 & \text{in } \Omega_i \\ \psi_i^{n+1} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\ \dfrac{\partial \psi_i^{n+1}}{\partial n_i} = \underbrace{\dfrac{\partial u_1^{n+1/2}}{\partial n_1} + \dfrac{\partial u_2^{n+1/2}}{\partial n_2}} & \text{on } \Gamma \end{cases} \qquad i = 1, 2$$

correction step

difference in normal derivatives

$$u_\Gamma^{n+1} = u_\Gamma^n - \underbrace{\left(\psi_1^{n+1} + \psi_2^{n+1} \right)}_{\text{correction}}$$

$$\left.\begin{aligned} -\Delta u_i^{n+1/2} &= f && \Omega_i \\ u_i^{n+1/2} &= 0 && \partial\Omega_i \setminus \Gamma \\ u_i^{n+1/2} &= u_\Gamma^n && \Gamma \end{aligned}\right\} D_i$$

$$A_{II}^{(i)} u_I^{(i)\,n+1/2} + A_{I\Gamma}^{(i)} u_\Gamma^n = f_I^{(i)} \qquad i = 1,2$$

$$\left.\begin{aligned} -\Delta \psi_i^{n+1} &= 0 && \Omega_i \\ \psi_i^{n+1} &= 0 && \partial\Omega_i \setminus \Gamma \\ \frac{\partial \psi_i^{n+1}}{\partial n_i} &= \frac{\partial u_1^{n+1/2}}{\partial n_1} + \frac{\partial u_2^{n+1/2}}{\partial n_2} && \Gamma \end{aligned}\right\} N_i$$

$$\begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} \psi_I^{(i)\,n+1} \\ \psi_\Gamma^{(i)\,n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ r_\Gamma \end{pmatrix} \qquad i = 1,2$$

$$r_\Gamma = \left(A_{\Gamma I}^{(1)} u_I^{(1)\,n+1/2} + A_{\Gamma\Gamma}^{(1)} u_\Gamma^n - f_\Gamma^{(1)}\right) + \left(A_{\Gamma I}^{(2)} u_I^{(2)\,n+1/2} + A_{\Gamma\Gamma}^{(2)} u_\Gamma^n - f_\Gamma^{(2)}\right)$$

= Local flux difference

$$u_\Gamma^{n+1} = u_\Gamma^n - \left(\psi_\Gamma^{(1)\,n+1} + \psi_\Gamma^{(2)\,n+1}\right)$$

N-N

Straightforward to show

$$r_\Gamma = -(g_\Gamma - S u_\Gamma^n)$$

i.e. -ve residual of Schur complement

Further simplification leads to :

$$\boxed{u_\Gamma^{n+1} = u_\Gamma^n + \left(S^{(1)^{-1}} + S^{(2)^{-1}}\right)\left(g_\Gamma - S u_\Gamma^n\right)}$$

N-N is preconditioned Richardson for $S u_\Gamma = g_\Gamma$.

Preconditioner : $S^{(1)^{-1}} + S^{(2)^{-1}}$

Preconditioned matrix : $\underbrace{\left(S^{(1)^{-1}} + S^{(2)^{-1}}\right)}_{F}\underbrace{\left(S^{(1)} + S^{(2)}\right)}_{S}$

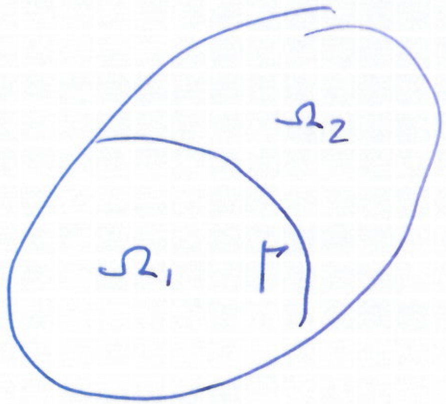
$\chi(FS) \le c$ optimal

Dirichlet - Dirichlet
(or preconditioned FETI)

(A method dual to N-N)

Given λ_Γ^n (approx. normal deiv.). Set $\lambda_1^n = -\lambda_2^n = \lambda^n$ (drop Γ)

$$
(N_i) \quad \begin{cases} -\Delta u_i^{n+1/2} = f & \text{in } \Omega_i \\ u_i^{n+1/2} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \qquad i = 1, 2 \\ \dfrac{\partial u_i^{n+1/2}}{\partial n_i} = \lambda_i^n & \text{on } \Gamma \end{cases}
$$



(D_i)

Correction step

$$
\begin{cases} -\Delta \psi_i^{n+1} = 0 & \text{in } \Omega_i \\ \psi_i^{n+1} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\ \psi_i^{n+1} = u_1^{n+1/2} - u_2^{n+1/2} & \text{on } \Gamma \end{cases}
$$

Update
$$
\lambda^{n+1} = \lambda^n - \underbrace{\left(\frac{\partial \psi_1^{n+1}}{\partial n_1} + \frac{\partial \psi_2^{n+1}}{\partial n_2} \right)}
$$

D–D

$$\left.\begin{aligned} -\Delta u_i^{n+1/2} &= f && \Omega_i \\ u_i^{n+1/2} &= 0 && \partial\Omega_i \setminus \Gamma \\ \frac{\partial u_i^{n+1/2}}{\partial n_i} &= \lambda_i^n && \Gamma \end{aligned}\right\} N_i \qquad \begin{pmatrix} A_I^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} u_I^{(i)\,n+1/2} \\ u_\Gamma^{(i)\,n+1/2} \end{pmatrix} = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} + \lambda_i^n \end{pmatrix} \qquad i=1,2$$

$$\left.\begin{aligned} -\Delta \psi_i^{n+1} &= 0 && \Omega_i \\ \psi_i^{n+1} &= 0 && \partial\Omega_i \setminus \Gamma \\ \psi_i^{n+1} &= u_1^{n+1/2} - u_2^{n+1/2} && \Gamma \end{aligned}\right\} D_i \qquad A_{II}^{(i)} \psi_I^{(i)\,n+1} + A_{I\Gamma}^{(i)} r_\Gamma = 0 \qquad i=1,2$$

$$r_\Gamma = u_\Gamma^{(1)\,n+1/2} - u_\Gamma^{(2)\,n+1/2}$$

= difference in value on Γ

$$\lambda^{n+1} = \lambda^n - \left(\eta_1^{n+1} + \eta_2^{n+2}\right)$$

η_i : local flux

$$= A_{\Gamma I}^{(i)} \psi_I^{(i)} + A_{\Gamma\Gamma}^{(i)} r_\Gamma$$

D-D

Straightforward to show:

$$r_\Gamma = -(d_\Gamma - F\lambda^n)$$

i.e. -ve residual of the system $F\lambda = d_\Gamma$

Further simplification:

$$\boxed{\lambda^{n+1} = \lambda^n + (S^{(1)} + S^{(2)})(d_\Gamma - F\lambda^n)}$$

D-D (PFETI) is preconditioned Richardson for $F\lambda = d_\Gamma$

Preconditioner : $S^{(1)} + S^{(2)}$

Preconditioned matrix : $(S^{(1)} + S^{(2)})(S^{(1)^{-1}} + S^{(2)^{-1}}) = SF$

$\kappa(SF) \leq c$

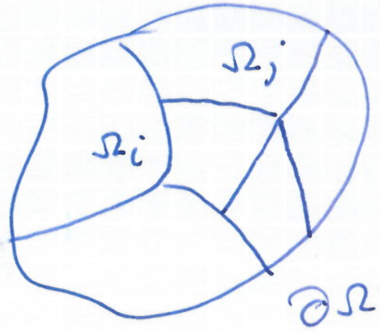
DD and N-N have same eigenvalues.

Many subdomains

$$\bar{\Omega} = \bigcup_i \bar{\Omega}_i \qquad \Omega_i \cap \Omega_j = \emptyset \quad i \neq j$$

$$\Gamma = \bigcup_i \Gamma_i \qquad \Gamma_i = \partial\Omega_i \setminus \partial\Omega$$

$$i = 1, \dots, N$$



- D-N is less widely used
- N-N, FETI widely used.

For D-N performance deteriorates for varying (highly) Coefficients.

$$\begin{pmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} f_I \\ f_\Gamma \end{pmatrix} \rightarrow \begin{pmatrix} A_{II} & A_{I\Gamma} \\ 0 & S \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} f_I \\ g_\Gamma \end{pmatrix}$$

where

$$S = \sum_i R_i^T S^{(i)} R_i$$

$$g_\Gamma = \sum_i R_i^T g_\Gamma^{(i)} R_i$$

R_i : Restriction matrix

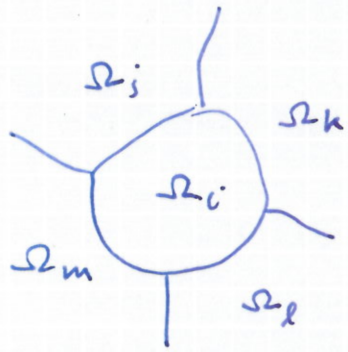
$R_i u_\Gamma$ returns $u_\Gamma^{(i)}$

N-N

$$S_{NN}^{-1} S = \sum_{i=1}^{N} R_i^T S^{(i)^{-1}} R_i \; S$$

Neumann prob. Dirichlet prob.

- $S^{(i)}$ is singular for floating subdomain
  $S^{(i)^{-1}}$ is understood as pseudo-inverse

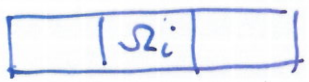


Ω_i : floating

$$\kappa\left(S_{NN}^{-1} S\right) \leq \frac{c}{H^2}\left(1 + \log\frac{H}{h}\right)^q$$

not scalable, quasi-optimal

$q = 0$ No cross pt.



$q = 3$ otherwise

$q = 2$ if suitable scaling matrix involved.

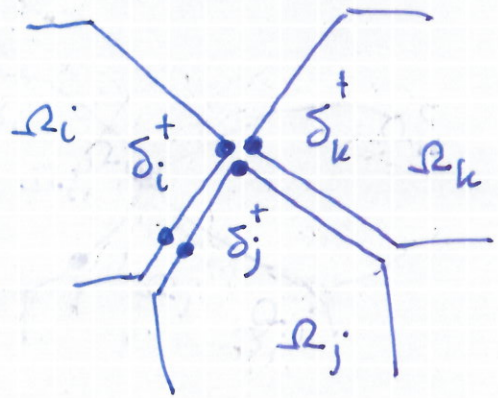
Scalability

rate of convergence does not deteriorate as # subdomain grows.

- Balancing N-N
  Le-Tallec (94)

Scaling

δ_i : weighted counting function associated with $\partial\Omega_i$



$$\delta_i(x) = \frac{\sum_{j\in N_x} \varrho_j}{\varrho_i}, \quad x \in \partial\Omega_{ih} \cap \Gamma_h$$

δ_i^+ : pseudoinverse

$$\delta_i^+(x) = \left(\delta_i(x)\right)^{-1} \quad x \in \partial\Omega_{ih} \cap \Gamma_h$$

They provide a partition of unity

$$\sum_i R_i^T \delta_i^+(x) = 1 \qquad x \in \Gamma_h$$

$D^{(i)}$: Diagonal matrix with elements $\delta_i^+(x)$, $x \in \partial\Omega_{ih}$.

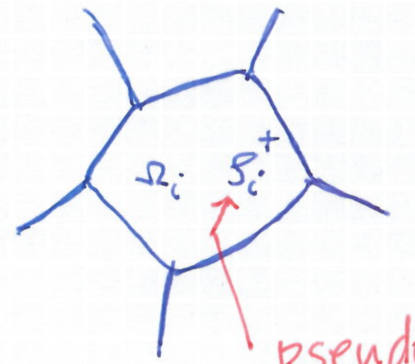
$$\boxed{S_{NN}^{-1} = \sum_{i=1}^{N} R_i^T D_i S^{(i)+} D_i R_i}$$

Convergence independent of jump in ϱ_i

A minimal coarse space for Neumann-Neumann

$$W_0 = \text{span}\left\{ R_i^T \delta_i^+ ,\ i = 1, \dots, N \right\}$$

basis function



pseudoinverse of the weighted counting func

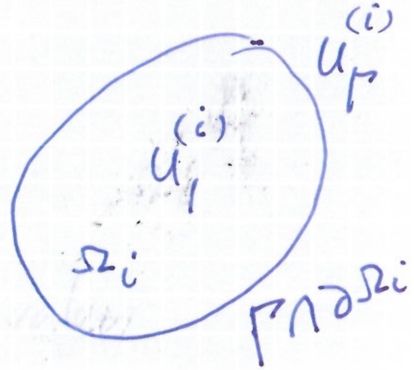
inside Ω_i the basis function is discrete harmonic $\mathcal{H}(R_i^T \delta_i^+)$.

Discrete Harmonic Extension : $\mathcal{H}$

directly related to
Schur-Complement methods

$$u^{(i)} = \begin{pmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{pmatrix} \quad \text{is Discrete harmonic}$$

if $\quad A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} = 0$



We write $\quad \boxed{u^{(i)} = \mathcal{H}(u_\Gamma^{(i)})}$ $\quad$ completely defined by values on $\Gamma \cap \partial\Omega_i$

Equivalently

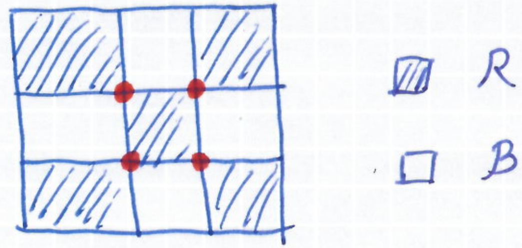
$$a_i\left(\mathcal{H} u_\Gamma^{(i)}, v\right) = 0 \qquad \forall v \in V^h \cap H_0^1(\Omega_i, \Gamma \cap \partial\Omega_i)$$

Lemma $\quad$ Discrete harmonic ext. $u^{(i)} = \mathcal{H}(u_\Gamma^{(i)})$ $\quad$ satisfies

$$a_i(u^{(i)}, u^{(i)}) = \left\langle S^{(i)} u_\Gamma^{(i)}, u_\Gamma^{(i)} \right\rangle = \min_{v|_{\Gamma \cap \partial\Omega_i} = u_\Gamma^{(i)}} a_i(v, v)$$

D–N

$$S = \sum_{i \in R} R_i^T S^{(i)} R_i + \sum_{i \in B} R_i^T S^{(i)} R_i$$



$$S_{DN}^{-1} = \left(\sum_{i \in R} R_i^T S^{(i)} R_i \right)^{-1}$$

- In general $\chi(S_{DN}^{-1} S)$ has the same logarithmic bound as N–N.

- Not scalable if no crosspts.
- scalable with crosspts.

S_{DN}^{-1} requires the solution of a global problem

Neumann problem on the union of Subdomains glued at crosspts.