Solution Methods for Nonlinear Finite Element Analysis (NFEA)

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Outline

- Linear versus nonlinear reponse
- Fundamental and secondary path
- Critical points
- Why Nonlinear Finite Element Analysis (NFEA) ?
- Sources of nonlinearities
- Solving nonlinear algebraic equations by Newton's method
- Line search procedures and convergence criteria
- Arc-length methods
- Implicit dynamics

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Linear vs Nonlinear Respons

- Numerical simulation of the response where both the LHS and RHS depends upon the primary unknown.

- Linear versus Nonlinear FEA:
  - LFEA:

    $$[\mathbf{K}]\{\mathbf{D}\}=\{\mathbf{R}\}$$

  - NFEA:

    $$[\mathbf{K}(\mathbf{D})]\{\mathbf{D}\}=\{\mathbf{R}(\mathbf{D})\}$$

- Field of Nonlinear FEA:
  - Continuum mechanics
  - FE discretization (FEM)
  - Numerical solution algorithms
  - Software considerations (engineering)

- Requirements for an effective NFEA:

Physical and theoretical understanding is most important

Physical insight ⟷ Mathematical formulation

Interaction and mutual enrichment

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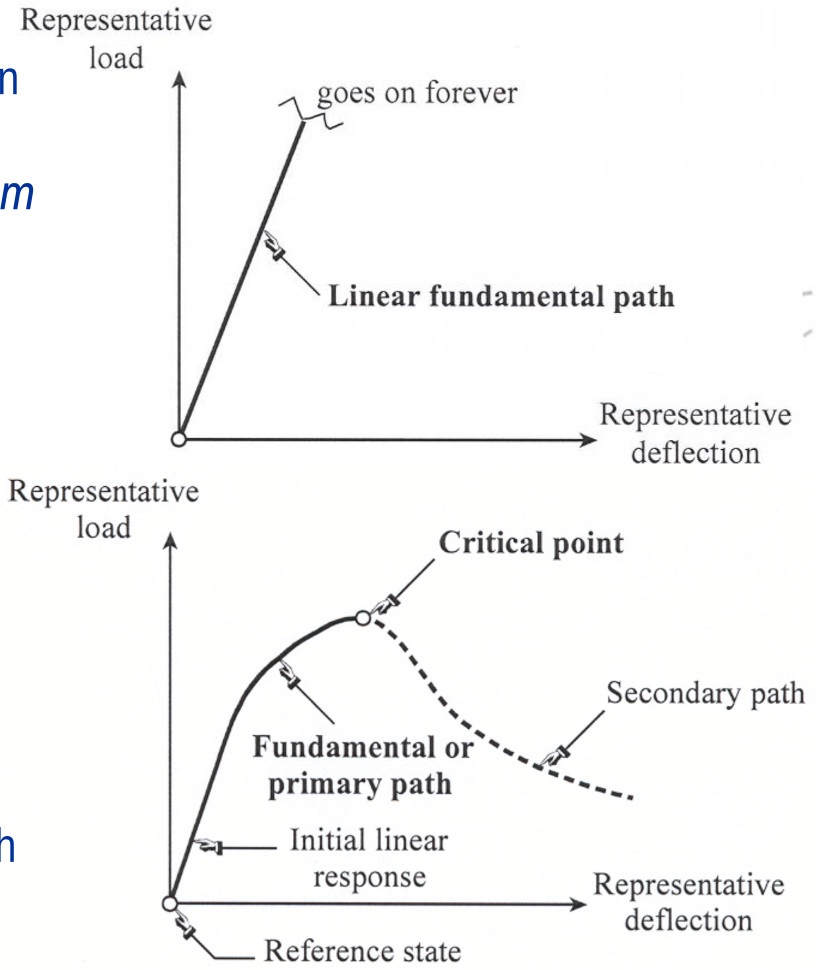
Equilibrium path

- The *equilibrium path* is a graphical representation of the *response* (load-deflection) *diagram* that *characterize* the overall *behaviour of the problem*
- *Each point* on the equilibrium path *represent* a *equilibrium point* or *equilibrium configuration*
- The *unstressed* and *undeformed configuration* from which loads and deflection are measured *is called* the *reference state*
- The equilibrium path that crosses the reference state is called the *fundamental* (or primary) *path*
- Any equilibrium path that is not a fundamental path but connects with it at a critical point is called a *secondary path*



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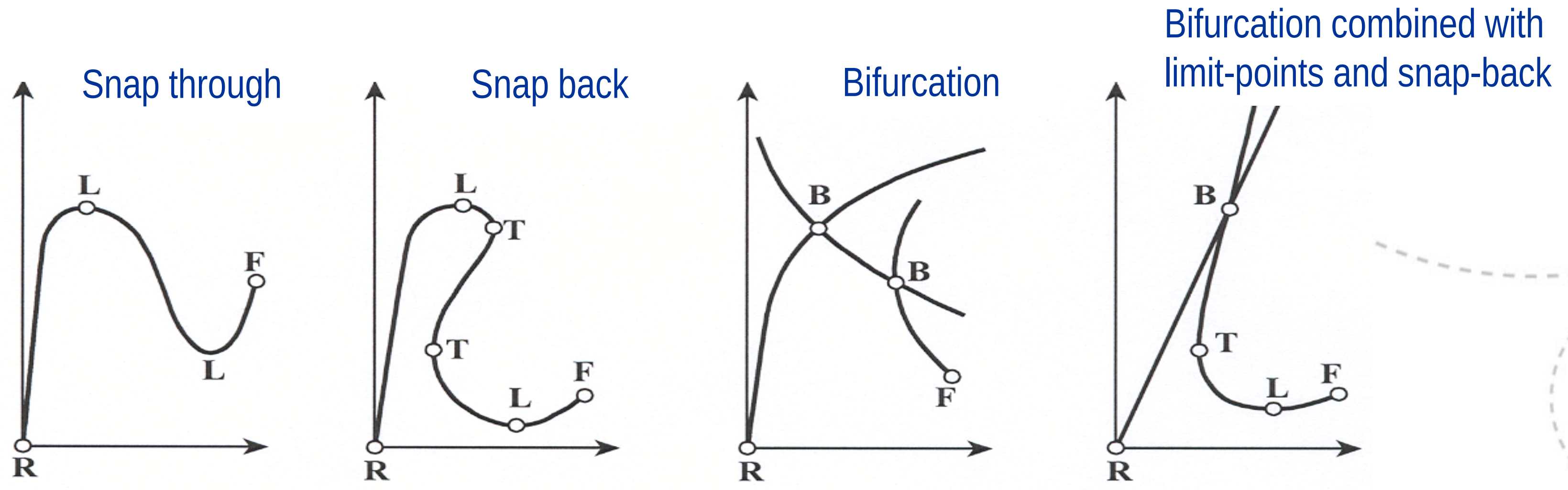
Critical points



- **Limit points (L)**, are points on the equilibrium path at which the tangent is horizontal
- **Bifurcation points (B)**, are points where two or more equilibrium paths cross
- **Turning points (T)**, are points where the tangent is vertical
- **Failure points (F)**, are points where the path suddenly stops because of physical failure

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Advantages of linear response

- A linear structure *can sustain any load whatsoever* and undergo any displacement magnitude
- There are *no critical* (limit, bifurcation, turning or failure) *points*
- *Solutions* for various load cases *may be superimposed*
- *Removing* all *loads returns* the *structure to* the *reference state*
- Simple *direct solution* of the structural stiffness relationship *without need for costly load incrementation and iterative schemes*

Reasons for Nonlinear FEA

- *Strength analysis* – how much load can the structure support before *global failure* occurs
- *Stability analysis* – finding *critical points* (limit points and bifurcation points) closest to operational range
- *Service configuration analysis* – finding the 'operational' equilibrium configuration of certain slender structures when the fabrication and service configurations are quite different (e.g. cable and inflatable structures)
- *Reserve strength analysis* – finding the load carrying capacity beyond critical points to assess safety under abnormal conditions
- *Progressive failure analysis* – a combined strength and stability analysis in which progressive detoriation (e.g. cracking) is considered

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Reasons for NFEA (2)

- Establish the *causes of* a *structural failure*
- *Safety and serviceability assessment* of existing infrastructure whose integrity may be in doubt due to:
  - *Visible damage* (cracking, etc)
  - *Special loadings* not envisaged at the design state
  - *Health–monitoring*
  - Concern over *corrosion or general aging*
- A shift towards *high performance materials* and more *efficient utilization* of structural components
- *Direct use* of NFEA *in design* for both ultimate load and serviceability limit states

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Reasons for NFEA (3)

- *Simulation* of materials processing and manufacturing (e.g. metal forming, extrusion and casting processes)
- In *research*:
  - To *establish simple* ‘code-based’ *methods* of analysis and design
  - To *understand basic structural behaviour*
  - To *test* the *validity of proposed ‘material models’*
- Computer *hardware becomes cheaper and faster and FE software becomes* more *robust and user-friendly*
- It will simply *become easier* for an engineer *to apply direct analysis rather than code-based checking*

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Consequences of NFEA

- For the analyst familiar with the use of LFEA, there are a number of *consequences of nonlinear behaviour* that have *to be recognized before embarking on a NFEA*:
  - The principle of *superposition cannot be applied*
  - *Results of several 'load cases' cannot be scaled, factored and combined* as is done with LFEA
  - Only *one load case* can be handled at a time
  - The *loading history* (i.e. sequence of application of loads) *may be important*
  - The *structural response can be* markedly *non-proportional to the applied loading*, even for simple loading states
  - Careful thought needs to be given to what is an appropriate measure of the behaviour
  - The *initial state of stress* (e.g. residual stresses from welding, temperature, or prestressing of reinforcement and cables) may be *extremely important for the overall response*

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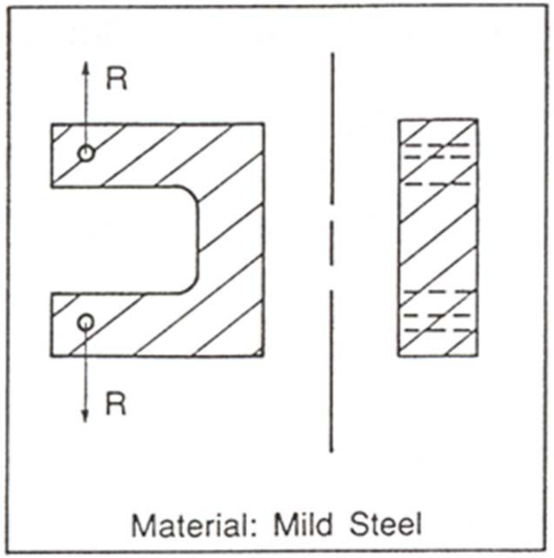
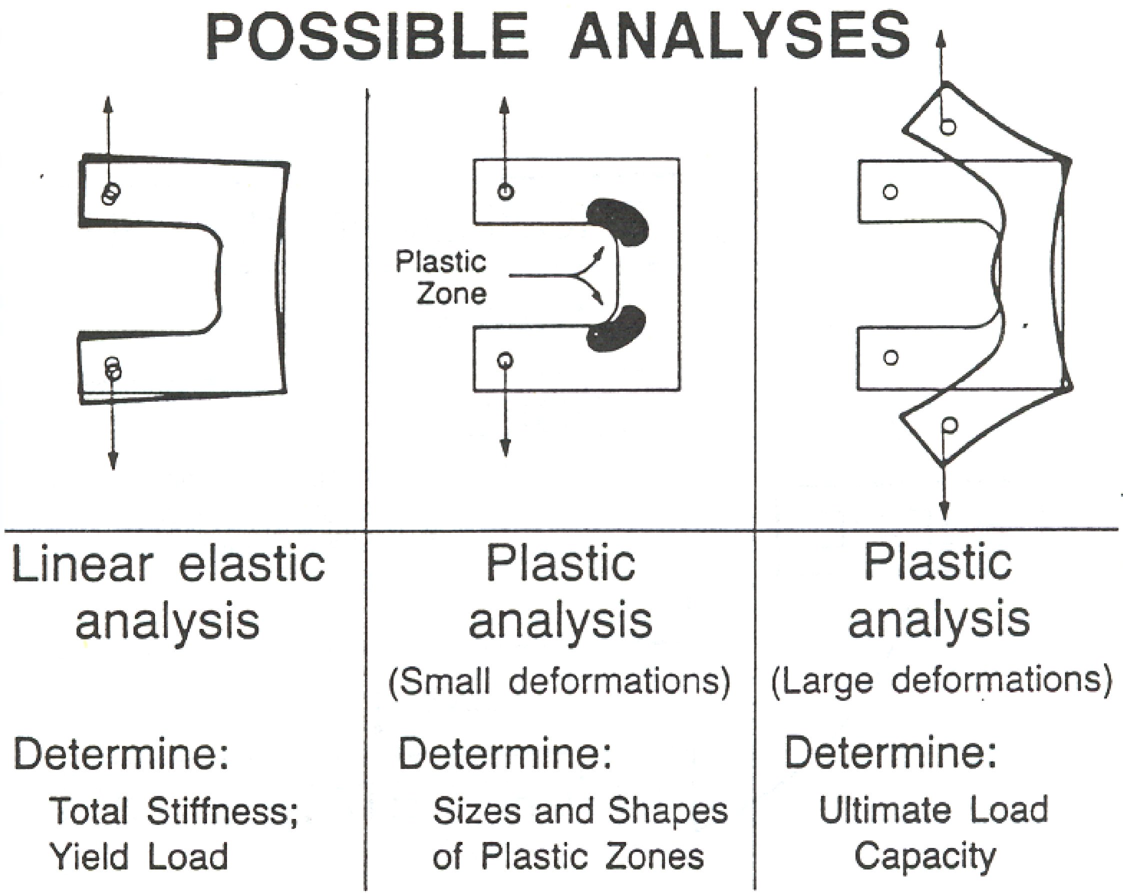
A typical Nonlinear Problem





Possible questions:

- Yield load
- Limit load
- Plastic zones
- Residual stresses
- Permanent deflections

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Sources of Nonlinearities

- Geometric Nonlinearity:
  - *Physical source:*

    Change in geometry as the structure deforms is **taken into account** in setting up the strain displacement (kinematic) and equilibrium equations.
  - *Applications:*
    - **Slender structures**
    - **Tensile structures** (cable structures and inflatable membranes)
    - **Metal and plastic forming**
    - **Stability of all types of structures**
  - *Mathematical source:*

    The **strain-displacement operator** $\bar{\boldsymbol{\partial}}$ **is nonlinear** when finite strains (as opposed to infinitesimal strains) are expressed in terms of displacements ***u***

    $$\boldsymbol{\varepsilon} = \bar{\partial}\boldsymbol{u}$$

    Considering geometric nonlinearities, the **operator applied to the stresses,** $\boldsymbol{\partial}^T$ for linear elasticity, is **not necessarily the transposed of the strain-displacement operator** $(\boldsymbol{\partial} \neq \bar{\boldsymbol{\partial}})$

    $$\partial^T \boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0}$$

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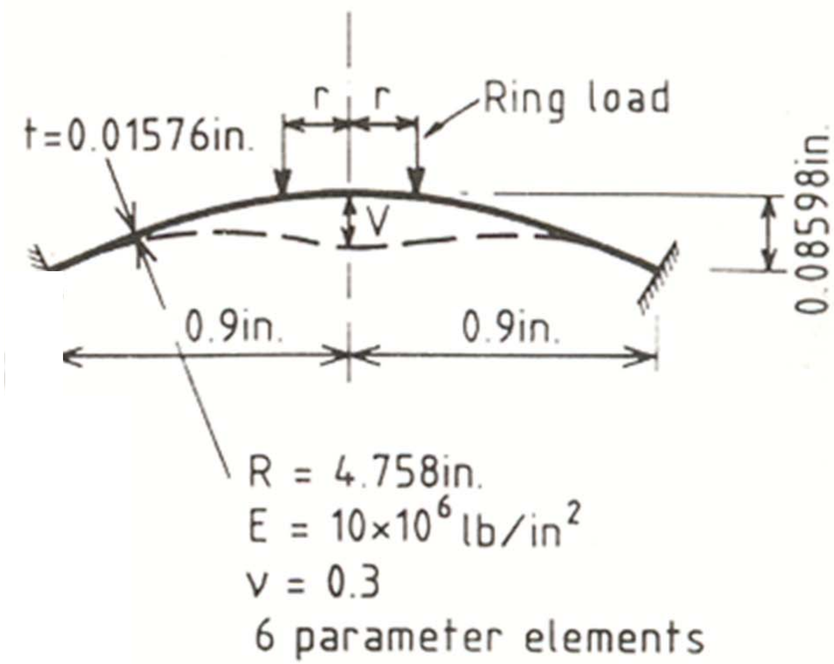
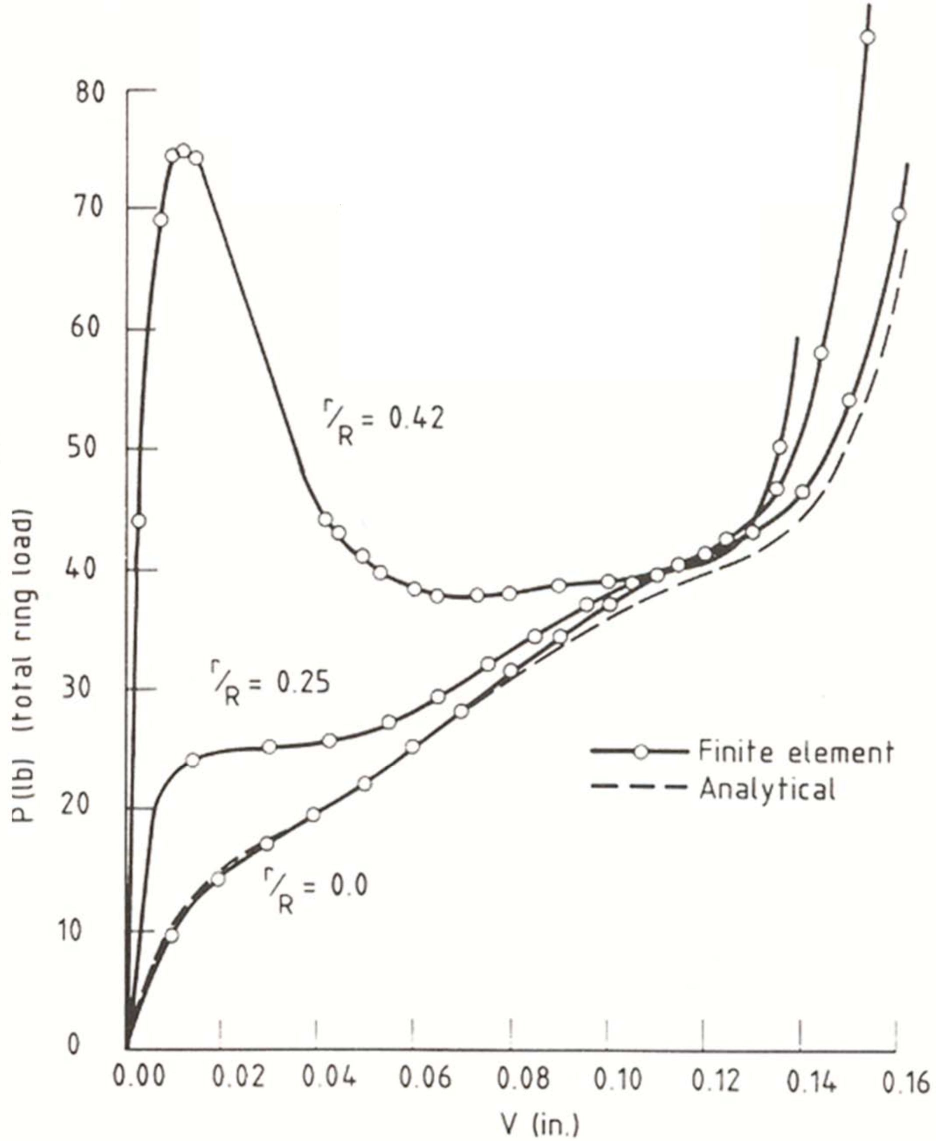
Example – Geometric Nonlin.

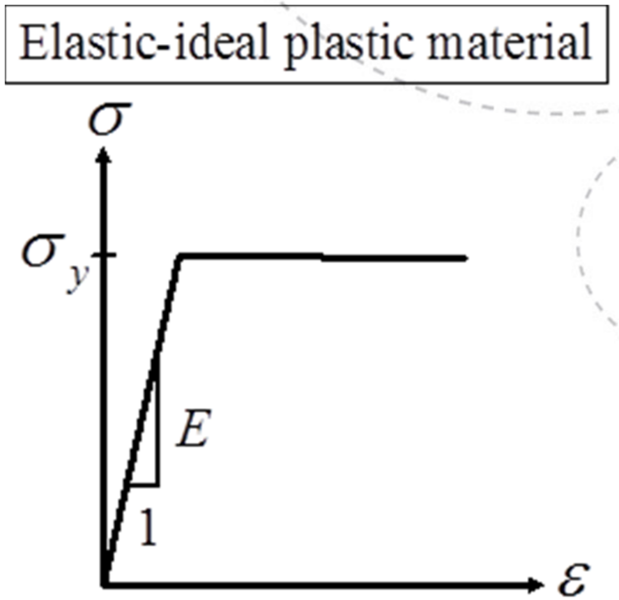
- Snap-through behavior of a shallow spherical cap with various ring loads





Sources of Nonlinearities (2)

- Material Nonlinearity:
  - *Physical source:*
    Material behavior depends on current deformation state and possibly past history of the deformation.
    The constitutive relation may depend on other variables (prestress, temperature, time, moisture, electromagnetic fields, etc)
  - *Applications:*
    - Nonlinear elasticity
    - Plasticity
    - Viscoelasticity
    - Creep, or inelastic rate effects
  - *Mathematical source:*
    The **constitutive relation** that relates strain and stresses, *C*, **is nonlinear** when the material no longer may be expressed in terms of e.g. Hooke's generalized law:



$$\sigma = C\left(\varepsilon - \varepsilon_0\right)$$

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Sources of Nonlinearities (3)

- Force Boundary Condition Nonlinearity:
  - *Physical source:*
    Applied **forces depend** on the **deformation**.
  - *Applications:*
    - **Hydrostatic loads** (submerged tubular bridges)
    - Aerodynamic or hydrodynamic loads
    - Non-conservative follower forces
  - *Mathematical source:*
    The **applied forces**, prescribed surface tractions $\bar{\boldsymbol{t}}$
    and/or body forces *b*, **depend on** the unknown **displacements** *u*:

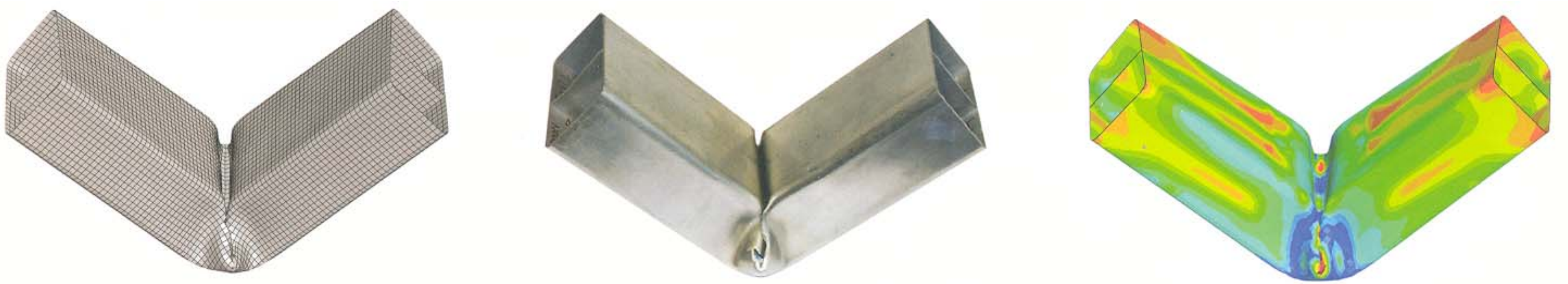
$$\bar{\boldsymbol{t}} = \bar{\boldsymbol{t}}(\boldsymbol{u})$$

$$\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{u})$$

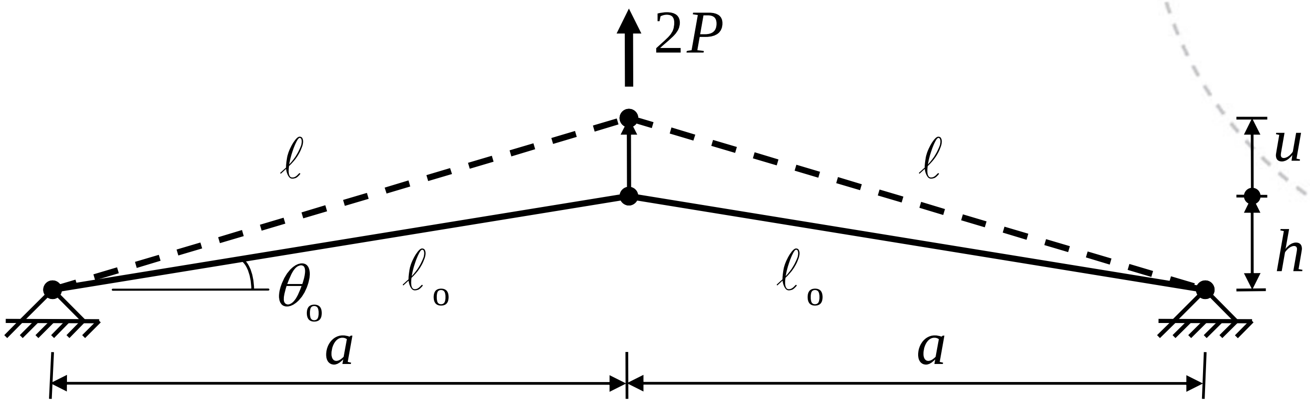
Sources of Nonlinearities (4)

- Displacement Boundary Condition Nonlinearity:
  - *Physical source:*
    Displacement boundary conditions depend on the deformation.
  - *Applications:*
    The most important application is the **contact problem**, in which no interpenetration conditions are enforced on flexible bodies while the extent of contact area is unknown.
  - *Mathematical source:*
    The prescribed displacements $\bar{\boldsymbol{u}}$ **depend on** unknown displacements, *u*:

$$\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}(\boldsymbol{u})$$



Example – Geometric Nonlin.

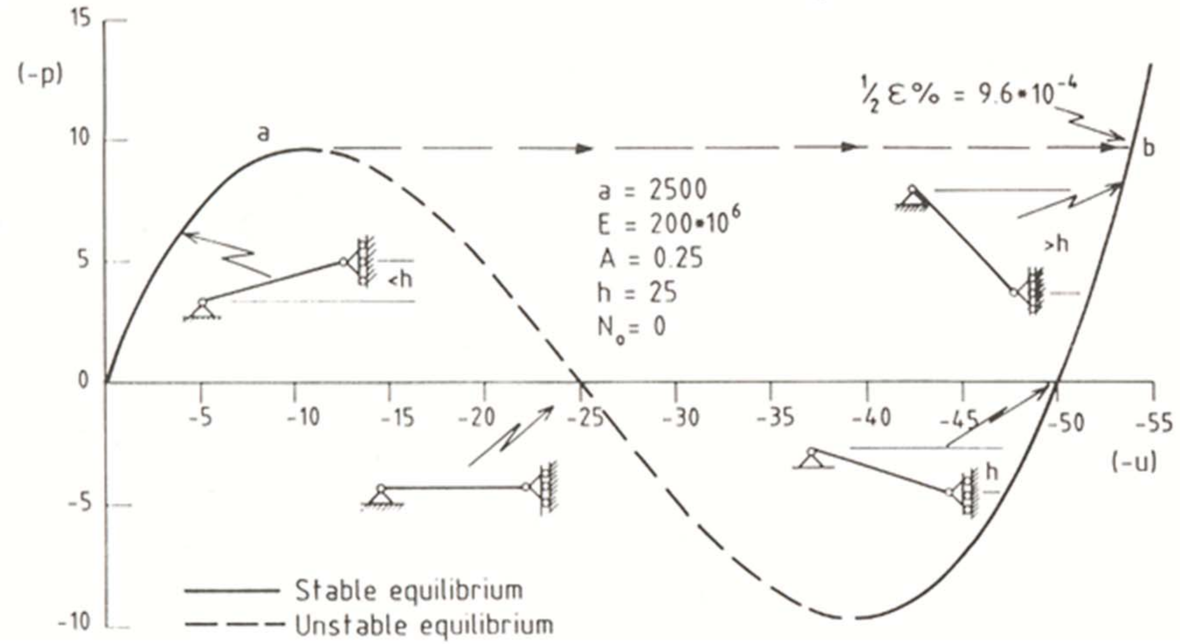


- A two-element truss model with constant axial stiffness *EA* and initial axial force N_o is considered to illustrate some basic features of geometric nonlinear behavior.
- From the three fundamental laws:
  - *Compatibility*
  - *Material law*
  - *Equilibrium*

$$\Rightarrow \boxed{P = \frac{EA}{2\ell_o^3}\left[u^3 + 3hu^2 + 2h^2u\right] + N_o\left(\frac{u+h}{\ell_o}\right)}$$

(nonlinear load - displacement relationship)

Example – Geometric Nonlin.



- Equilibrium path representing the solution of the nonlinear load-displacement relationship
  - As the load increases (downward) an **initial maximum load**, called the **limit load**, is **reached at** the **limit point** (a)
  - Further increase of the load would lead to **snap-through to** the new **equilibrium state** at (b). The snap-through is an **unstable dynamic process**

  $\Rightarrow$ the straight line from (a) to (b) does not represent the true equilibrium path.
  - In order to trace the true unstable equilibrium branch between (a) and (b), the displacement *u* has to be prescribed rather than prescribing the load *P*.

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Solving the Nonlinear Equations

- From conservation of linear momentum, we may establish the equations of motion.
- Substituting the FE approximations (and neglecting time dependent terms), the global equilibrium equations on discretized form is obtained:

$$\underbrace{\left\{\mathbf{R}^{\text{ext}}\right\}}_{\substack{\text{externally}\\\text{applied loads}}} = \underbrace{\left\{\mathbf{R}^{\text{int}}\right\}}_{\substack{\text{nodal forces}\\\text{from internal}\\\text{element stresses}}} \quad \Leftrightarrow \quad \left\{\mathbf{R}^{\text{res}}\right\} = \left\{\mathbf{R}^{\text{ext}}\right\} - \left\{\mathbf{R}^{\text{int}}\right\} = \{\mathbf{0}\}$$

where $\left\{\mathbf{R}^{\text{ext}}\right\}$ and $\left\{\mathbf{R}^{\text{int}}\right\}$ denote sum of externally applied loads and sum of internal element nodal forces, respectively :

$$\left\{\mathbf{R}^{\text{ext}}\right\} = \sum_{i=1}^{N_{els}} \left\{\mathbf{r}_e\right\}_i + \{\mathbf{P}\} = \sum_{i=1}^{N_{els}} \left(\int_{V_i} [\mathbf{N}]^T \{\boldsymbol{b}\}\, dV + \int_{S_i} [\mathbf{N}]^T \{\bar{\boldsymbol{t}}\}\, dS \right) + \{\mathbf{P}\}$$

$$\left\{\mathbf{R}^{\text{int}}\right\} = \sum_{i=1}^{N_{els}} \left\{\mathbf{r}^{\text{int}}\right\}_i = \sum_{i=1}^{N_{els}} \left(\int_{V_i} [\mathbf{B}]^T \{\boldsymbol{\sigma}\} dV \right)$$

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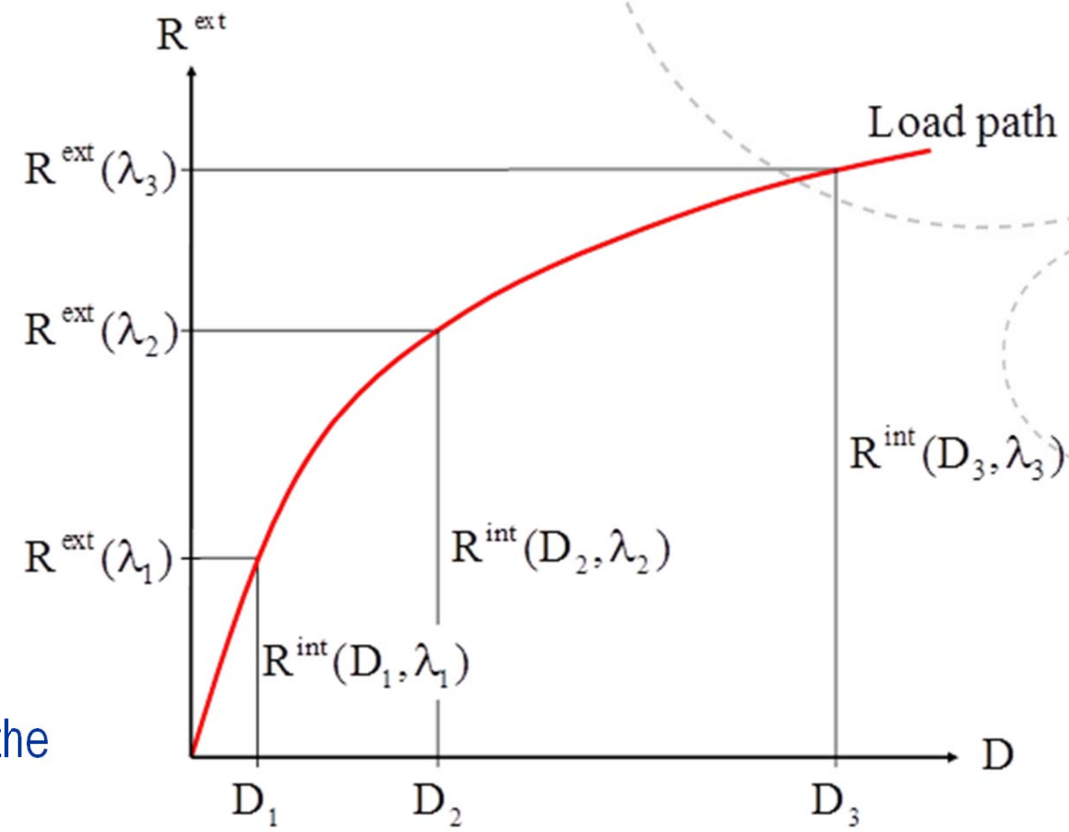
Solving the Nonlinear Equations

- In order to satisfy equilibrium, external $\{\mathbf{R}^{ext}\}$ and internal forces $\{\mathbf{R}^{int}\}$ have to be in balance

$$\Rightarrow \{\mathbf{R}^{res}\} = \{\mathbf{R}^{ext}\} - \{\mathbf{R}^{int}\} = \{\mathbf{0}\}.$$

- Consider the solution of nonlinear equilibrium equations for prescribed values of the load or time parameter λ.

- The problem consists of finding the displacement vector $\{\mathbf{D}\}$ which produces an internal force vector $\{\mathbf{R}^{int}(\mathbf{D},\lambda)\}$ balancing externally applied loads $\{\mathbf{R}^{ext}(\lambda)\}$.



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Load incrementation

- Our purpose is to **trace** the fundamental (primary) **equilibrium path** while **travers-ing critical points** (limit, turning and bifurcation points)

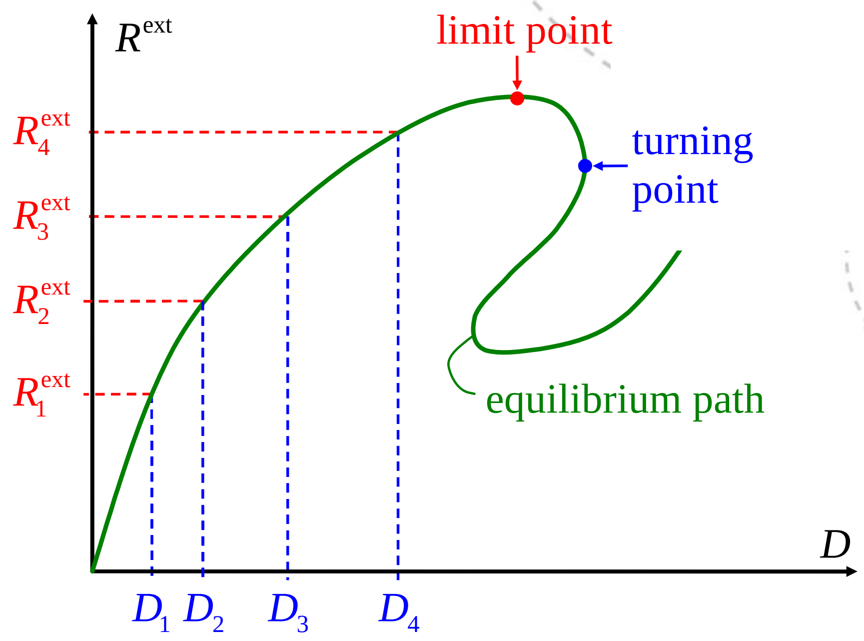
  $\Rightarrow$ we want to **calculate** a series of **solutions**:

$$\{\mathbf{D}_n\}, \lambda_n \quad \text{for} \quad n = 0, 1, 2, \cdots, n_{\text{step}}$$

that within prescribed accuracy satisfy the **equilibrium equations**:

$$\{\mathbf{R}^{\text{res}}\} = \{\mathbf{R}^{\text{ext}}\} - \{\mathbf{R}^{\text{int}}\} = \{\mathbf{0}\}$$

- A **major problem** in tracing a nonlinear solution path is how to **choose the size** of the **load increments** $\Delta\lambda_n$.



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Incremental-Iterative solution

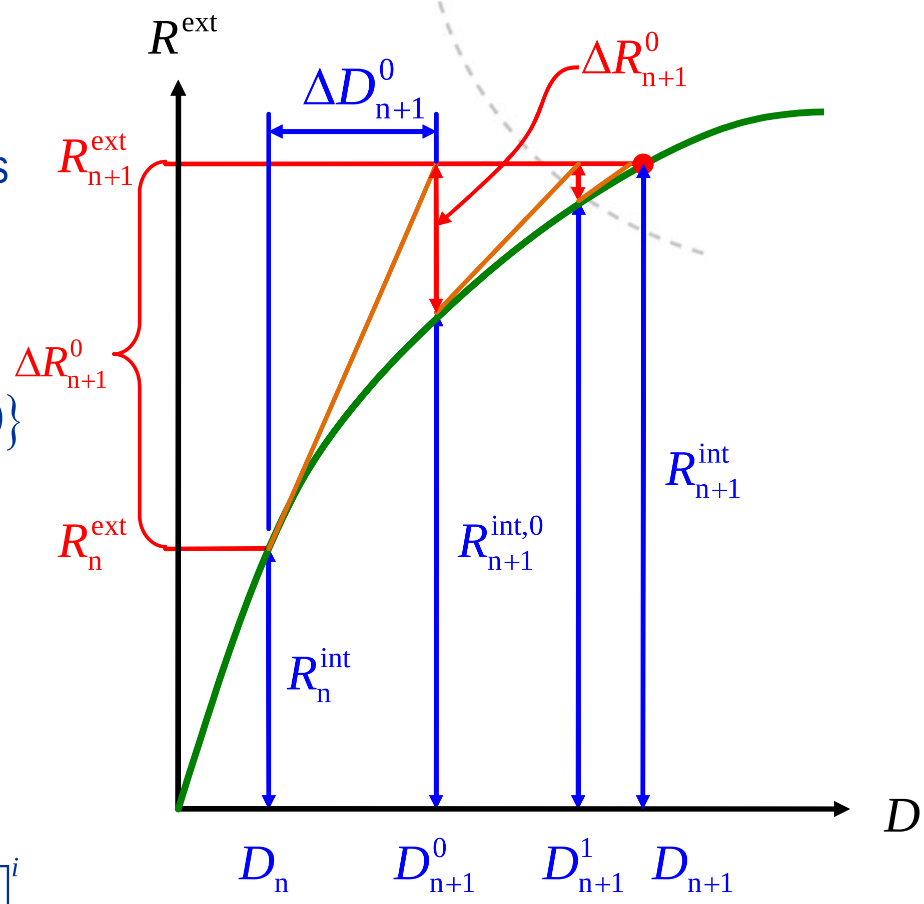
- By linearizing the residual of the global equilibrium equations the incremental form of the equations of motion expressed in terms of the incremental nodal displacements $\{\Delta \mathbf{D}\}$, is obtained as:

$$\{\mathbf{R}^{res}\}_{n+1}^{i+1} = \{\mathbf{R}^{res}\}_{n+1}^{i} + \left[\frac{\partial \mathbf{R}^{res}}{\partial \mathbf{D}}\right]_{n+1}^{i} \{\Delta \mathbf{D}\}_{n+1}^{i} = \{\mathbf{0}\}$$

$$\Rightarrow \qquad \boxed{[\mathbf{K}_T]_{n+1}^{i} \{\Delta \mathbf{D}\}_{n+1}^{i} = \{\mathbf{R}^{res}\}_{n+1}^{i}}$$

where

$$[\mathbf{K}_T]_{n+1}^{i} = -\left[\frac{\partial \mathbf{R}^{res}}{\partial \mathbf{D}}\right]_{n+1}^{i} = \left[\frac{\partial \mathbf{R}^{int}}{\partial \mathbf{D}}\right]_{n+1}^{i} - \left[\frac{\partial \mathbf{R}^{ext}}{\partial \mathbf{D}}\right]_{n+1}^{i}$$



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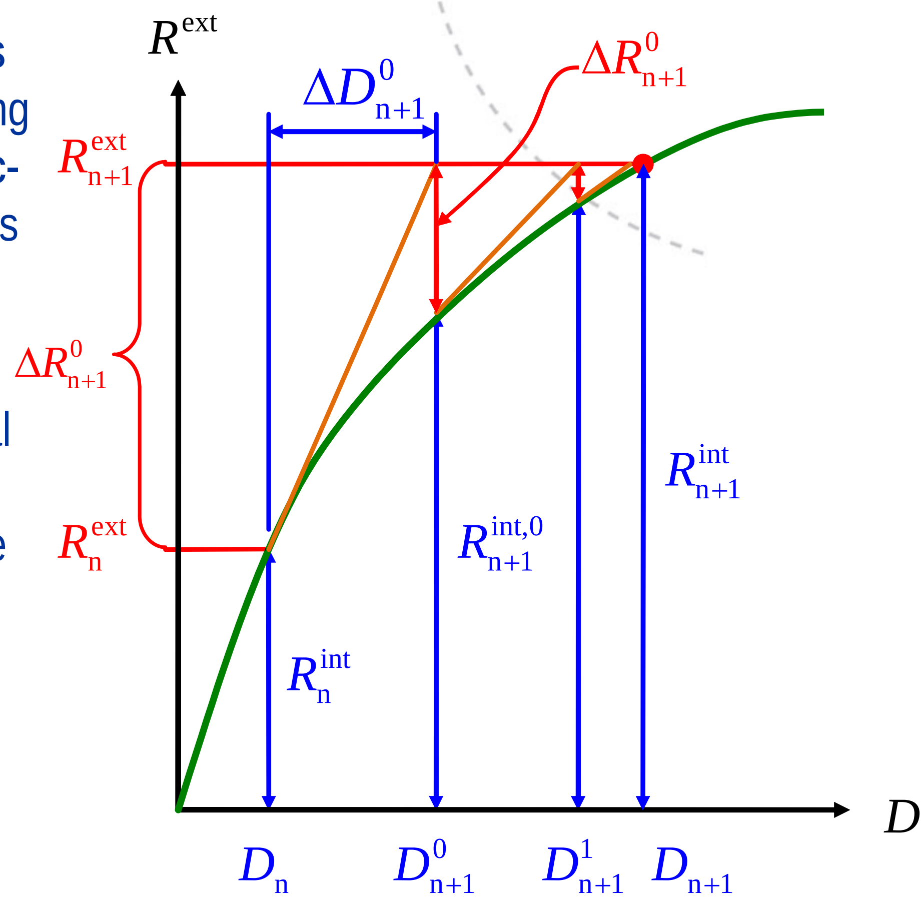
Incremental-Iterative solution

- The most frequently used **solution procedures for NFEA consists of a predictor step** involving forward Euler load incrementation **and a corrector step** in which some kind of Newton iterations are used to enforce equilibrium.
- The **incremental-iterative procedure** that advances the solution while satisfying the global equilibrium equations at each iteration *'i'*, within each time (load) step *'n*+1*'* , is **governed by** the **incremental equations:**

$$\left[\mathbf{K}_T\right]_{n+1}^{i}\left\{\Delta\mathbf{D}\right\}_{n+1}^{i}=\left\{\mathbf{R}^{\text{res}}\right\}_{n+1}^{i}$$

- A series of successive approximations gives:

$$\left\{\mathbf{D}\right\}_{n+1}^{i+1}=\left\{\mathbf{D}\right\}_{n+1}^{i}+\left\{\Delta\mathbf{D}\right\}_{n+1}^{i}$$



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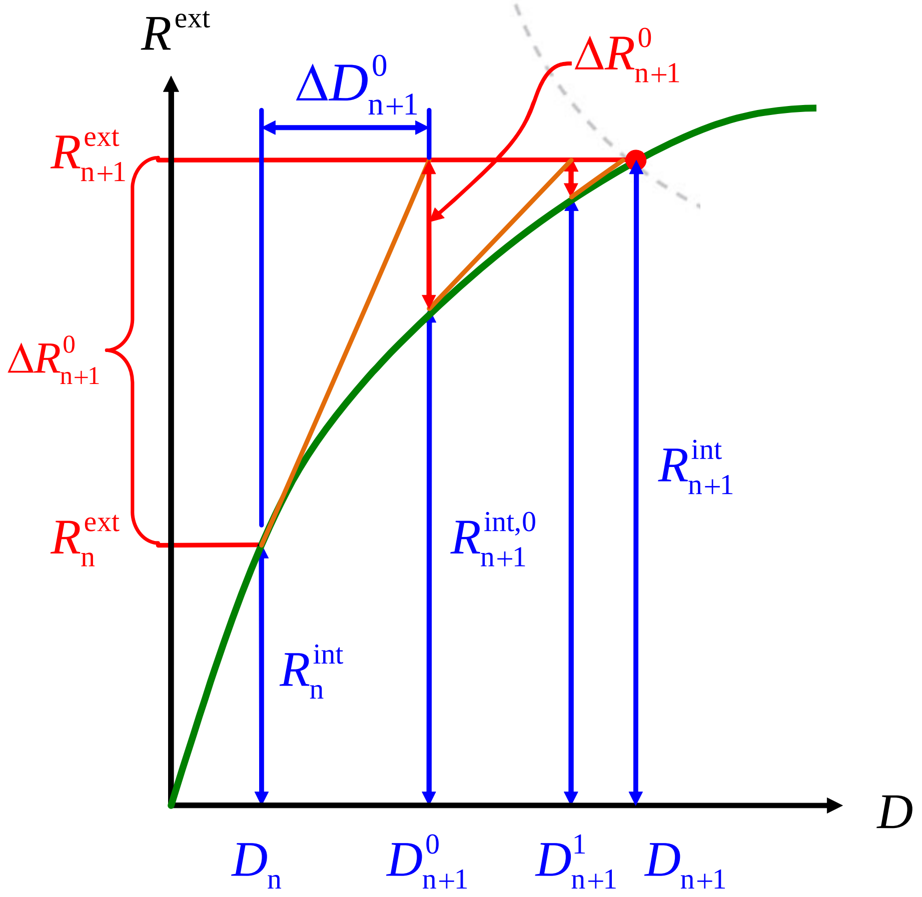
Newton's method

- Newton's method is the most rapidly convergent process for solution of problems in which only one evaluation of the residual is made in each iteration.
- Indeed, it is the only method, provided that the initial solution is within the "ball of convergence", in which the **asymptotic rate of convergence** is **quadratic.**
- Newton's method illustrated in the Figure shows the very rapid convergence that can be achieved.



Weaknesses of Newton's method

- The standard (true) Newton's method, although effective in most cases, is not necessarily the most economical solution method and does not always provide rapid and reliable convergence.

- Weaknesses of the method:
  - Computational expense:
    - Tangent stiffness has to be computed and assembled at each iteration within each load step
    - If a direct solver is employed K_T also needs to be factored at each iteration within each load step
  - Increment size:
    - If the time stepping algorithm used is not robust (self-adaptive), a certain degree of trial and error may be required to determine the appropriate load increments
  - Divergence:
    - If the equilibrium path include critical points negative load increments must be prescribed to go beyond limit points
    - If the load increments are too large such that the solution falls outside "the ball of convergence" analysis may fail to converge

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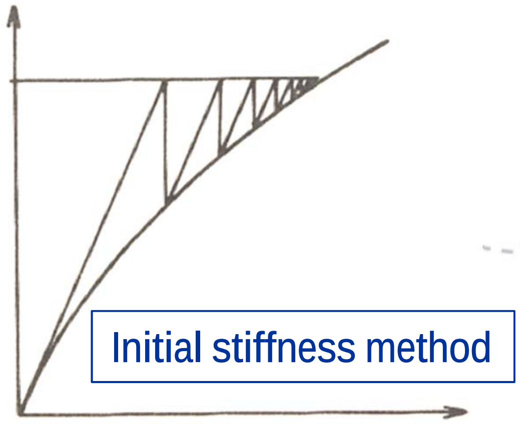
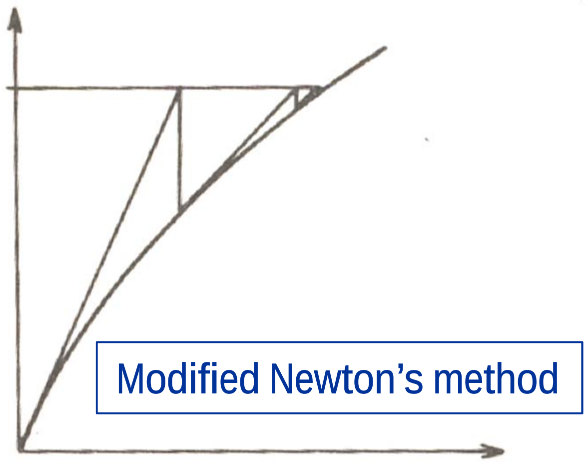
Modified Newton methods

- **Modified Newton methods** differ from the standard method in that the **tangent stiffness** K_T is only **updated occasionally.**
- **Initial stiffness method:**
  - Tangent stiffness K_T **updated only once**
  - The method may result in a **slow rate of convergence**
- **Modified Newton's method:**
  - Tangent stiffness K_T **updated occasionally** (but not for every iteration)
  - **More rapid convergence** than the initial stiffness method (but not quadratic)
- **Quasi (secant) Newton methods:**
  - The inverse of the tangent stiffness obtained by a secant approximation rather than recomputing and factorizing K_T at every iteration





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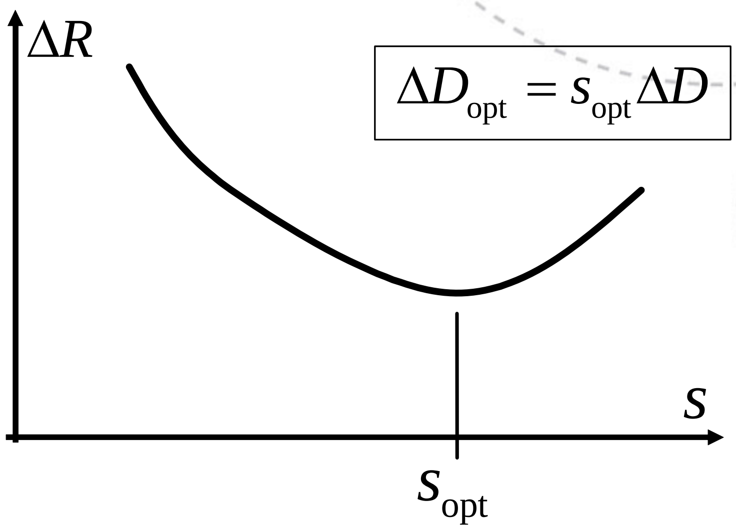
Line search procedures

- By line searches (LS) an *optimal incremental step* length is obtained by minimizing the residual $\{\mathbf{R}^{\text{res}}\}$ in the direction of $\{\Delta\mathbf{D}\}$.
- LS can be *particularly useful for problems involving rapid changes in tangent stiffness*, such as in reinforced concrete analysis when concrete cracks or steel yields.
- LS not only accelerate the iterative process, they *can provide convergence where none is obtainable without LS*, especially if the predictor increment lies outside the “ball of convergence”.
- LS is *highly recommended and may be used in all type of Newton methods*; standard, modified, and quasi Newton methods.



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Convergence criteria

- A convergence criteria measures how well the obtained solution satisfies equilibrium.
- In NFEA of the convergence criteria are usually based on some norm of the:
  - Displacements (total or incremental)
  - Residuals
  - Energy (product of residual and displacement)
- Although displacement based criteria seem to be the most natural choice they are not advisable in general as they can be misleadingly satisfied by a slow convergence rate.
- Residual based criteria are far more reliable as they check that equilibrium has been achieved within a specified tolerance in the current increment.
- Alternatively energy based criteria that use both displacements and residuals may be applied. However, energy criteria should not be used together with LS.
- In general NFEA it is recommended that a combination of the three criteria is applied.
- The convergence criteria and tolerances must be carefully chosen so as to provide accurate yet economical solutions.
  - If the convergence criterion is too loose inaccurate results are obtained.
  - If the convergence criterion is too tight too much effort spent in obtaining unnecessary accuracy.

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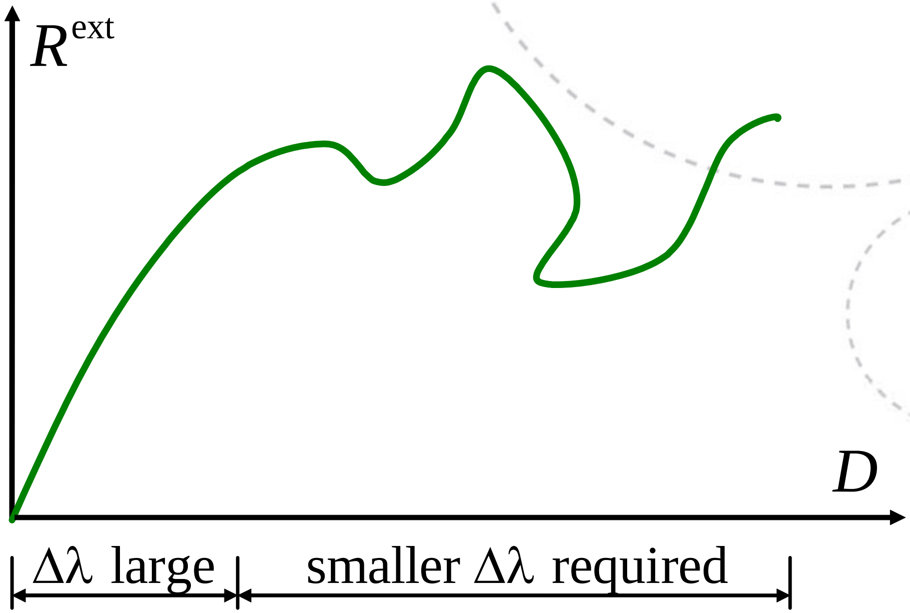
Choosing step length

- The optimal choice of the incremental step depends on:
  - The shape of the equilibrium path:
    Large increments may be used were the path is almost linear and smaller ones where the curve is highly nonlinear
  - The objective of the analysis:
    If it is necessary to trace the entire equilibrium path accurately, small increments are needed, while if only the failure load is of interest, larger steps can be used until the load is close to the limit value



  - The solution algorithm employed:
    The initial stiffness method require smaller increments than the modified Newton's method that again require smaller increments than the standard Newton's method
- It is desirable that the solution algorithm includes a solution monitoring device that on basis of:
  - Certain user prescribed input, and
  - Degree of nonlinearity of the equilibrium path is able to adjust the size of the load increment

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Load incrementation

- For monotonic loading, the **load increment** can be **based on number of iterations:**

$$\Delta\lambda_n = \Delta\lambda_{n-1}\sqrt{\frac{N_d}{N_{n-1}}} \qquad \left(\Delta\lambda^{\min} \le \Delta\lambda_n \le \Delta\lambda^{\max}\right)$$

where N_d is a '**desired number of iterations**' selected by the analyst, N_{n-1} is the **number of iterations required for convergence at increment** '$n-1$', while $\Delta\lambda^{\max}$ and $\Delta\lambda^{\min}$ are **upper and lower limit of** the **increment** prescribed by the analyst

- However, the **initial load increment** still **have to be selected** by the analyst

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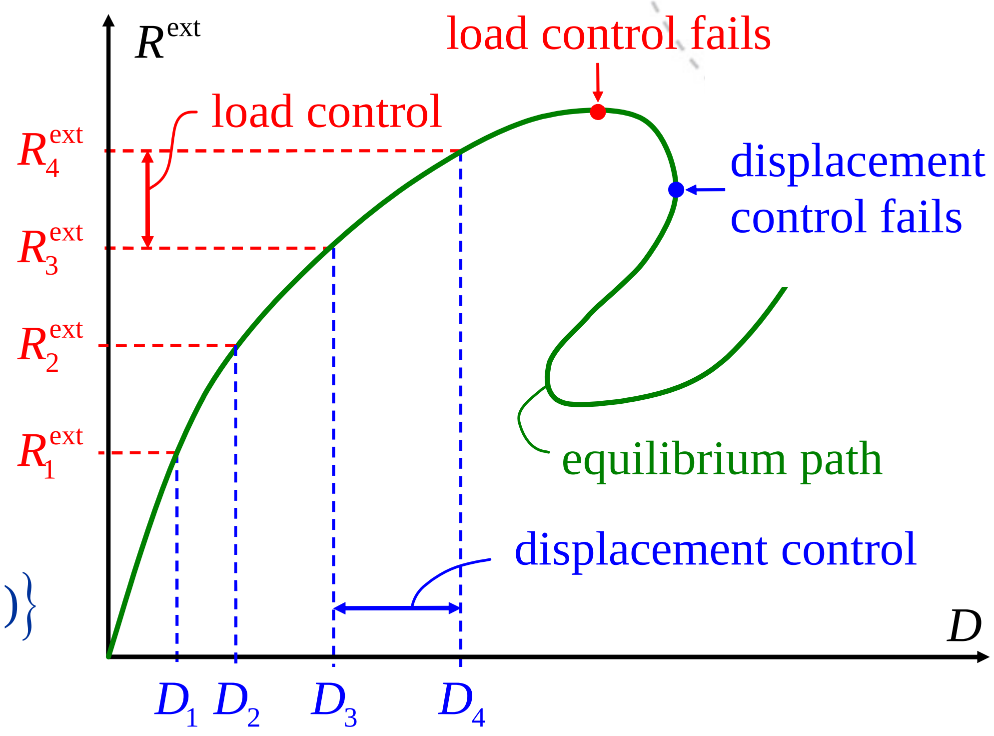
Automatic load incrementation

- Even though you may find more sophisticated incremental load control methods, they can only work effectively if nonlinearity spreads gradually.
- Such methods cannot predict a sudden change in the stiffness.
- Solution methods based on prescribed load $\{\mathbf{R}_n^{\text{ext}}\} = \{\mathbf{R}^{\text{ext}}(\lambda_n)\}$ or prescribed displacements $\{\mathbf{D}_n\} = \{\mathbf{D}(\lambda_n)\}$ are not able to trace the equilibrium path beyond limit and turning points, respectively.



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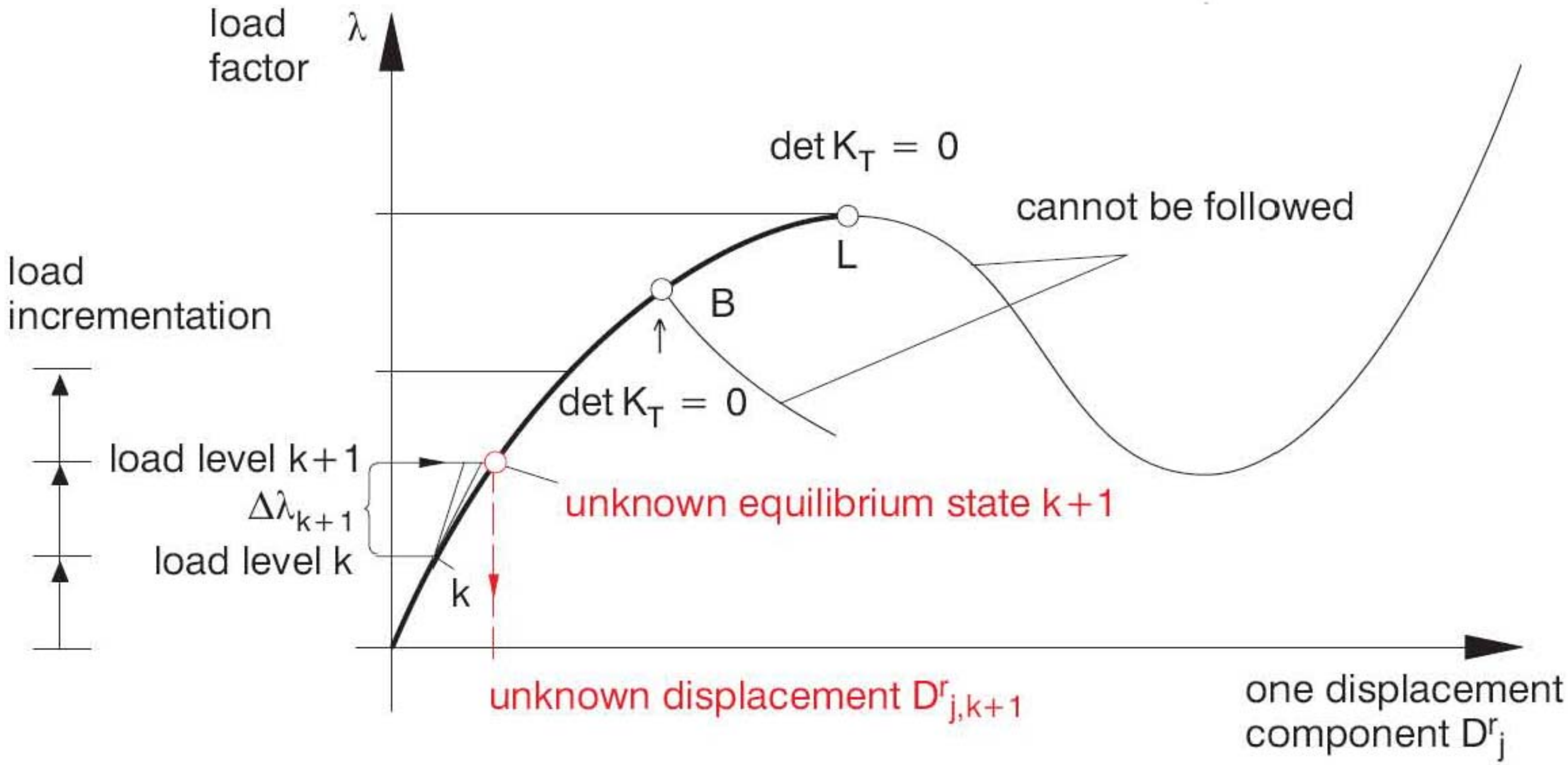
Example – Load control fails



- At limit (L) and bifurcation (B) points the tangent stiffness K_T becomes singular $\Rightarrow$ the solution of the nonlinear equilibrium equations is not unique at this point

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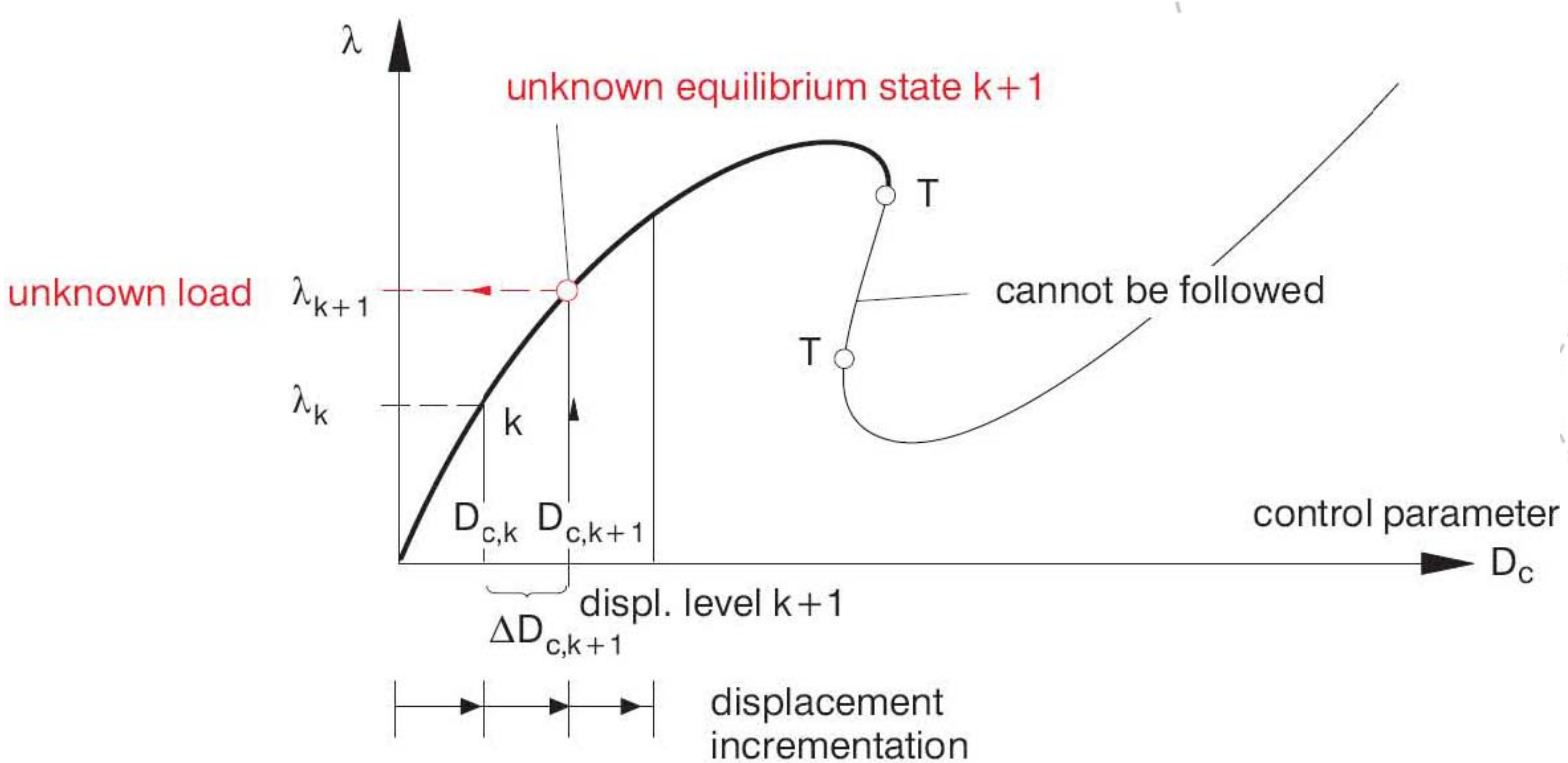
Example – Displ. control fails



- Cannot go beyond turning (T) points $\Rightarrow$ have to prescribe negative displacement increments

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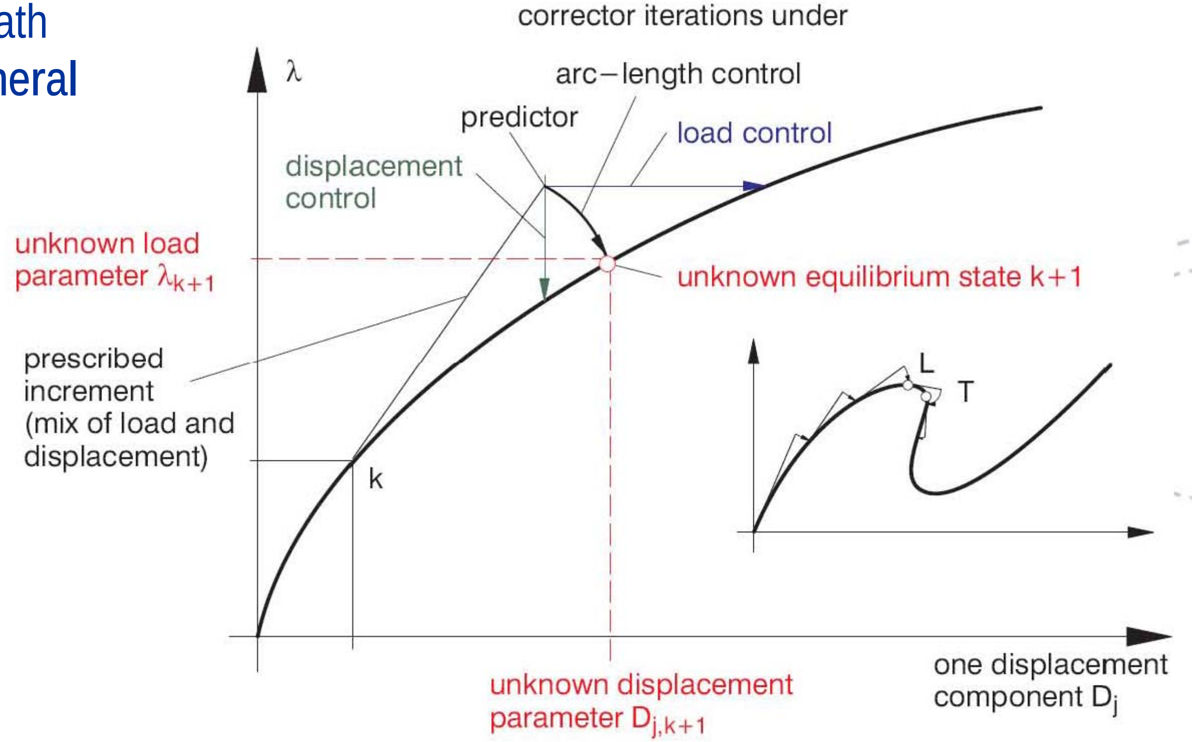
Arc-length methods

- In order to trace the equilibrium path beyond critical points, a more **general incremental control strategy** is needed, **in which** displacement $\{\Delta\mathbf{D}\}$ **and** load $\Delta\lambda$ increments are **controlled simultaneously**
- Such methods are known as "**arc-length methods**" in which the 'arc length' ℓ of the combined **displacement-load increment** is **controlled during equilibrium iterations**



⇒ we **introduce** an **additional unknown** $\Delta\lambda$ to the n_{dof} incremental displacements $\{\Delta\mathbf{D}\}$

⇒ an **additional equation** is **required** to obtain a unique solution to $\{\Delta\mathbf{D}\}$ and $\Delta\lambda$

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Arc-length methods (2)

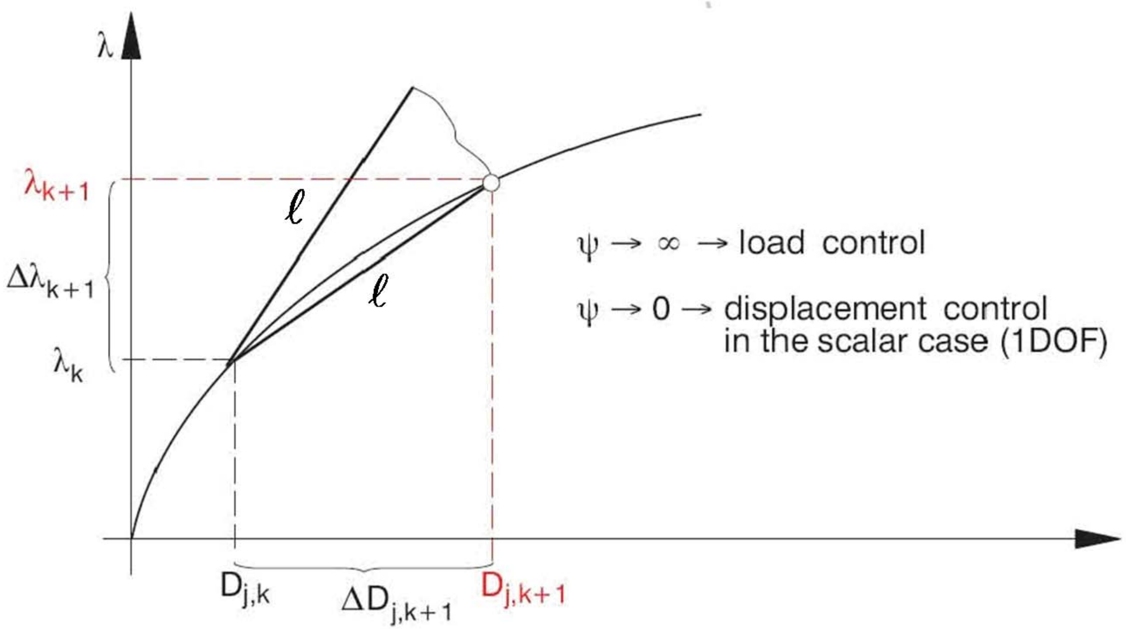
- In arc length methods a **constraint scalar equation** is introduced

$$\{\mathbf{C}(\Delta\mathbf{Z})\} = \{\mathbf{C}(\Delta\lambda, \Delta\mathbf{D})^T\} = 0$$

in which the ‘**length**’ ℓ **of the combined displacement-load increment** is **prescribed**

$$\ell^2 = \{\Delta\mathbf{D}\}^T\{\Delta\mathbf{D}\} + \psi^2\Delta\lambda^2$$

where ψ **is a scaling parameter**
($\{\Delta\mathbf{D}\}$ and $\Delta\lambda$ have different dimension)



- The **basic idea behind arc length methods** is that **instead of keeping** the **load** (or the displacement) **fixed during** an **incremental step**, **both** the **load and displacement increments** are **modified during iterations**

⇒ **Limit and turning points may be passed** with this method

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Arc-length methods (3)

- All variants of the arc length method consists of a prediction phase and a correction phase:
  1. Prediction phase:

     During the prediction phase, an estimate for the next point on the equilibrium path $\{\mathbf{Z}_n^0\} = (\lambda_n^0, \{\mathbf{D}_n^0\})^T$, is established from a known converged solution on the equilibrium path $\{\mathbf{Z}_{n-1}\} = (\lambda_{n-1}, \{\mathbf{D}_{n-1}\})^T$

  2. Correction phase:

     From this estimate, Newton iterations are employed during the correction phase to find a new point on the equilibrium curve based on the incremental form of the equations of motion and the constraint equation

     $$[\widehat{\mathbf{K}}_{T,n}^i]\{\Delta\mathbf{Z}_n^i\} = \{\Delta\mathbf{R}_n^i\} \quad \text{and} \quad \{\mathbf{C}(\mathbf{Z}_n^i)\} = 0$$

     where the augmented tangent stiffness matrix $[\widehat{\mathbf{K}}_{Tn}^i]$ and the incremental force vector $\{\Delta\mathbf{R}_n^i\}$ is obtained from

     $$\left[\hat{\mathbf{K}}_{T,n}^i\right] = \left[\hat{\mathbf{K}}_T(\mathbf{Z}_n^{i-1})\right] = \left[\left\{\frac{\partial\mathbf{R}^{\text{int}}(\mathbf{Z}_n^{i-1})}{\partial\lambda}\right\} \quad \left\{\frac{\partial\mathbf{R}^{\text{int}}(\mathbf{Z}_n^{i-1})}{\partial\mathbf{D}}\right\}\right]$$

     $$\left\{\Delta\mathbf{R}_n^i\right\} = \left\{\mathbf{R}^{\text{ext}}(\mathbf{Z}_n^{i-1})\right\} - \left\{\mathbf{R}^{\text{int}}(\mathbf{Z}_n^{i-1})\right\}$$

     and where '*n*' and '*i*' signifies the incremental load step and iteration number.

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Arc-length methods (4)

1. Normal plane arc-length method
   Newton iterations are forced to follow a **hyperplane** that is normal to the initial tangent $\{\bar{\mathbf{Z}}_{n-1}\}$ at a 'distance' ℓ from the previous obtained solution at step '*n*-1':

   $$\{\mathbf{C}(\mathbf{Z}_n^i)\} = \{\bar{\mathbf{Z}}_{n-1}\}^T\{\mathbf{Z}_n^i - \mathbf{Z}_{n-1}\} - \ell = 0$$

2. Updated normal plane arc-length method
   Hyperplane is normal to the updated tangent $\{\bar{\mathbf{Z}}_n^i\}$ instead of $\{\bar{\mathbf{Z}}_{n-1}\}$:

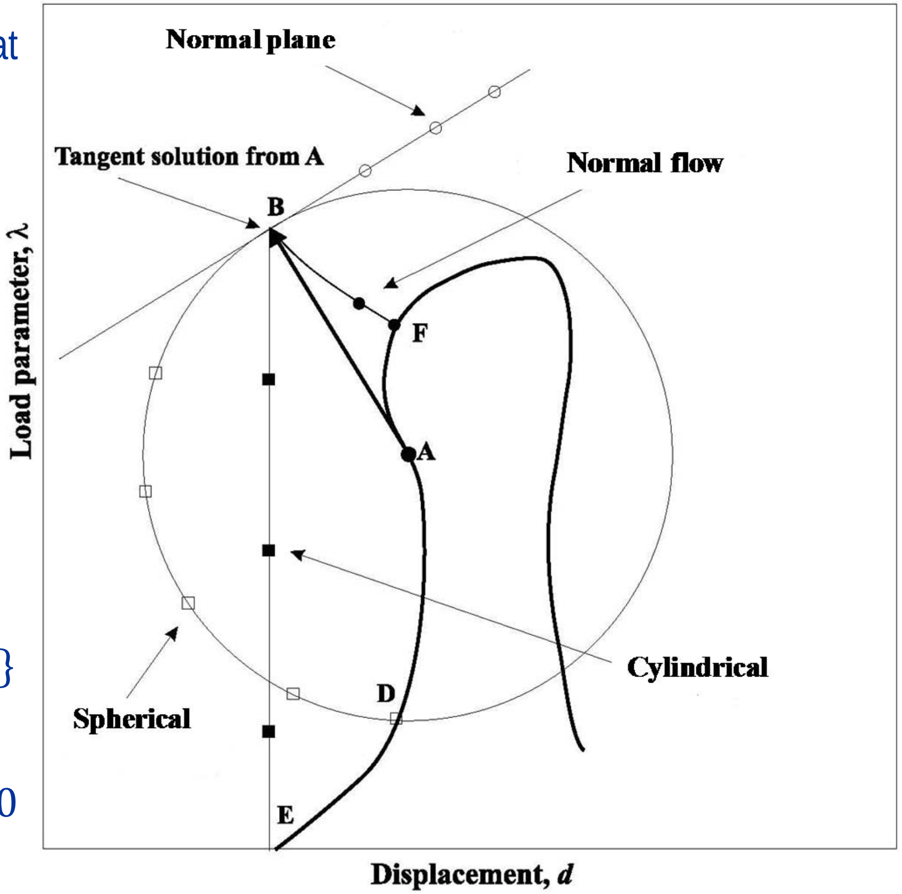
   $$\{\mathbf{C}(\mathbf{Z}_n^i)\} = \{\bar{\mathbf{Z}}_n^i\}^T\{\mathbf{Z}_n^i - \mathbf{Z}_{n-1}\} - \ell = 0$$

3. Spherical arc-length method
   Newton iterations are forced to follow a **hypersphere of radius** ℓ centered at the converged solution $\{\mathbf{Z}_{n-1}\}$ of the previous step '*n*-1':

   $$\{\mathbf{C}(\mathbf{Z}_n^i)\} = \{\mathbf{Z}_n^i - \mathbf{Z}_{n-1}\}^T\{\mathbf{Z}_n^i - \mathbf{Z}_{n-1}\} - \ell^2 = 0$$

4. Cylindrical arc-length method

   $$\psi = 0 \Rightarrow \{\mathbf{C}(\mathbf{D}_n^i)\} = \{\mathbf{D}_n^i - \mathbf{D}_{n-1}\}^T\{\mathbf{D}_n^i - \mathbf{D}_{n-1}\} - \ell^2 = 0$$



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Implicit dynamics algorithm

- The main **advantage of an implicit method** over an explicit method **is the large time step permitted by unconditionally stable time integration methods**
- However, **unconditional stability in** a **linear problem does not guarantee unconditional stability in** a **nonlinear problem**
- The **incremental strategy for dynamic problems** is provided by a temporal discretization algorithm that **transforms the ordinary differential equation system into a time-stepping sequence of nonlinear algebraic equations.**
- Hence, a **unified treatment of nonlinear static and implicit dynamic algorithms** may be employed:

  ⇒ The **solution algorithms** that have been **presented for nonlinear static problems may also be applied to nonlinear dynamic problems**

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Implicit dynamics algorithm (2)

- Substituting a linearized (first-order) approximation to the internal forces $\{\mathbf{R}^{\text{int}}\}_{n+1}$, the **equation of motion** at time t_{n+1} becomes:

$$[\mathbf{M}]\{\ddot{\mathbf{D}}\}_{n+1}+[\mathbf{C}]\{\dot{\mathbf{D}}\}_{n+1}+[\mathbf{K}_T]_n\{\Delta\mathbf{D}\}=\{\mathbf{R}^{\text{ext}}\}_{n+1}-\{\mathbf{R}^{\text{int}}\}_n$$

- **Substituting** the **updated** values for the nodal **accelerations and velocities** at time t_{n+1} , that with **Newmark approximations** may be obtained from:

$$\{\ddot{\mathbf{D}}\}_{n+1}=\frac{1}{\beta\Delta t^2}\left(\{\mathbf{D}\}_{n+1}-\{\mathbf{D}\}_n-\Delta t\{\dot{\mathbf{D}}\}_n\right)-\left(\frac{1}{2\beta}-1\right)\{\ddot{\mathbf{D}}\}_n$$

$$\{\dot{\mathbf{D}}\}_{n+1}=\frac{\gamma}{\beta\Delta t}\left(\{\mathbf{D}\}_{n+1}-\{\mathbf{D}\}_n\right)-\left(\frac{\gamma}{\beta}-1\right)\{\dot{\mathbf{D}}\}_n-\Delta t\left(\frac{\gamma}{2\beta}-1\right)\{\ddot{\mathbf{D}}\}_n$$

we obtain the **equation of motion on incremental form**

$$[\mathbf{K}^{\text{eff}}]_n\{\Delta\mathbf{D}\}=\{\Delta\mathbf{R}^{\text{eff}}\}_{n+1}$$

where

$$[\mathbf{K}^{\text{eff}}]_n=\frac{1}{\beta\Delta t^2}[\mathbf{M}]+\frac{\gamma}{\beta\Delta t}[\mathbf{C}]+[\mathbf{K}_T]_n\ \ddot{x}$$

$$\{\Delta\mathbf{R}^{\text{eff}}\}_{n+1}=\{\mathbf{R}^{\text{ext}}\}_{n+1}-\{\mathbf{R}^{\text{int}}\}_n+[\mathbf{M}]\left[\frac{1}{\beta\Delta t}\{\dot{\mathbf{D}}\}_n+\left(\frac{1}{2\beta}-1\right)\{\ddot{\mathbf{D}}\}_n\right]+[\mathbf{C}]\left[\left(\frac{\gamma}{\beta}-1\right)\{\dot{\mathbf{D}}\}_n+\Delta t\left(\frac{\gamma}{2\beta}-1\right)\{\ddot{\mathbf{D}}\}_n\right]$$