Lecture 12:

Formulation of Geometrically Nonlinear FE

Review of Continuum Mechanics

- In the following the necessary background in the theory of the mechanics of continuous media (***continuum mechanics***) for derivation of ***geometrically nonlinear finite elements*** is presented

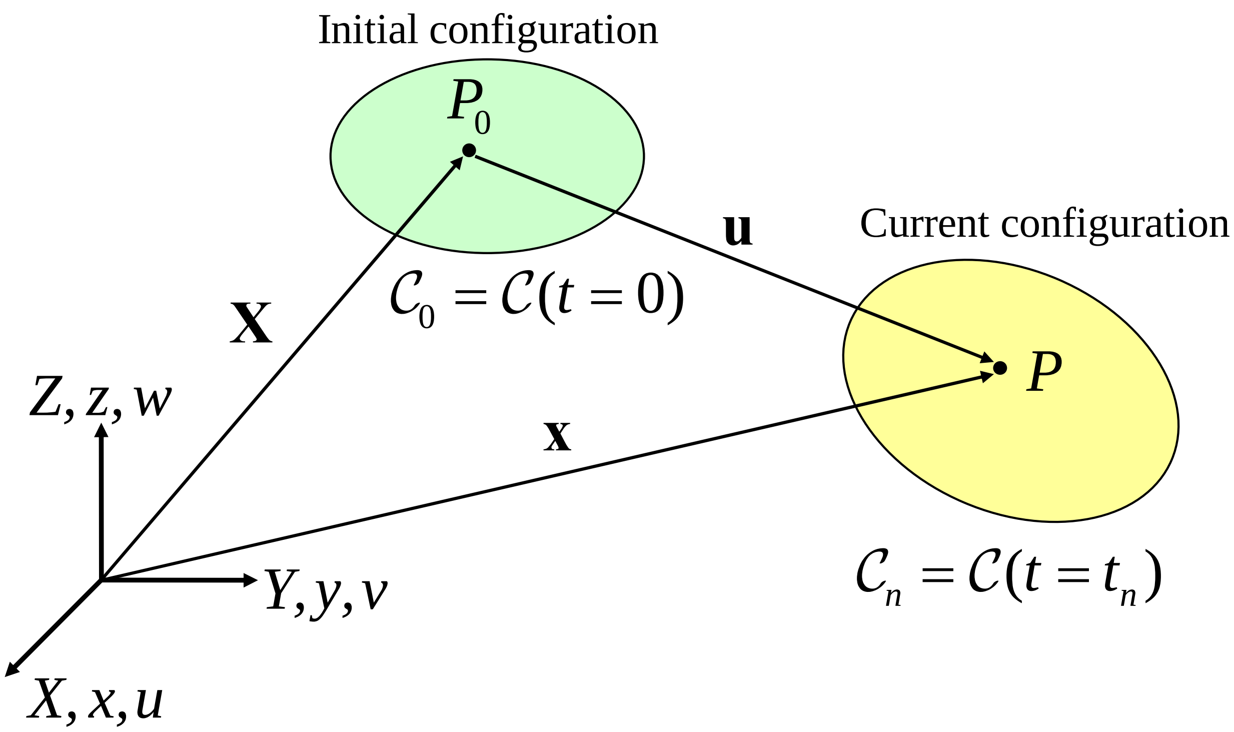
- In continuum mechanics a solid ***structure*** is mathematically ***treated as a continuum body*** being formed by a set of material particles

- The ***position of all material particles*** comprising the body at a given time *t* is ***called*** the ***configuration*** of the body, and denoted $\mathcal{C}$

- A ***sequence of configurations*** for all times *t* ***defines*** the ***motion of the body***

- In previous lectures we have seen that the motion of a body or structure is often represented by a load-displacement diagram, starting from an initial, usually undeformed, state at time $t = 0$, called ***initial configuration***, $\mathcal{C}_0$, to which displacements $\{\mathbf{u}\}$ are referred

- Each individual point on the equilibrium path corresponds to an instantaneous actual or ***current*** (deformed) ***configuration***, $\mathcal{C}_n$, at time $t = t_n$



- The ***reference configuration***, is the ***configuration to which state variables*** (e.g. strains and stresses) ***are referred***

- It is important to ***note that*** the ***time t*** is not necessarily the physical time; in this context *t* ***should be viewed as a state or load parameter*** or simply a pseudotime λ

- Three ***basic choices*** need to be made ***in*** developing ***a large displacement (deformation) analysis scheme***:

  1. The ***kinematic description***; i.e. how the body move and how the local deformations and strains are measured

  2. The ***balance law***; i.e. the definition of linear and angular momentum and the definition of (conjugate) stresses

  3. The ***constitutive equations***; i.e. an appropriate material relation that is objective and defines the stresses in terms of strains or rate of strains

Description of Motion:

- To describe the deformation of a body requires knowledge of the ***position occupied by the material particles*** comprising the body ***at all time***

- Two sets of coordinates may be used:

  i) ***Material (Lagrangian) coordinates;*** $\{\mathbf{X}\}$

  ii) ***Spatial (Eulerian) coordinates;*** $\{\mathbf{x}\} = \{\mathbf{x}(\mathbf{X}, t)\}$

- $\{\mathbf{x}\}$ defines the ***current coordinates of material particles*** in terms of ***material coordinates*** $\{\mathbf{X}\}$, the latter ***being the initial coordinates of the particles*** at time $t = 0$

- In the ***Lagrangian approach***, all physical ***quantities*** (displacements, strains and stresses) are ***expressed as functions of time*** t ***and*** their ***initial position*** $\{\mathbf{X}\}$, in the ***Eulerian approach*** they are ***functions of time and*** their ***current position***

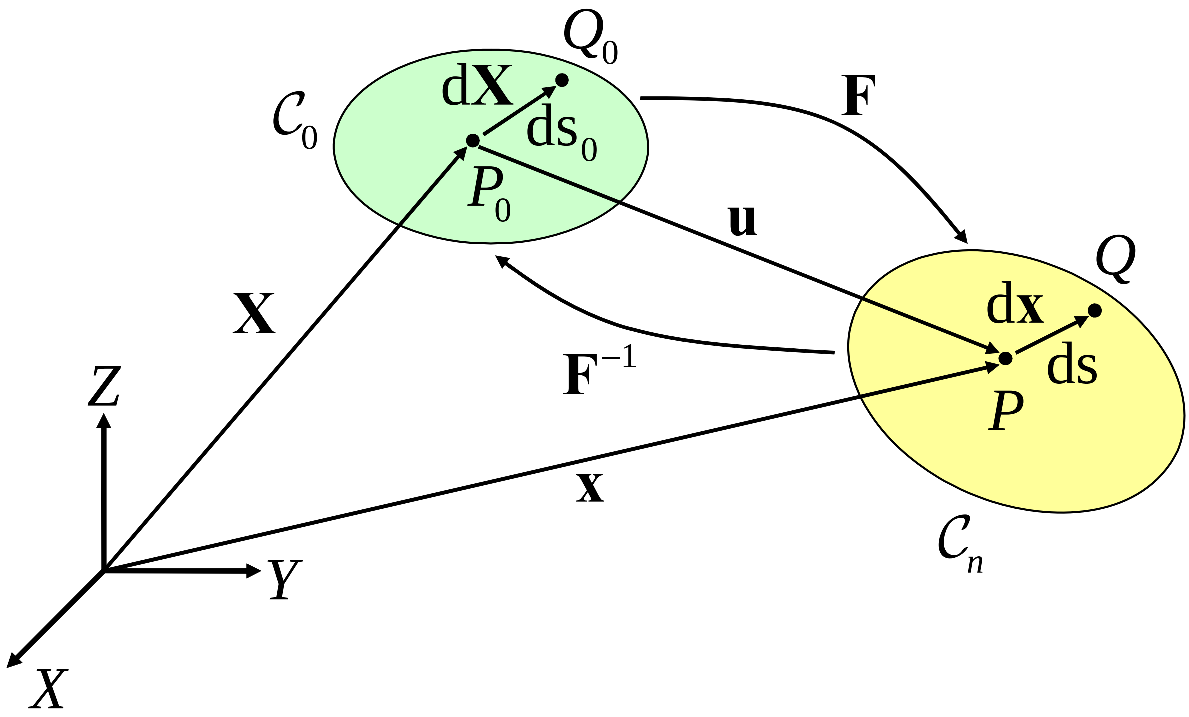
- Although both approaches may be used, the ***Lagrangian approach*** turns out to be the ***most attractive in solid and structural mechanics*** problems

- The ***Lagrangian description of motion*** is ***referred to a fixed global***, ***Cartesian coordinate system*** (X, Y, Z)

- In the Lagrangian description ***displacements of any material point*** in the solid is given by:

$$\{\mathbf{x}(\mathbf{X}, t)\} = \{\mathbf{X}\} + \{\mathbf{u}(\mathbf{X}, t)\} \quad \Leftrightarrow \quad \{\mathbf{u}(\mathbf{X}, t)\} = \{\mathbf{x}(\mathbf{X}, t)\} - \{\mathbf{X}\}$$

Deformation Gradient and Strain Measures:



- In order to define the strain we need to know the ***relative motion of two neighbouring particles***. Two such particles (P and Q) are shown in the Figure above where at time $t = 0$ the relative position is $\{d\mathbf{X}\}$ and at time $t = t_n$ the relative position is $\{d\mathbf{x}\}$

- The ***deformation gradient*** $[\mathbf{F}]$, ***describes*** the ***mapping*** (deformation) ***of*** the ***infinitesimal material 'fibre'*** $\{d\mathbf{X}\}$, with length ds_0, ***in*** $\mathcal{C}_0$ (the initial configuration) ***to its new position*** $\{d\mathbf{x}\}$, with length ds, ***in*** $\mathcal{C}_n$ (the current configuration):

$$\{d\mathbf{x}\} = [\mathbf{F}]\{d\mathbf{X}\}$$

where

$$[\mathbf{F}] = \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right] = \left[\frac{\partial(\mathbf{X}+\mathbf{u})}{\partial \mathbf{X}}\right] = [\mathbf{I}] + \left[\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right] = [\mathbf{I}] + [\mathbf{G}]$$

$[\mathbf{I}]$ is the ***unit tensor*** and $[\mathbf{G}]$ is called the ***displacement gradient tensor***

- The ***components of*** the ***deformation gradient*** $[\mathbf{F}]$, ***and*** the ***displacement gradient tensor*** $[\mathbf{G}]$, thus becomes:

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \quad \text{and} \quad [\mathbf{G}] = \begin{bmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial X} & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial Z} \\ \frac{\partial w}{\partial X} & \frac{\partial w}{\partial Y} & \frac{\partial w}{\partial Z} \end{bmatrix}$$

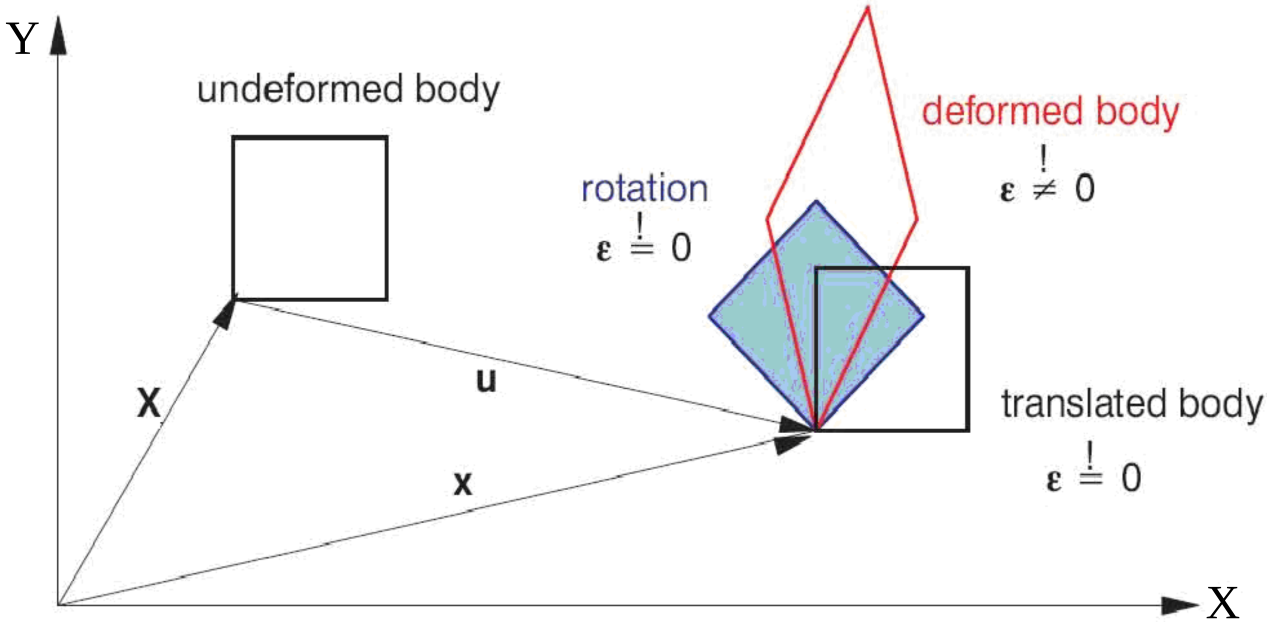
- The ***deformation gradient*** $[\mathbf{F}]$ ***describes stretches and rigid body motion of*** the ***material fibers from*** $\mathcal{C}_0$ ***to*** $\mathcal{C}_n$

- In contrast to a linear analysis, where we may apply a linear strain measure (e.g. the engineering strain), ***a finite strain measure*** is used to ***represent local deformations in a large deformation nonlinear analysis***

- ***In large deformation nonlinear analysis***, ***a body may be subjected to both large rigid body motion and large deformations***

  $\Rightarrow$ ***An important feature of a finite strain measure is that it vanish for arbitrary rigid body translations and rotations***

- Another property of the ***finite strain measure*** is that it ***must reduce to*** the ***infinitesimal strains if it is linearized*** (i.e. when the nonlinear strain terms are neglected)



- One finite strain measure that has these desired properties is the ***Green strain tensor*** $[\boldsymbol{\varepsilon}_G]$, which is a ***symmetric tensor defining*** the ***relationship between*** the ***squares of the length of the material 'fibre' vector*** $\{d\mathbf{X}\}$ with length ds_0 ***in*** $\mathcal{C}_0$ ***to its deformed vector*** $\{d\mathbf{x}\}$ with length ds ***in*** $\mathcal{C}_n$***:***

$$ds^2 - ds_0^2 = 2\{d\mathbf{X}\}^T [\boldsymbol{\varepsilon}_G]\{d\mathbf{X}\}$$

- ***Green strain tensor*** $[\boldsymbol{\varepsilon}_G]$ can also be ***expressed in terms of*** the ***deformation gradient*** $[\mathbf{F}]$ through:

$$[\boldsymbol{\varepsilon}_G] = \frac{1}{2}\left([\mathbf{F}]^T [\mathbf{F}] - [\mathbf{I}]\right)$$

with components:

$$[\boldsymbol{\varepsilon}_G] = \begin{bmatrix} \varepsilon_{GXX} & \varepsilon_{GXY} & \varepsilon_{GXZ} \\ \varepsilon_{GYX} & \varepsilon_{GYY} & \varepsilon_{GYZ} \\ \varepsilon_{GZX} & \varepsilon_{GZY} & \varepsilon_{GZZ} \end{bmatrix}$$

- The six strain ***components of*** the ***Green strain tensor*** may be expressed in terms of the displacement gradients:

$$\varepsilon_{GXX} = \frac{\partial u}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2\right]$$

$$\varepsilon_{GYY} = \frac{\partial v}{\partial Y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2 + \left(\frac{\partial w}{\partial Y}\right)^2\right]$$

$$\varepsilon_{GZZ} = \frac{\partial w}{\partial Z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial v}{\partial Z}\right)^2 + \left(\frac{\partial w}{\partial Z}\right)^2\right]$$

$$\varepsilon_{GXY} = \frac{1}{2}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right) + \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)\left(\frac{\partial u}{\partial Y}\right) + \left(\frac{\partial v}{\partial X}\right)\left(\frac{\partial v}{\partial Y}\right) + \left(\frac{\partial w}{\partial X}\right)\left(\frac{\partial w}{\partial Y}\right)\right]$$

$$\varepsilon_{GYZ} = \frac{1}{2}\left(\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y}\right) + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)\left(\frac{\partial u}{\partial Z}\right) + \left(\frac{\partial v}{\partial Y}\right)\left(\frac{\partial v}{\partial Z}\right) + \left(\frac{\partial w}{\partial Y}\right)\left(\frac{\partial w}{\partial Z}\right)\right]$$

$$\varepsilon_{GZX} = \frac{1}{2}\left(\frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z}\right) + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Z}\right)\left(\frac{\partial u}{\partial X}\right) + \left(\frac{\partial v}{\partial Z}\right)\left(\frac{\partial v}{\partial X}\right) + \left(\frac{\partial w}{\partial Z}\right)\left(\frac{\partial w}{\partial X}\right)\right]$$

- ***Green strain tensor*** is ***symmetric***:

$$\Rightarrow \quad \varepsilon_{GYX} = \varepsilon_{GXY}, \quad \varepsilon_{GZY} = \varepsilon_{GYZ} \quad \text{and} \quad \varepsilon_{GXZ} = \varepsilon_{GZX}$$

- ***If*** the ***nonlinear portion*** (that enclosed in square brackets) is ***neglected***, ***we obtain*** the ***infinitesimal strains***:

$$\varepsilon_{xx} = \varepsilon_{GXX}, \quad \varepsilon_{yy} = \varepsilon_{GYY}, \quad \varepsilon_{zz} = \varepsilon_{GXX}$$
$$\gamma_{xy} = 2\varepsilon_{GXY}, \quad \gamma_{yz} = 2\varepsilon_{GYZ}, \quad \gamma_{zx} = 2\varepsilon_{GZX}$$

- ***Green strain tensor*** is often ***used for problems with large displacements but small strains***

- ***Several other finite strain measures are used in nonlinear continuum mechanics***, however, ***they all have to satisfy the constraints*** of finite strain measures:
  - They ***must predict zero strains for arbitrarily rigid-body motions***, and
  - They ***must reduce to the infinitesimal strains if the nonlinear terms are neglected***

- ***For*** the ***uniaxial case of a stretched bar*** that has ***initial length*** L_0 ***in*** $\mathcal{C}_0$ ***and length*** L ***in*** $\mathcal{C}_n$, the ***Green strain becomes:***

$$\varepsilon_G = \varepsilon_{GXX} = \frac{L^2 - L_0^2}{2L_0^2}$$

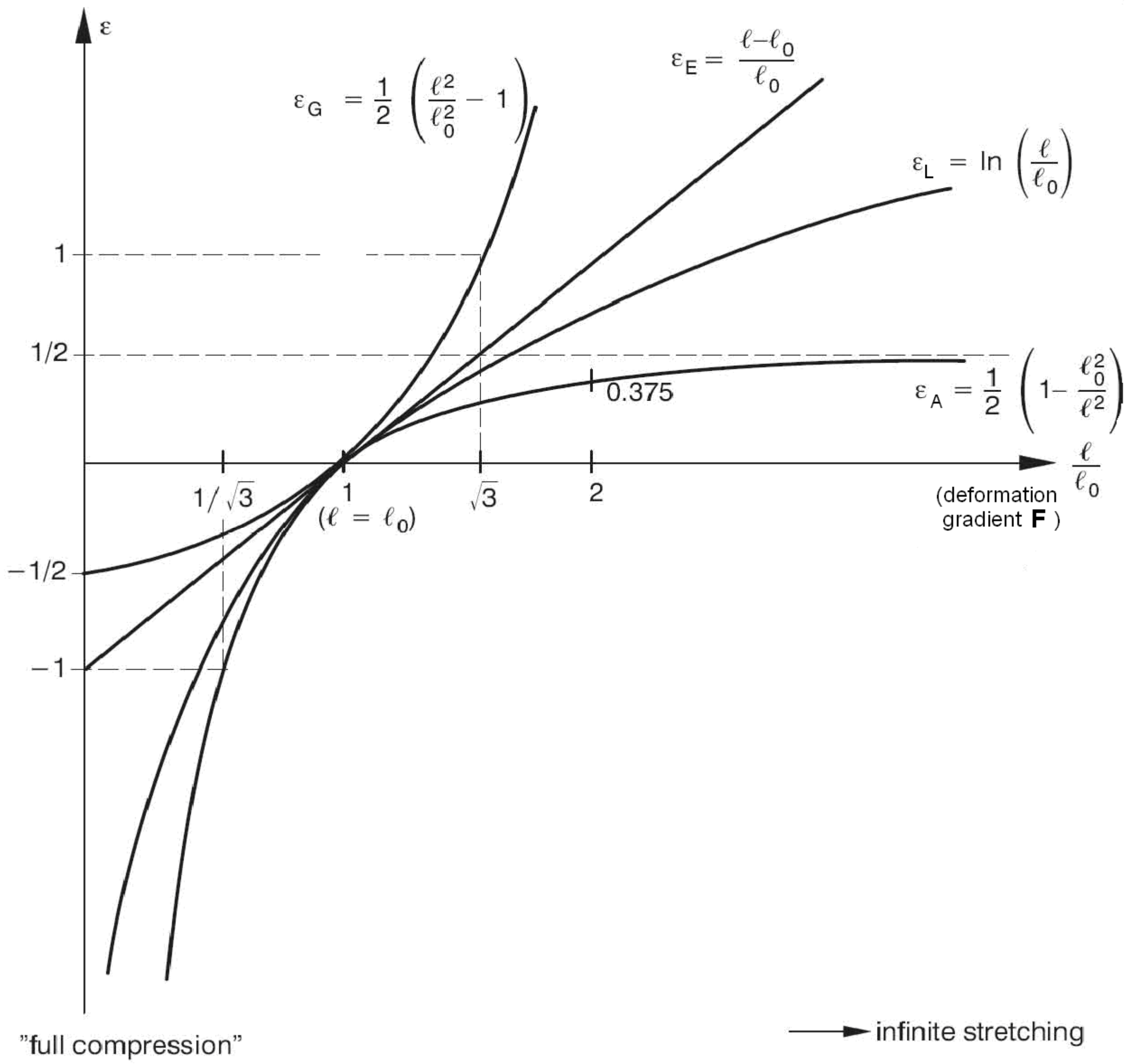
- Other ***uniaxial strain measures*** that are ***frequently used*** in nonlinear structural and solid mechanics:

$$\text{Almansi strain:} \quad \varepsilon_A = \frac{L^2 - L_0^2}{2L^2}$$

$$\text{Logarithmic strain:} \quad \varepsilon_L = \log\left(\frac{L}{L_0}\right)$$

$$\text{Engineering strain:} \quad \varepsilon_E = \frac{L - L_0}{L_0}$$

- ***Almansi strains*** are, in contrast to the Green strains that are referred to the material coordinates $\{\mathbf{X}\}$, ***referred to the spatial coordinates*** $\{\mathbf{x}\}$ and ***used together with*** an ***Euler description***, while ***logarithmic*** (also called natural or "true") ***strains*** are ***useful for large strain problems*** (e.g. metal forming)



- When choosing ***a proper finite strain measure*** we have to judge whether the strain measure ***predicts a realistic finite strain value*** or not

- ***If we want to model large strain deformations***, the ***chosen strain measure should tend to –∞ for "full compression" and ∞ for "infinite stretching"***, otherwise it could become difficult to describe a sensible constitutive law

- In the ***Figure above*** that ***shows*** the ***behaviour of*** the different ***strain measures*** introduced ***for large strains***, we observe that both the ***Green and*** the ***Engineering strains remain finite for "infinite" compression***, while the ***Almansi strain predicts a finite strain for "infinite" tension***

$\Rightarrow$ The ***only strain measure*** which is ***suitable in*** the ***entire range*** is the ***logarithmic*** (natural) ***strain***

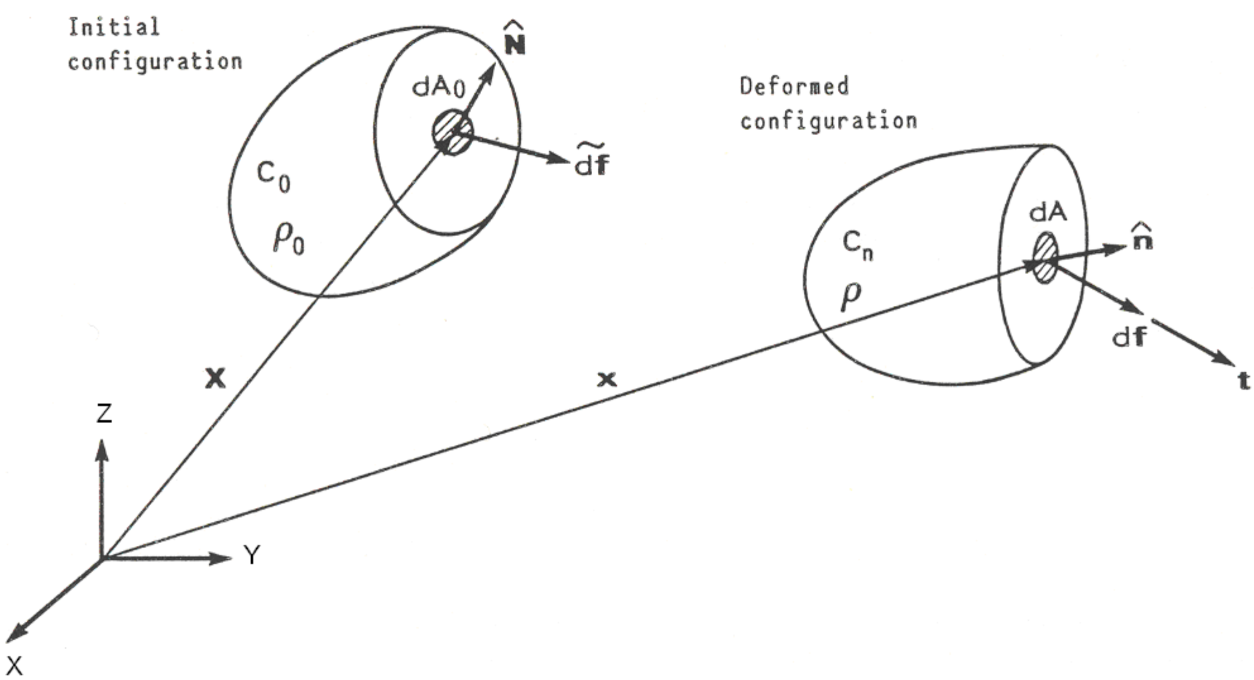
- However, ***if*** $\left|\frac{L - L_0}{L_0}\right| < 0.05$ the ***deviation between*** the ***finite strain measures and*** the ***Engineering strain*** is ***of*** the ***order*** $2 - 3\%$

Stress Measures:

- The ***surface traction*** $\{\mathbf{t}\}$ is defined as:

$$\{\mathbf{t}\} = \frac{\{d\mathbf{f}\}}{dA}$$

where $\{d\mathbf{f}\}$ is the infinitesimal force vector that acts on the infinitesimal area element dA in deformed configuration.



- The ***Cauchy*** or ***true stress tensor*** $[\boldsymbol{\sigma}]$, energy conjugate to the Almansi strain tensor $[\boldsymbol{\varepsilon}_A]$, gives the current force per unit area in deformed configuration, consequently:

$$\{\mathbf{t}\} = [\boldsymbol{\sigma}]\{\hat{\mathbf{n}}\}$$

where $\{\hat{\mathbf{n}}\}$ is the unit outward normal to the infinitesimal area element dA in deformed configuration.

- Multiplying $[\boldsymbol{\sigma}]$ by the determinant of $[\mathbf{F}]$ ($J = \det[\mathbf{F}]$) gives the ***Kirchhoff stress tensor*** $[\boldsymbol{\tau}]$

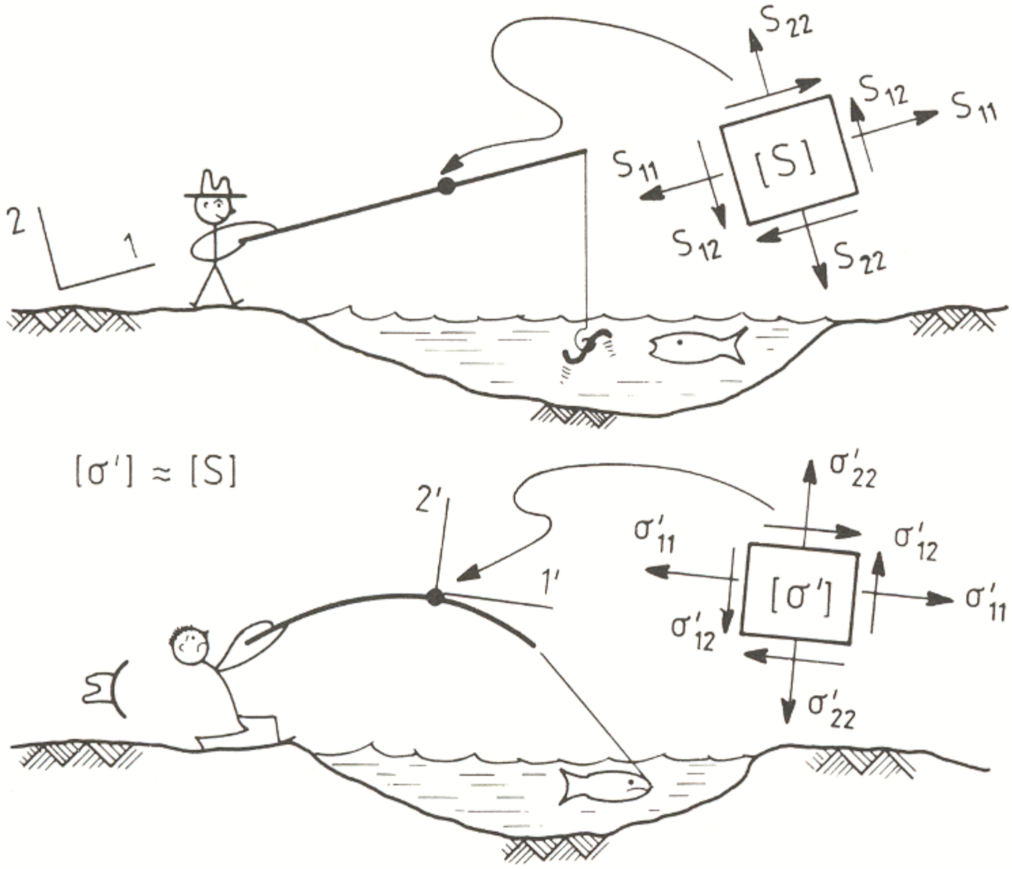
$$[\boldsymbol{\tau}] = J[\boldsymbol{\sigma}]$$

- A ***stress tensor work conjugate to*** the ***Green strain tensor*** $[\boldsymbol{\varepsilon}_G]$ must be referred to the initial (undeformed) configuration as is the Green strain tensor.

- It may be shown that the ***2nd Piola-Kirchhoff (PK) stress tensor*** $[\mathbf{S}]$ that gives the transformed current force $\{d\tilde{\mathbf{f}}\}$ per unit undeformed area dA_o is work conjugate to $[\boldsymbol{\varepsilon}_G]$ and related to $[\boldsymbol{\sigma}]$ through

$$[\mathbf{S}] = J[\mathbf{F}]^{-1}[\boldsymbol{\sigma}][\mathbf{F}]^{-T}$$

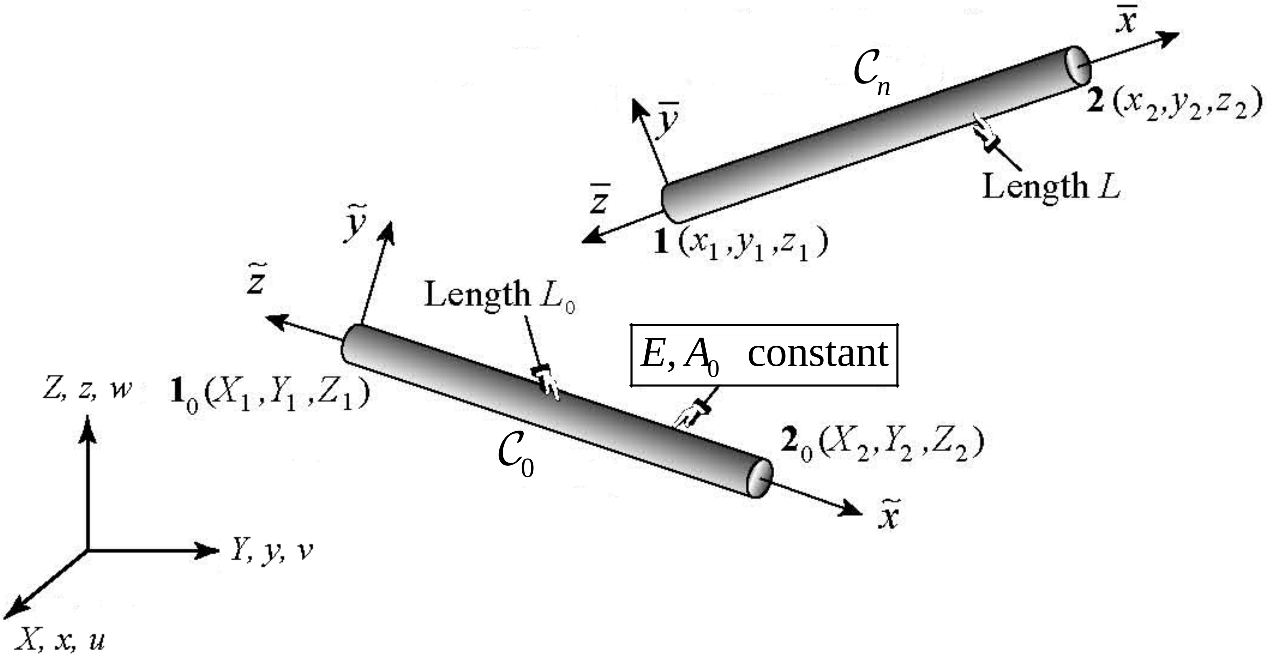
- While the ***Cauchy stress tensor*** $[\boldsymbol{\sigma}]$ ***and the Kirchhoff stress tensor*** $[\boldsymbol{\tau}]$ are ***preferable in general NFEA involving large deformations***, the ***2nd PK stress tensor*** $[\mathbf{S}]$ is ***a good approximation when*** the ***deformational*** (strain giving) ***displacement components are small*** (i.e. large rigid body displacements, but small strains).



Total and Updated Lagrangian Formulations:

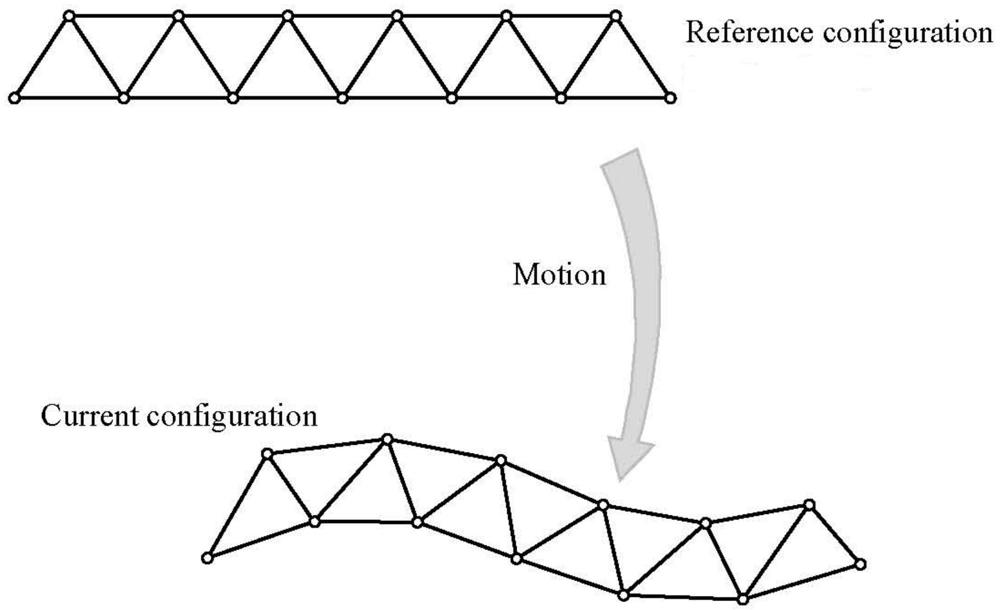
- In a ***Total Lagrangian (TL) formulation strain and stress measures*** are ***referred to the initial*** (undeformed) ***configuration***, $\mathcal{C}_0$
- Alternatively if a known ***deformed configuration***, $\mathcal{C}_n$, is ***taken as the initial state and continuously updated*** as the calculation proceeds this is called an ***Updated Lagrangian (UL) formulation***
- In a ***CoRotational (CR) formulation*** a ***local reference frame***, $\mathcal{C}_R$, is ***attached to each element*** and ***translates and rotates with*** the ***element as a rigid body***. In a CR formulation, the ***total deformation*** is ***decomposed into*** a ***rigid-body motion***, which is ***identical to rigid-body motion of*** the ***local reference frame***, ***and local deformations*** (strains and stresses), that are ***measured relative to*** the ***local reference frame***

2-node TL Bar Element in 3D Space[1]



- In the following the ***key concepts of nonlinear continuum mechanics*** are ***applied to establish*** the ***internal forces*** $\{\mathbf{r}^{\text{int}}\}$ ***and tangential stiffness*** $[\mathbf{k}_t]$ of a 2-node three dimensional bar element based on the Total Lagrangian formulation
- The 2-node bar element may be used to model truss structures as shown in the Figure on the next page
- It is assumed that the ***material behaviour*** is ***linearly elastic*** with elasticity modulus E, ***such that we may consider geometric nonlinear effects only***
- ***In*** the initial configuration $\mathcal{C}_0$, ***which is*** the ***reference configuration for*** the ***TL formulation***, the ***element has cross section area*** A_0 (assumed constant along the element) ***and length*** L_0
- ***In*** the current configuration $\mathcal{C}_n$, the ***cross section area and length*** become A ***and*** L, respectively

[1] Carlos Felippa, University of Colorado at Boulder: Chapter 14 of lecture notes in ASEN 5107 (NFEM).



Element Kinematics:

- Assume that the ***bar remains straight*** in any configuration

  $\Rightarrow$ The ***coordinates of a generic point*** $\{\mathbf{X}\}$ ***located on the longitudinal axis of*** the ***reference configuration*** $\mathcal{C}_0$ ***and*** the ***corresponding coordinates*** $\{\mathbf{x}\}$ ***in*** the ***current configuration*** $\mathcal{C}_n$, reads:

$$\{\mathbf{X}(\xi)\} = \begin{Bmatrix} X(\xi) \\ Y(\xi) \\ Z(\xi) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} = [\mathbf{N}]\{\mathbf{C}\}$$

$$\{\mathbf{x}(\xi)\} = \begin{Bmatrix} x(\xi) \\ y(\xi) \\ z(\xi) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{Bmatrix} = [\mathbf{N}]\{\mathbf{c}\}$$

where ξ is the ***dimensionless*** isoparametric ***coordinate*** that varies from $\xi_1 = -1$ at node 1 to $\xi_2 = 1$ at node 2, and $\boldsymbol{N_i}$ are the ***linear shape functions***:

$$N_i = \frac{1}{2}(1 + \xi_i \xi); \quad i = 1, 2$$

- The ***displacement field***, $\{\mathbf{u}\}$, is ***obtained by subtracting*** the two ***position vectors*** $\{\mathbf{X}\}$ ***and*** $\{\mathbf{x}\}$:

$$\{\mathbf{u}(\xi)\} = \begin{Bmatrix} u(\xi) \\ v(\xi) \\ w(\xi) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix} = [\mathbf{N}]\{\mathbf{d}\}$$

Strain Energy:

- ***Denoting*** the ***axial strain and stress*** measures by ***e and s***, respectively, with s being the energy conjugate of e

- Because of the ***linear displacement*** assumptions

  $\Rightarrow$ ***Strain e and stress s become constant*** over the element length (volume)

- The ***axial strain e*** is ***assumed to be zero in $\mathcal{C}_0$ and e in $\mathcal{C}_n$***

  $\Rightarrow$ ***The stresses in $\mathcal{C}_0$ and $\mathcal{C}_n$ become***:

$$\begin{aligned} s &= s_0 && \text{in } \mathcal{C}_0 \\ s &= s_0 + Ee && \text{in } \mathcal{C}_n \end{aligned}$$

- Similarly, the ***axial forces N_0 in $\mathcal{C}_0$ and N in $\mathcal{C}_n$ become***:

$$\begin{aligned} N_0 &= A_0 s_0 && \text{in } \mathcal{C}_0 \\ N &= A_0 s = N_0 + EA_0 e && \text{in } \mathcal{C}_n \end{aligned}$$

- The ***strain energy density U_0 in $\mathcal{C}_0$*** is ***assumed to be zero*** ($e_0 = 0$), while ***in $\mathcal{C}_n$ it becomes***:

$$U_0 = s_0 e + \frac{1}{2} E e^2$$

which is ***constant over*** the volume of the ***element***

- The ***total strain energy in $\mathcal{C}_n$***, thus ***becomes***:

$$U = \int_{V_0} U_0 dV = U_0 V_0 = A_0 L_0 \left(s_0 e + \frac{1}{2} E e^2 \right) = L_0 \left(N_0 e + \frac{1}{2} EA_0 e^2 \right)$$

Internal Forces and Tangential Stiffness:

- The ***FE equilibrium equations*** are ***obtained by making*** the ***total potential energy U_0 stationary***

$\Rightarrow$ The ***internal force vector*** $\{\mathbf{r}^{\text{int}}\}$ is ***obtained as*** the ***gradient of*** the ***internal strain energy U with respect to*** the ***nodal displacements*** $\{\mathbf{d}\}$

$$\{\mathbf{r}^{\text{int}}\} = \left\{ \frac{\partial U}{\partial \mathbf{d}} \right\}$$

- It is assumed that the ***strain measure*** $\boldsymbol{e}$ is ***a function of the element lengths*** $\boldsymbol{L_0}$ ***in*** $\boldsymbol{\mathcal{C}_0}$ ***and*** $\boldsymbol{L}$ ***in*** $\boldsymbol{\mathcal{C}_n}$ (where L_0 is fixed):

$$e = e(L) \quad \Rightarrow \quad U = U(e) = U(L)$$

- The ***derivatives of*** the ***strain energy*** $\boldsymbol{U_0}$ ***with respect to nodal displacements*** $\{\mathbf{d}\}$ are ***obtained by*** the ***chain rule***:

$$\left\{\frac{\partial U}{\partial \mathbf{d}}\right\} = \frac{\partial U}{\partial e}\left\{\frac{\partial e}{\partial \mathbf{d}}\right\} = L_0\left(N_0 + EA_0 e\right)\left\{\frac{\partial e}{\partial \mathbf{d}}\right\} = L_0 N \frac{\partial e}{\partial L}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}$$

- The ***element length*** $\boldsymbol{L}$ ***in*** $\boldsymbol{\mathcal{C}_n}$ is defined by:

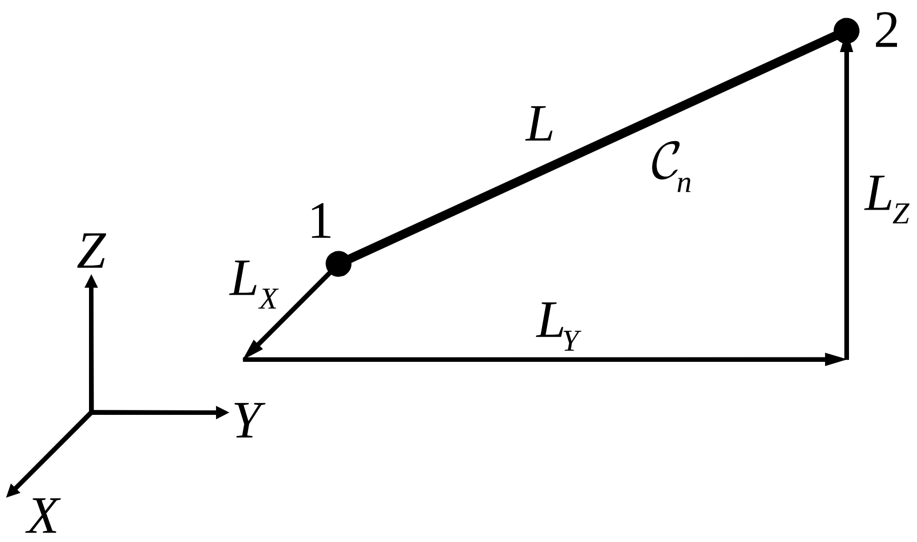
$$L = \sqrt{L_X^2 + L_Y^2 + L_Z^2}$$

where the ***projected lengths onto*** the ***global axes in*** $\boldsymbol{\mathcal{C}_n}$ reads:

$$\begin{aligned} L_X &= x_2 - x_1 = \left(X_2 + u_2\right) - \left(X_1 + u_1\right) \\ L_Y &= y_2 - y_1 = \left(Y_2 + v_2\right) - \left(Y_1 + v_1\right) \\ L_Z &= z_2 - z_1 = \left(Z_2 + w_2\right) - \left(Z_1 + w_1\right) \end{aligned}$$

- The ***partial derivatives of*** $\boldsymbol{L}$ ***with respect to*** the ***nodal displacements*** $\{\mathbf{d}\}$, thus become:

$$\left\{\frac{\partial L}{\partial \mathbf{d}}\right\} = \begin{Bmatrix} \partial L/\partial u_1 \\ \partial L/\partial v_1 \\ \partial L/\partial w_1 \\ \partial L/\partial u_2 \\ \partial L/\partial v_2 \\ \partial L/\partial w_2 \end{Bmatrix} = \frac{1}{L}\begin{Bmatrix} -L_X \\ -L_Y \\ -L_Z \\ L_X \\ L_Y \\ L_Z \end{Bmatrix} = \begin{Bmatrix} -\mathbf{L} \\ \mathbf{L} \end{Bmatrix} = \left\{\hat{\mathbf{L}}\right\}$$



where $\{\mathbf{L}\}$ ***contains*** the ***direction cosines of*** the ***length segment*** L:

$$\{\mathbf{L}\} = \frac{1}{L}\lfloor L_X \quad L_Y \quad L_Z \rfloor^T$$

- Hence, the ***internal force vector*** $\{\mathbf{r}^{\text{int}}\}$ may be ***expressed in terms of*** the ***direction cosines*** contained ***in*** $\{\hat{\mathbf{L}}\}$:

$$\{\mathbf{r}^{\text{int}}\} = \left\{\frac{\partial U}{\partial \mathbf{d}}\right\} = L_0 N \frac{\partial e}{\partial L}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\} = L_0 N \frac{\partial e}{\partial L}\{\hat{\mathbf{L}}\}$$

- Similarly, it may easily be shown that the ***second derivatives of*** L ***with respect to*** the ***nodal displacements*** $\{\mathbf{d}\}$, become:

$$\left[\frac{\partial^2 L}{\partial \mathbf{d} \partial \mathbf{d}}\right] = \frac{1}{L}\left(\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix} - \left\{\frac{\partial L}{\partial \mathbf{d}}\right\}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}^T\right) = \frac{1}{L}\left[\hat{\mathbf{I}} - \hat{\mathbf{L}}\hat{\mathbf{L}}^T\right]$$

- The ***tangent stiffness*** $[\mathbf{k}_t]$ is ***obtained*** simply ***by differentiating*** the ***internal force vector*** $\{\mathbf{r}^{\text{int}}\}$ ***with respect to*** the ***nodal displacements*** $\{\mathbf{d}\}$:

$$
\begin{aligned}
[\mathbf{k}_t] &= \left\{\frac{\partial \mathbf{r}^{\text{int}}}{\partial \mathbf{d}}\right\} \\
&= L_0\left(\left\{\frac{\partial N}{\partial \mathbf{d}}\right\}\frac{\partial e}{\partial L}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}^T + N\frac{\partial^2 e}{\partial L^2}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}^T + N\frac{\partial e}{\partial L}\left[\frac{\partial^2 L}{\partial \mathbf{d}\partial \mathbf{d}}\right]\right) \\
&= L_0\left(EA_0\left(\frac{\partial e}{\partial L}\right)^2\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}^T + N\left(\frac{\partial^2 e}{\partial L^2}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}\left\{\frac{\partial L}{\partial \mathbf{d}}\right\}^T + \frac{\partial e}{\partial L}\left[\frac{\partial^2 L}{\partial \mathbf{d}\partial \mathbf{d}}\right]\right)\right)
\end{aligned}
$$

- ***Substituting*** the ***expressions for*** the ***first and second partial derivatives of*** the ***element length from above***, we obtain:

$$
\begin{aligned}
[\mathbf{k}_t] &= L_0\left(EA_0\left(\frac{\partial e}{\partial L}\right)^2\left[\hat{\mathbf{L}}\hat{\mathbf{L}}^T\right] + N\left(\frac{\partial^2 e}{\partial L^2}\left[\hat{\mathbf{L}}\hat{\mathbf{L}}^T\right] + \frac{1}{L}\frac{\partial e}{\partial L}\left[\hat{\mathbf{I}} - \hat{\mathbf{L}}\hat{\mathbf{L}}^T\right]\right)\right) \\
&= L_0\left(EA_0\left(\frac{\partial e}{\partial L}\right)^2\left[\hat{\mathbf{L}}\hat{\mathbf{L}}^T\right] + N\left(\frac{1}{L}\frac{\partial e}{\partial L}\left[\hat{\mathbf{I}}\right] + \left(\frac{\partial^2 e}{\partial L^2} - \frac{1}{L}\frac{\partial e}{\partial L}\right)\left[\hat{\mathbf{L}}\hat{\mathbf{L}}^T\right]\right)\right) \\
&= [\mathbf{k}_m] + [\mathbf{k}_g]
\end{aligned}
$$

where the ***material stiffness*** $[\mathbf{k}_m]$ ***and*** the ***geometrical stiffness*** $[\mathbf{k}_g]$ reads:

$$
[\mathbf{k}_m] = EA_0L_0\left(\frac{\partial e}{\partial L}\right)^2\left[\hat{\mathbf{L}}\hat{\mathbf{L}}^T\right]
$$

$$
[\mathbf{k}_g] = NL_0\left(\frac{1}{L}\frac{\partial e}{\partial L}\left[\hat{\mathbf{I}}\right] + \left(\frac{\partial^2 e}{\partial L^2} - \frac{1}{L}\frac{\partial e}{\partial L}\right)\left[\hat{\mathbf{L}}\hat{\mathbf{L}}^T\right]\right)
$$

- The ***above expressions for*** the internal force vector $\{\mathbf{r}^{\text{int}}\}$ ***and*** the tangent stiffness $[\mathbf{k}_t]$ ***are general and made independent of*** the ***choice of strain measure***
- The ***appropriate choice of strain measure should be made to get*** the ***final form of*** the internal force vector $\{\mathbf{r}^{\text{int}}\}$ ***and*** tangent stiffness $[\mathbf{k}_t]$
- ***The values of*** the ***partial derivatives with respect to*** L ***and*** the ***final form of*** the internal force vector $\{\mathbf{r}^{\text{int}}\}$, the material stiffness $[\mathbf{k}_m]$, ***and*** the geometric stiffness $[\mathbf{k}_g]$ ***for some*** specific ***strain measures are collected in*** the ***Table below***:

Strain Measure	$\dfrac{\partial e}{\partial L}$	$\dfrac{\partial^2 e}{\partial L^2}$	$\{\mathbf{r}^{\text{int}}\}$	$[\mathbf{k}_m]$	$[\mathbf{k}_g]$
$\varepsilon_E = \dfrac{L-L_0}{L_0}$	$\dfrac{1}{L_0}$	0	$N\{\hat{\mathbf{L}}\}$	$\dfrac{EA_0}{L_0}[\hat{\mathbf{L}}\hat{\mathbf{L}}^T]$	$\dfrac{N}{L}[\hat{\mathbf{I}}-\hat{\mathbf{L}}\hat{\mathbf{L}}^T]$
$\varepsilon_G = \dfrac{L^2-L_0^2}{2L_0^2}$	$\dfrac{L}{L_0^2}$	$\dfrac{1}{L_0^2}$	$\dfrac{NL}{L_0}\{\hat{\mathbf{L}}\}$	$\dfrac{EA_0L^2}{L_0^3}[\hat{\mathbf{L}}\hat{\mathbf{L}}^T]$	$\dfrac{N}{L_0}[\hat{\mathbf{I}}]$
$\varepsilon_L = \log\left(\dfrac{L}{L_0}\right)$	$\dfrac{1}{L}$	$-\dfrac{1}{L^2}$	$\dfrac{NL_0}{L}\{\hat{\mathbf{L}}\}$	$\dfrac{EA_0L_0}{L^2}[\hat{\mathbf{L}}\hat{\mathbf{L}}^T]$	$\dfrac{NL_0}{L^2}[\hat{\mathbf{I}}-2\hat{\mathbf{L}}\hat{\mathbf{L}}^T]$

- The ***internal force vector and*** the ***geometric stiffness matrix for*** the ***Green strain measure*** thus becomes:

$$\{\mathbf{r}^{\text{int}}\} = \frac{N}{L_0}\begin{Bmatrix} -L_X \\ -L_Y \\ -L_Z \\ L_X \\ L_Y \\ L_Z \end{Bmatrix}$$

and

$$[\mathbf{k}_g] = \frac{N}{L_0}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$