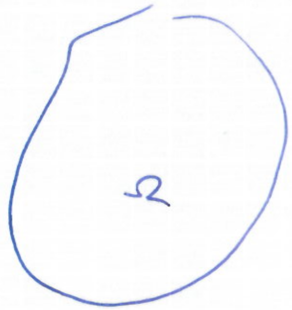
Substructuring methods.

Variational problem:

Find $u \in H_0^1(\Omega)$: $\quad a(u,v) = f(v) \quad \forall v \in H_0^1(\Omega)$

$$a(u,v) = \int_\Omega \rho(x)\, \nabla u \cdot \nabla v \qquad f(v) = \int_\Omega f v$$



Discrete problem:

Find $u \in V^h$: $\quad a(u,v) = f(v) \quad \forall v \in V^h$

$\overline{\Omega} = \bigcup_i \overline{\Omega}_i$

$\Omega_i \cap \Omega_j = \emptyset$

$i \neq j$

Linear system:

$$Au = f$$

or

$$\begin{pmatrix} A_{II} & A_{I\Gamma} \\ 0 & S \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} f_I \\ g_\Gamma \end{pmatrix}$$

Schur Complement System:

$$S u_\Gamma = g_\Gamma$$

$$S = \sum R_i^T S^{(i)} R_i \qquad \text{where} \qquad S^{(i)} = A^{(i)}_{\Gamma\Gamma} - A^{(i)}_{\Gamma I} A^{(i)^{-1}}_{II} A^{(i)}_{I\Gamma}$$

Dirichlet problem

- A is sparse but S is not.
- S is never formed.
  Action of S is calculated using $A^{(i)^{-1}}_{II}$, $A^{(i)}_{\Gamma I}$, $A^{(i)}_{\Gamma\Gamma}$
- Iterative Substructuring preconditioner are based on inverses of certain Schur complements.
  These inverses are never calculated, e.g.

$$\boxed{u^{(i)}_\Gamma = S^{(i)^{-1}} \omega^{(i)}_\Gamma}$$

can be found by solving

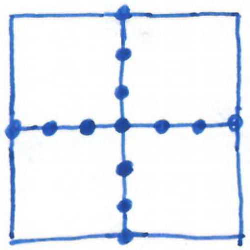
$$\boxed{A^{(i)} \begin{pmatrix} * \\ u^{(i)}_\Gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \omega^{(i)}_\Gamma \end{pmatrix}}$$

- $\kappa(S) \sim H^{-1} h^{-1}$

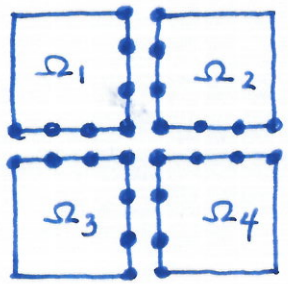
One-level FETI

Farhat - Roux (1991), Tezaur (1998)

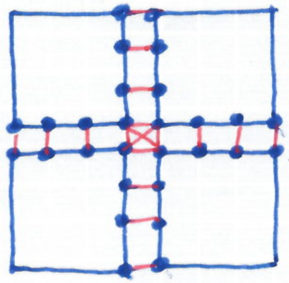
Finite Element Tearing & Interconnecting



$\hat{W}$



$W = \prod_i W_i$



W

constraints

$\boxed{u_i(x) = u_j(x)}$

at nodes of Γ_{ij}

$\hat{W} \subset W$ s.t.

$u \in \hat{W}$ is continous across Γ

$u \in W$ can be discontinuous across Γ.

$\bar{\Omega} = \bigcup_i \bar{\Omega}_i$ nonoverlapp

$\Gamma = \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$

$W^h(\Omega_i)$: continuous piecewise lin. function on Ω_i

$W_i = W^h(\Omega_i)\big|_{\partial\Omega_i \cap \Gamma}$

<u>Idea</u>: Enforce continuity constraints using Lagrange multiplier.

Reformulate the finite element problem as constraint minimization problem on Γ,

$$W = \prod_{i=1}^{N} W_i$$

$$\boxed{\begin{aligned} &\text{Find } u \in W: \\ &\qquad \left.\begin{aligned} J(u) := \tfrac{1}{2}\langle S u, u\rangle - \langle f, u\rangle \to \min \\ B u = 0 \end{aligned}\right\} \end{aligned}}$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} S^{(1)} & & \\ & \ddots & \\ & & S^{(N)} \end{pmatrix}$$

The matrix $B = [B^{(1)} \dots B^{(N)}]$,

where $B^{(i)}$ contains elements $\{-1, 0, 1\}$, $B^{(i)}: W_i \to U$

such that $Bu = \sum B_i u_i = 0$

Space of Lagrange multiplier

$\lambda \in U$, vector of Lagrange multiplier (continuity)

Equivalent Saddle point formulation:

$$\boxed{\begin{array}{l}\text{Find } (u,\lambda) \in W \times U \\ \qquad\qquad Su + B^T\lambda = f \\ \qquad\qquad\qquad Bu = 0\end{array}}$$

or
$$\begin{pmatrix} S^{(1)} & & & B^{(1)T} \\ & \ddots & & \vdots \\ & & S^{(N)} & B^{(N)T} \\ B^{(1)} & \cdots & B^{(N)} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \\ \lambda \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \\ 0 \end{pmatrix}$$

- $S^{(i)}$ pos. semi-definite, and singular for floating subdomain.
- The problem is solvable since $null(S) \cap null(B) = \{0\}$.
- λ is unique upto an additive element of $null(B^T)$

Choose $\lambda \in U$ where $U := range(B)$.

Introduce $\boxed{R_i : \mathbb{R} \to W_i}$ s.t. $\boxed{\text{range}(R_i) = \text{null}(S^{(i)})}$

$$R = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_N \end{pmatrix} \qquad R_i = 0 \text{ if } \Omega_i \text{ is nonfloating}$$

Solution exists iff, $\left(f_i - B^{(i)T}\lambda\right) \in \text{range}(S^{(i)})$

Because of which, $u_i = S^{(i)+}\left(f_i - B^{(i)T}\lambda\right) + \underbrace{R_i \alpha_i}_{\text{null}(S^{(i)})}$

can use, e.g. Moore-Penrose

$\left(f_i - B^{(i)T}\lambda\right) \in \text{range}(S^{(i)})$ is equivalent to

$$R_i^T\left(f_i - B^{(i)T}\lambda\right) = 0$$

Compatibility condition

Using compatibility and $Bu = 0$, we get a new saddle pt. system:

$$\boxed{\begin{aligned} F\lambda + G\alpha &= d \\ G^T\lambda &= e \end{aligned}}$$

← Solved using projection onto $\mathrm{null}(G^T)$

where

$$F = BS^+B^T, \quad G = BR, \quad d = BS^+f, \quad e = R^Tf$$

Define projection, $P: U \to \mathrm{null}(G^T)$

$$P = I - G(G^TG)^{-1}G$$

$P^T = P$
$P^2 = P$

Look for $\lambda \in \lambda_0 + \mathrm{null}(G^T)$ s.t. $G^T\lambda_0 = e$ and

$$P^TF\lambda = P^Td.$$

Once λ is found α can be calculated as

$$\alpha = (G^T G)^{-1} G^T (d - F\lambda)$$

Dirichlet preconditioner M^{-1} Farhat-Mandel-Roux (1994)

$$M^{-1} = BSB^T = \sum B_i S^{(i)} B_i^T$$

Preconditioned FETI :

Coarse problem

$$PM^{-1}P^T F \lambda = PM^{-1}P^T d$$

Dirichlet Neumann

Using scaling matrices $D^{(i)}$

$$M^{-1} = \sum_i D^{(i)} B_i S^{(i)} B_i^T D^{(i)T}$$

$$\boxed{\varkappa(PM^{-1}P^T F) \leq c\left(1 + \log\left(H/h\right)\right)^2}$$