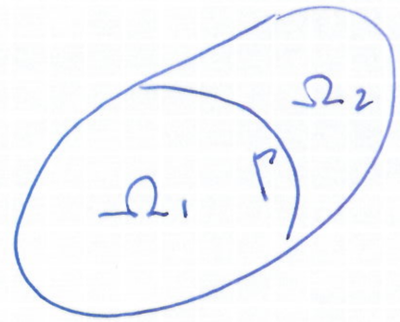
Neumann–Neumann

Bourgat–Glowinski–Le Tallec–Vidrascu ('88)
Le-Tallec – De Roeck – Vidrasu ('91)

Given u_Γ^n on Γ



$$(D_i)\quad \begin{cases} -\Delta u_i^{n+1/2} = f & \text{in } \Omega_i \\ u_i^{n+1/2} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\ u_i^{n+1/2} = u_\Gamma^n & \text{on } \Gamma \end{cases} \qquad i=1,2$$

$$(N_i)\quad \begin{cases} -\Delta \psi_i^{n+1} = 0 & \text{in } \Omega_i \\ \psi_i^{n+1} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\ \dfrac{\partial \psi_i^{n+1}}{\partial n_i} = \underbrace{\dfrac{\partial u_1^{n+1/2}}{\partial n_1} + \dfrac{\partial u_2^{n+1/2}}{\partial n_2}} & \text{on } \Gamma \end{cases} \qquad i=1,2$$

correction step

difference in normal derivatives

$$u_\Gamma^{n+1} = u_\Gamma^n - \underbrace{(\psi_1^{n+1} + \psi_2^{n+1})}_{\text{correction}}$$

$$\left.\begin{aligned} -\Delta u_i^{n+1/2} &= f && \Omega_i \\ u_i^{n+1/2} &= 0 && \partial\Omega_i \setminus \Gamma \\ u_i^{n+1/2} &= u_\Gamma^n && \Gamma \end{aligned}\right\} D_i \qquad A_{II}^{(i)} u_I^{(i)\,n+1/2} + A_{I\Gamma}^{(i)} u_\Gamma^n = f_I^{(i)} \qquad i = 1,2$$

$$\left.\begin{aligned} -\Delta \Psi_i^{n+1} &= 0 && \Omega_i \\ \Psi_i^{n+1} &= 0 && \partial\Omega_i \setminus \Gamma \\ \frac{\partial \Psi_i^{n+1}}{\partial n_i} &= \frac{\partial u_1^{n+1/2}}{\partial n_1} + \frac{\partial u_2^{n+1/2}}{\partial n_2} && \Gamma \end{aligned}\right\} N_i \qquad \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} \Psi_I^{(i)\,n+1} \\ \Psi_\Gamma^{(i)\,n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ r_\Gamma \end{pmatrix} \qquad i = 1,2$$

$$r_\Gamma = \left(A_{\Gamma I}^{(1)} u_I^{(1)\,n+1/2} + A_{\Gamma\Gamma}^{(1)} u_\Gamma^n - f_\Gamma^{(1)} \right) + \left(A_{\Gamma I}^{(2)} u_I^{(2)\,n+1/2} + A_{\Gamma\Gamma}^{(2)} u_\Gamma^n - f_\Gamma^{(2)} \right)$$

= Local flux difference

$$u_\Gamma^{n+1} = u_\Gamma^n - \left(\Psi_\Gamma^{(1)\,n+1} + \Psi_\Gamma^{(2)\,n+1} \right)$$

N-N

Straightforward to show

$$r_\Gamma = -(g_\Gamma - S u_\Gamma^n)$$

i.e. -ve residual of Schur complement

Further simplification leads to :

$$\boxed{u_\Gamma^{n+1} = u_\Gamma^n + \left({S^{(1)}}^{-1} + {S^{(2)}}^{-1}\right)(g_\Gamma - S u_\Gamma^n)}$$

N-N is preconditioned Richardson for $S u_\Gamma = g_\Gamma$.

Preconditioner : ${S^{(1)}}^{-1} + {S^{(2)}}^{-1}$

Preconditioned matrix : $\underbrace{\left({S^{(1)}}^{-1} + {S^{(2)}}^{-1}\right)}_{F} \underbrace{\left(S^{(1)} + S^{(2)}\right)}_{S}$

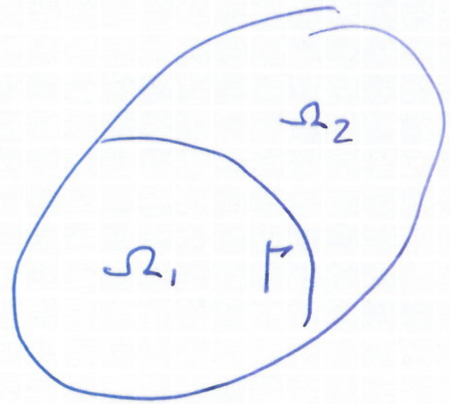
$\chi(FS) \leq c$ optimal

Dirichlet - Dirichlet
(or preconditioned FETI)

(A method dual to N-N)

Given λ_Γ^n (approx. normal deiv.). Set $\lambda_1^n = -\lambda_2^n = \lambda^n$ (drop Γ)

N_i
$$\begin{cases} -\Delta u_i^{n+1/2} = f & \text{in } \Omega_i \\ u_i^{n+1/2} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\ \dfrac{\partial u_i^{n+1/2}}{\partial n_i} = \lambda_i^n & \text{on } \Gamma \end{cases} \qquad i = 1, 2$$



D_i

Correction step
$$\begin{cases} -\Delta \psi_i^{n+1} = 0 & \text{in } \Omega_i \\ \psi_i^{n+1} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \\ \psi_i^{n+1} = u_1^{n+1/2} - u_2^{n+1/2} & \text{on } \Gamma \end{cases}$$

Update
$$\lambda^{n+1} = \lambda^n - \left(\underbrace{\frac{\partial \psi_1^{n+1}}{\partial n_1} + \frac{\partial \psi_2^{n+1}}{\partial n_2}}_{} \right)$$

D–D

$$\left.\begin{aligned} -\Delta u_i^{n+1/2} &= f & &\Omega_i \\ u_i^{n+1/2} &= 0 & &\partial\Omega_i\setminus\Gamma \\ \frac{\partial u_i^{n+1/2}}{\partial n_i} &= \lambda_i^n & &\Gamma \end{aligned}\right\} N_i \qquad \begin{pmatrix} A_{I}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} u_I^{(i)\,n+1/2} \\ u_\Gamma^{(i)\,n+1/2} \end{pmatrix} = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} + \lambda_i^n \end{pmatrix} \qquad i=1,2$$

$$\left.\begin{aligned} -\Delta \psi_i^{n+1} &= 0 & &\Omega_i \\ \psi_i^{n+1} &= 0 & &\partial\Omega_i\setminus\Gamma \\ \psi_i^{n+1} &= u_1^{n+1/2} - u_2^{n+1/2} & &\Gamma \end{aligned}\right\} D_i \qquad A_{II}^{(i)} \psi_I^{(i)\,n+1} + A_{I\Gamma}^{(i)} \underbrace{r_\Gamma}_{r_\Gamma = u_\Gamma^{(1)\,n+1/2} - u_\Gamma^{(2)\,n+1/2}} = 0 \qquad i=1,2$$

$$r_\Gamma = u_\Gamma^{(1)\,n+1/2} - u_\Gamma^{(2)\,n+1/2}$$

= difference in value on Γ

$$\lambda^{n+1} = \lambda^n - \left(\eta_1^{n+1} + \eta_2^{n+2}\right)$$

η_i : local flux

$$= A_{\Gamma I}^{(i)} \psi_I^{(i)} + A_{\Gamma\Gamma}^{(i)} r_\Gamma$$

D-D

Straightforward to show:

$$r_\Gamma = -(d_\Gamma - F\lambda^n)$$

i.e. -ve residual of the system $F\lambda = d_\Gamma$

Further simplification:

$$\boxed{\lambda^{n+1} = \lambda^n + (S^{(1)} + S^{(2)})(d_\Gamma - F\lambda^n)}$$

D-D (PFETI) is preconditioned Richardson for $F\lambda = d_\Gamma$

Preconditioner : $S^{(1)} + S^{(2)}$

Preconditioned matrix : $(S^{(1)} + S^{(2)})(S^{(1)^{-1}} + S^{(2)^{-1}}) = SF$

$\kappa(SF) \leq c$

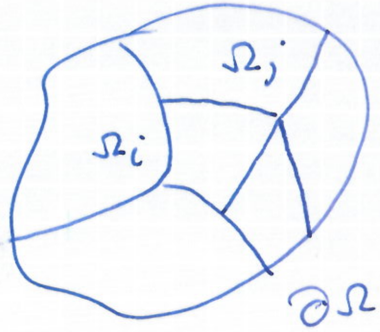
DD and N-N have same eigenvalues.

Many subdomains

$$\bar{\Omega} = \bigcup_i \bar{\Omega}_i \qquad \Omega_i \cap \Omega_j = \emptyset \quad i \neq j$$

$$\Gamma = \bigcup_i \Gamma_i \qquad \Gamma_i = \partial\Omega_i \setminus \partial\Omega$$

$$i = 1, \dots, N$$



- D-N is less widely used
- N-N, FETI widely used.

For D-N performance deteriorates for varying (highly) Coefficients.

$$\begin{pmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} f_I \\ f_\Gamma \end{pmatrix} \rightarrow \begin{pmatrix} A_{II} & A_{I\Gamma} \\ 0 & S \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} f_I \\ g_\Gamma \end{pmatrix}$$

where

$$S = \sum_i R_i^T S^{(i)} R_i$$

$$g_\Gamma = \sum_i R_i^T g_\Gamma^{(i)} R_i$$

R_i : Restriction matrix

$R_i u_\Gamma$ returns $u_\Gamma^{(i)}$

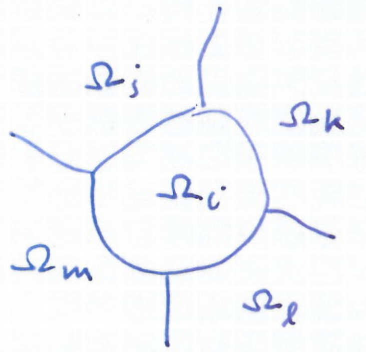
N-N

$$S_{NN}^{-1} S = \sum_{i=1}^{N} R_i^T S^{(i)^{-1}} R_i \; S$$

Neumann prob.

Dirichlet prob.

- $S^{(i)}$ is singular for floating subdomain
  $S^{(i)^{-1}}$ is understood as pseudo-inverse

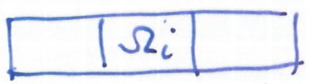


Ω_i : floating

$$\chi\left(S_{NN}^{-1} S\right) \leq \frac{c}{H^2}\left(1 + \log\frac{H}{h}\right)^q$$

not scalable,
quasi-optimal

$q = 0$ No cross pt.



$q = 3$ otherwise

$q = 2$ if suitable scaling matrix involved.

Scalability

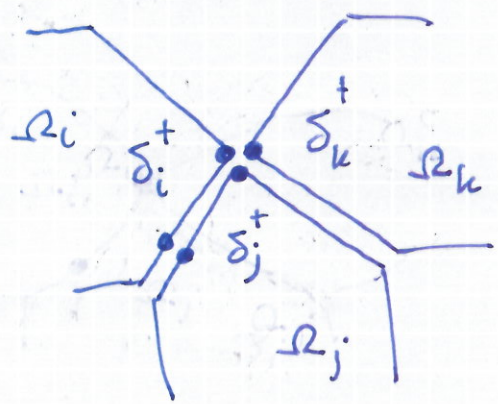
rate of convergence does not deteriorate as # subdomain grows.

- Balancing N-N
  Le-Tallec (94)

Scaling

δ_i : weighted counting function associated with $\partial\Omega_i$

$$\delta_i(x) = \frac{\sum_{j \in N_x} \rho_j}{\rho_i}, \quad x \in \partial\Omega_{ih} \cap \Gamma_h$$



δ_i^+ : pseudoinverse

$$\delta_i^+(x) = \left(\delta_i(x)\right)^{-1} \quad x \in \partial\Omega_{ih} \cap \Gamma_h$$

They provide a partition of unity

$$\sum_i R_i^T \delta_i^+(x) = 1 \qquad x \in \Gamma_h$$

$D^{(i)}$: Diagonal matrix with elements $\delta_i^+(x)$, $x \in \partial\Omega_{ih}$.

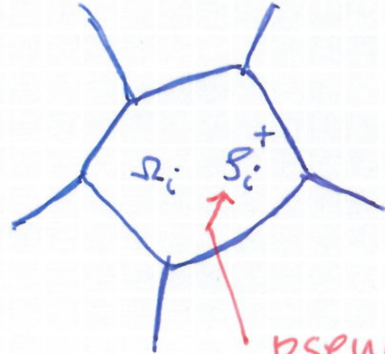
$$\boxed{S_{NN}^{-1} = \sum_{i=1}^{N} R_i^T D_i S^{(i)+} D_i R_i}$$

Convergence independent of jump in ρ_i ←

A minimal coarse space for Neumann-Neumann

$$W_0 = \text{span}\left\{ R_i^T \delta_i^+ \,,\; i = 1, \dots, N \right\}$$

basis function



pseudoinverse of the weighted counting func

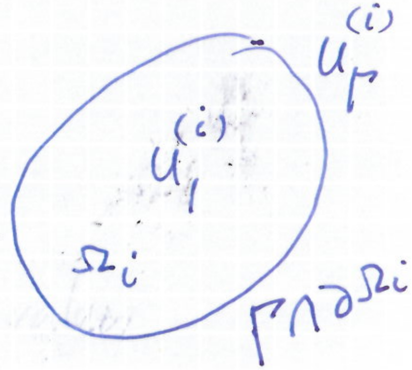
inside Ω_i the basis function is discrete harmonic $\mathcal{H}\left(R_i^T \delta_i^+\right)$.

Discrete Harmonic Extension : $\mathcal{H}$

directly related to Schur-Complement methods

$u^{(i)} = \begin{pmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{pmatrix}$ is Discrete harmonic

if $\quad A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} = 0$



We write $\quad \boxed{u^{(i)} = \mathcal{H}(u_\Gamma^{(i)})}$ completely defined by values on $\Gamma \cap \partial\Omega_i$

Equivalently

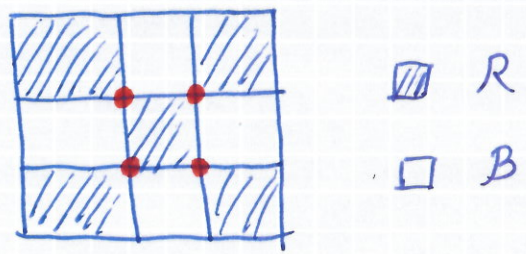
$$a_i(\mathcal{H} u_\Gamma^{(i)}, v) = 0 \quad \forall v \in V^h \cap H_0^1(\Omega_i, \Gamma \cap \partial\Omega_i)$$

Lemma Discrete harmonic ext. $u^{(i)} = \mathcal{H}(u_\Gamma^{(i)})$ satisfies

$$a_i(u^{(i)}, u^{(i)}) = \langle S^{(i)} u_\Gamma^{(i)}, u_\Gamma^{(i)} \rangle = \min_{v|_{\Gamma \cap \partial\Omega_i} = u_\Gamma^{(i)}} a_i(v, v)$$

D-N

$$S = \sum_{i \in R} R_i^T S^{(i)} R_i + \sum_{i \in B} R_i^T S^{(i)} R_i$$



$$S_{DN}^{-1} = \left(\sum_{i \in R} R_i^T S^{(i)} R_i \right)^{-1}$$

- In general $\chi(S_{DN}^{-1} S)$ has the same logarithmic bound as N-N.

- Not scalable if no crosspts.
- scalable with crosspts.

S_{DN}^{-1} requires the solution of a global problem

Neumann problem on the union of Subdomains glued at crosspts.