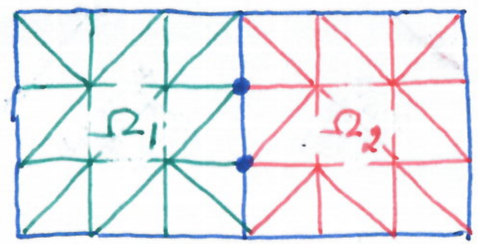
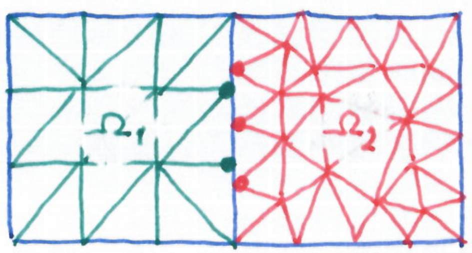
Mortar FEM

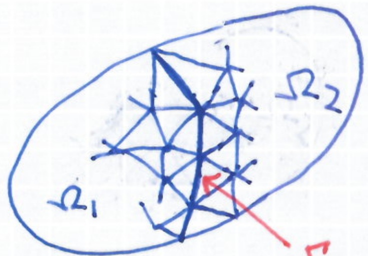




Variational problem:

Find $u \in H_0^1$: $\quad a(u,v) = f(v) \quad \forall v \in H_0^1$

Discrete formulation (standard)



matching mesh

$\mathcal{T}_h(\Omega)$: Triangulation

$W^h(\Omega)$: continuous piecewise linear function space.

Find $u \in W^h$: $\quad a(u,v) = f(v) \quad \forall v \in W^h$

$u \in W^h$ is continuous, even across the interface as long as the triangulation is matching along Γ.

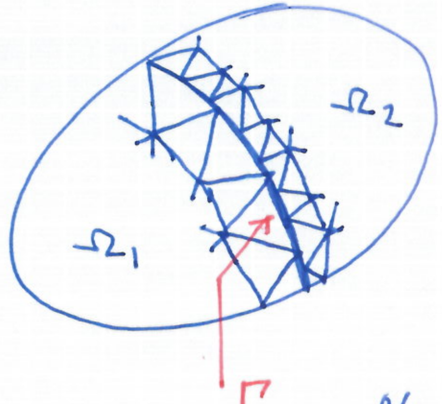
System: $\quad A_W \, u_W = f_W$

For each Ω_i (Locally)

$\mathcal{T}_h(\Omega_i)$: Triangulation

$X^h(\Omega_i)$: continuous piecewise linear function space

h_i : mesh size



Non matching mesh

Product space $X^h = \prod_i X^h(\Omega_i)$, $u \in X^h$ is discontinuous across Γ.

Mortar discretization :

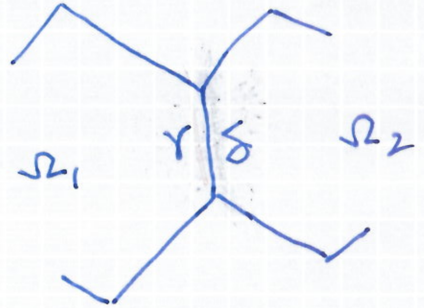
Replace pointwise continuity with a weak continuity

So that we still have the same error estimate as in the matching mesh case.

Bernardi - Maday - Patera (1994)

For each Γ,

- Identify master and slave side, γ and δ



- Define Lagrange multiplier space $L^h(\delta)$ associated with $X^h(\delta)$.

- We say $u \in X^h$ salisfy mortar condition if

$$\boxed{\int_\Gamma u_\gamma \xi \, ds = \int_\Gamma u_\delta \xi \, ds \qquad \xi \in L^h}$$

γ: master side

δ: slave side

$\Gamma = \gamma = \delta$

$$V^h = \{ u \in X^h : \ u \text{ satisfies mortar condition} \}$$

Mortar discrete problem:

Find $u_h \in V^h$: $\quad a(u_h, v) = f(v) \quad \forall v \in V^h$.

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$V^h \not\subset H_0^1$ non conforming

V^h : Hilbert space with inner prod. $a(\cdot,\cdot)$

Error estimate :

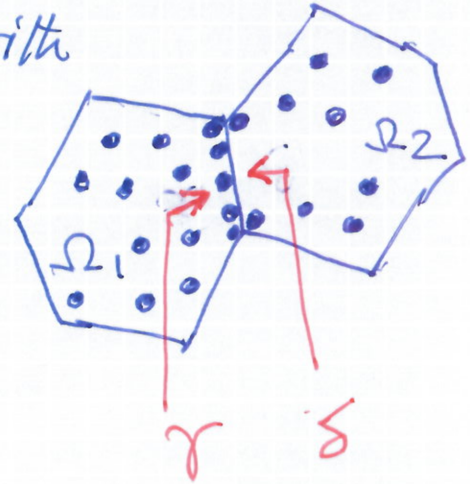
$$|u - u_h|^2_{H_h^1(\Omega)} \leq c \sum_{i=1}^{N} h_i^2 \, |u|^2_{H^2(\Omega_i)}$$

$|\cdot|_{H_h^1}$: broken seminorm

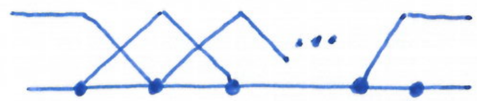
Mortar discrete problem:

- Unique solution, and same error estimate as in the conforming discretization.

- Each basis function of V^h is associated with
  - a node interior to a subdomain
  - a node on the master side
  - a subdomain vertex (multiple values)



- Lagrange multiplier space L^h

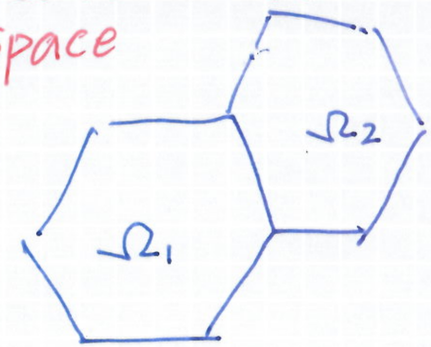


(standard)

$$L^h(\delta) = \{ \xi \in X^h(\delta) : \xi \text{ is constant on the element touching } \partial\delta \}$$

On X^h

$$X^h = \prod_i X^h(\Omega_i)$$ unconstrained space



Stiffness matrix:

$$A = \begin{pmatrix} A^{(1)} & & & \\ & A^{(2)} & & \\ & & \ddots & \\ & & & A^{(N)} \end{pmatrix}$$

rhs:

$$f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(N)} \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(N)} \end{pmatrix}.$$

How do we get from A to A_v on V^h ← constrained space

Matrix formulation

Let $u_\Gamma = \begin{pmatrix} u_\gamma \\ u_\delta \\ u_c \end{pmatrix}$ master, slave, corner

Mortar condition :

$$M u_\gamma = S u_\delta + C u_c$$

whereby, $u_\gamma = S^{-1}(M u_\gamma - C u_c)$

slave values determined by mortar + corner values .

Define $Q = \{ Q_\Gamma, I \}$:

$$u_\delta = Q_\Gamma \begin{pmatrix} u_\gamma \\ u_c \end{pmatrix} \quad \text{for each } \Gamma;$$

I is identity matrix.

Then $\qquad Q^T A Q u_v = Q^T f$

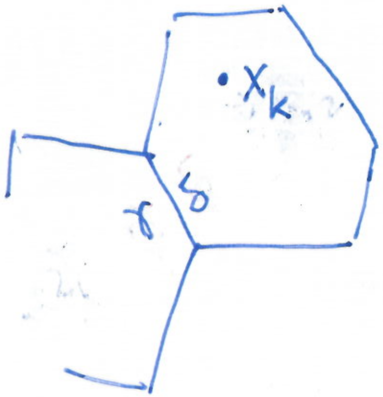
or $\qquad A_v u_v = f_v$

CR – mortar FE (Approximate version)

Assumption on I_γ :

① $I_\gamma \phi_k|_\gamma = 0$ for any basis function ϕ_k inside a subdomain.

② $I_\gamma p = p$ for any linear function p



③ For any $\tau \in \mathcal{T}_h(\gamma)$ and $\text{dist}(\tau, c_k) \sim O(h)$

$$\| I_\gamma u \|_{L^\infty} \lesssim \max_{c_k} |u(c_k)|$$

④ $$\boxed{\int_\tau I_\gamma u = \int_\tau u , \quad \text{for } \tau \in \mathcal{T}_h(\gamma)}$$