# Isogeometric Analysis of Finite Deformation Solids

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# Outline

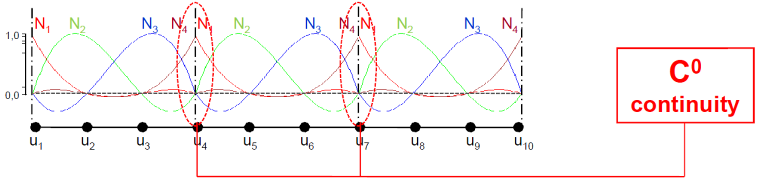
- Motivation
- NURBS vs. Lagrange FEM
- Nonlinear elasticity
- Mixed formulation
- Implementational issues
- Numerical examples
- Concluding remarks

# Motivation

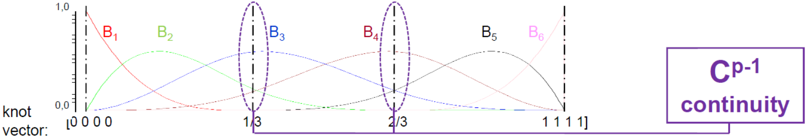
- "Locking" – a challenge in linear as well as nonlinear problems
- Volumetric locking – a challenge where nearly incompressible behavior is prevalent
- NFEA has been dominated by use of low-order elements designed to avoid volumetric or incompressible locking
- Recently the isogeometric approach has formed the basis for overcoming the incompressibility problem
  - Hughes and co-workers has addressed this by the $\bar{B}$ and $\bar{F}-$projection methods
  - Taylor improved the performance of mixed elements by using NURBS
- We have implemented two classes of mixed elements into *IFEM*, an object-oriented toolbox for performing isogeometric NFEA with splines and NURBS as basis functions

# B-splines vs Lagrange shape functions in 1D

$C^0-$ continuity $\Rightarrow$ discontinuous stress/strain fields between elements




$C^{p-1}-$ continuity $\Rightarrow$ continuous stress/strain fields for $p \geq 2$




NURBS can exactly represent conical sections

**Note:** Number of control points less than number of nodal points
$\Rightarrow$ B-Splines obtain higher accuracy vs dofs invested for Lagrange

# Comparison of FEA and IGA:

**Finite Element Analysis:**

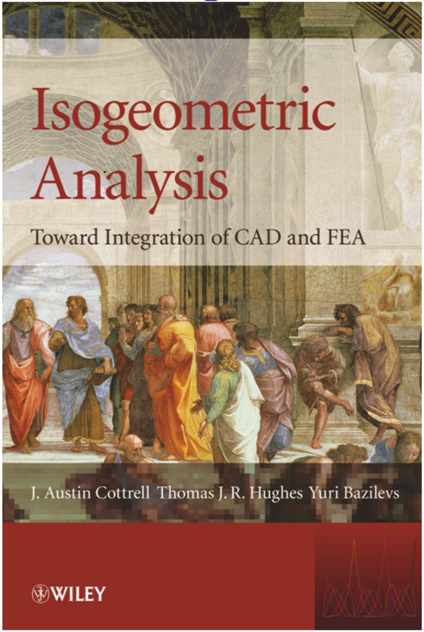
- Nodal points
- Nodal variables
- Mesh
- Lagrange basis functions
- Basis interpolate nodal points and variables
- $h-$refinement
- $p-$refinement
- Approximate geometry
- Subdomains

**Isogeometric Analysis:**

- Control points
- Control variables
- Knots
- NURBS basis functions
- Basis does not interpolate control points and variables
- Knot insertion
- Order elevation
- Exact geometry
- Patches

- Partition of unity
- Isoparametric concept
- Patch test satisfied

# Textbook on Isogeometric Analysis




ICES

Institute for Computational Engineering and Sciences

Austin, Texas, U.S.A.

The authors, which are the originators of IGA, provide us with a systematic and comprehensive coverage on how to add isogeometric capabilities to FE programs

# Constitutive equations for finite hyperelasticity

# Hyperelasticity

- We assume hyperelastic homogeneous isotropic material behavior for which there exist a free-energy function[1] $\Psi$ that depends on the left Cauchy-Green deformation tensor[2] $\mathbf{b}$

$$\Psi = \Psi(\mathbf{b}) \quad \text{with} \quad \mathbf{b} = \mathbf{F}\mathbf{F}^T \quad \text{and} \quad \mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$

  $\mathbf{F}$ is the deformation gradient, $\mathbf{u}$ is the displacement and $\mathbf{I}$ is the 2nd order unit tensor

- Cauchy stresses $\boldsymbol{\sigma}$ may be derived from the invariants of $\mathbf{b}$

$$\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial \Psi}{\partial \mathbf{b}} = \frac{2}{J}\left(\Psi_I \mathbf{b} + 2\Psi_{II}\mathbf{b}^2 + J^2\Psi_{III}\mathbf{I}\right)$$

  $\Psi_I$, $\Psi_{II}$ and $\Psi_{III}$ are the derivatives of $\Psi$ with respect to the invariants of $\mathbf{b}$ and $J = \det \mathbf{F}$; the determinant of the deformation gradient

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[1] Also called stored energy or strain energy function

[2] Also referred to as the *finger* tensor

# Compressible neo-Hookean material model

- For hyperelastic materials exhibiting a completely different volumetric and isochoric response, the free-energy function may be additively decomposed into a volume-changing (dilatational part), and a volume-preserving (isochoric part)

$$\Psi(J, \mathbf{b}) = \Psi^{\mathrm{dil}}(J) + \Psi^{\mathrm{iso}}(J, \mathbf{b})$$

- The dilatational part is expressed in terms of $J$

$$\Psi^{\mathrm{dil}}(J) = \lambda U(J) = \frac{1}{2}\lambda\left(\ln J\right)^2$$

- The isochoric part is expressed in terms of $J$ and $\mathbf{b}$

$$\Psi^{\mathrm{iso}}(J, \mathbf{b}) = \frac{1}{2}\mu\left(\mathrm{tr}\mathbf{b} - 3\right) - \mu \ln J$$

- $\lambda$ and $\mu$ are the Lame's constants that may be derived from Young's modulus, $E$, and Poisson's ratio, $\nu$

# Compressible neo-Hookean material model

- Cauchy stresses are obtained from the first derivatives of $\Psi^{\mathrm{dil}}$ and $\Psi^{\mathrm{iso}}$ w.r.t. $J$ and the first invariant of $\mathbf{b}$; $I = \mathrm{tr}\mathbf{b} = b_{kk}$

$$\begin{aligned}\sigma_{ij} =& \sigma_{ij}^{\mathrm{dil}} + \sigma_{ij}^{\mathrm{iso}} = \left(\lambda \frac{\partial U}{\partial J} + \frac{\partial \Psi^{\mathrm{iso}}}{\partial J}\right)\delta_{ij} + \frac{2}{J} b_{ij} \frac{\partial \Psi^{\mathrm{iso}}}{\partial I} \\ =& \frac{1}{J}\left[\mu b_{ij} + \left(\lambda \ln J - \mu\right)\delta_{ij}\right]\end{aligned}$$

- Spatial tangent moduli are similarly obtained from the second derivatives

$$c_{ijkl} = c_{ijkl}^{\mathrm{dil}} + c_{ijkl}^{\mathrm{iso}} = \frac{1}{J}\left[\lambda \delta_{ij}\delta_{kl} + 2\left(\mu - \lambda \ln J\right)\mathcal{I}_{ijkl}\right]$$

where $\mathcal{I}_{ijkl} = \frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]$

# Modified neo-Hookean material model

- Rubber-like materials are characterized by relatively low shear modulus and high bulk modulus $\Rightarrow$ they are nearly incompressible while highly deformable when sheared
- Multiplicative split of the deformation gradient into a volume-changing (dilatational) and volume-preserving (isochoric) part

$$\mathbf{F} = \mathbf{F}^{\mathrm{dil}}\mathbf{F}^{\mathrm{iso}} \qquad \begin{cases} \mathbf{F}^{\mathrm{dil}} = J^{1/3}\mathbf{I} & \Rightarrow \quad \det \mathbf{F}^{\mathrm{dil}} = \det \mathbf{F} = J \\ \mathbf{F}^{\mathrm{iso}} = J^{-1/3}\mathbf{F} & \Rightarrow \quad \det \mathbf{F}^{\mathrm{iso}} = 1 \end{cases}$$

- A modified deformation gradient $\bar{\mathbf{F}}$ is obtained by replacing $J$ with the scalar parameter $\overline{J}$ in the dilatational part

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}^{\mathrm{dil}}\mathbf{F}^{\mathrm{iso}} = \left(\frac{\overline{J}}{J}\right)^{1/3}\mathbf{F} \quad \text{where} \quad \bar{\mathbf{F}}^{\mathrm{dil}} = \overline{J}^{1/3}\mathbf{I} \Rightarrow \det \bar{\mathbf{F}} = \overline{J}$$

# Modified neo-Hookean material model

- Using the multiplicative split the isochoric part of the finger tensor becomes

$$\bar{\mathbf{b}} = \mathbf{b}^{\mathrm{iso}} = \mathbf{F}^{\mathrm{iso}}(\mathbf{F}^{\mathrm{iso}})^T = J^{-2/3}\mathbf{b}$$

- The isochoric part of the free-energy function may know be written in terms of the modified invariant $\bar{I} = \mathrm{tr}\bar{\mathbf{b}} = J^{-2/3}\mathrm{tr}\mathbf{b}$

$$\Psi(J, \bar{I}) = \Psi^{\mathrm{dil}}(J) + \Psi^{\mathrm{iso}}(\bar{I})$$

where

$$\begin{aligned}\Psi^{\mathrm{dil}}(J) &= \kappa U(J) = \tfrac{1}{4}\kappa\left(J^2 - 1 - 2\ln J\right)\\ \Psi^{\mathrm{iso}}(\bar{I}) &= \tfrac{1}{2}\mu\left(\bar{I} - 3\right)\end{aligned}$$

- $\kappa$ and $\mu$ are equivalent to the small strain bulk and shear modulus, respectively

# Modified neo-Hookean material model

- The volumetric part of the Cauchy stresses for the above volumetric behavior gives rise to the hydrostatic pressure

$$\sigma_{ij}^{\mathrm{dil}} = \kappa \frac{\partial U}{\partial J} \delta_{ij} = \frac{\kappa}{2J}(J^2 - 1)\delta_{ij}$$

- The deviatoric part now may be expressed in terms of the modified deformation tensor $\bar{b}_{ij}$

$$\sigma_{ij}^{\mathrm{iso}} = \frac{\mu}{J} \bar{b}_{ij}^{\mathrm{d}} \quad \text{where} \quad \bar{b}_{ij}^{\mathrm{d}} = \bar{b}_{ij} - \frac{1}{3}\delta_{ij}\bar{b}_{kk}$$

- Current spatial tangent moduli for the modified neo-Hookean material model

$$c_{ijkl} = c_{ijkl}^{\mathrm{dil}} + c_{ijkl}^{\mathrm{iso}}$$

where

$$c_{ijkl}^{\mathrm{dil}} = \tfrac{\kappa}{J}\left[J^2\delta_{ij}\delta_{kl} + (1 - J^2)\mathcal{I}_{ijkl}\right]$$

$$c_{ijkl}^{\mathrm{iso}} = \tfrac{2\mu}{3J}\left[\bar{b}_{mm}(\mathcal{I}_{ijkl} - \tfrac{1}{3}\delta_{ij}\delta_{kl}) - \delta_{ij}\bar{b}_{kl}^{\mathrm{d}} - \bar{b}_{ij}^{\mathrm{d}}\delta_{kl}\right]$$

# Variational and discrete formulation of the finite deformation problem

# Mixed formulation

- A three-field mixed approximation has led to successful lower-order solid elements that can be used in finite deformation problems that exhibit compressible and/or nearly incompressible behavior for a large class of materials

$$\Pi(\mathbf{u}, p, \theta) = \int_\Omega \Psi(J, \bar{\mathbf{b}})\mathrm{d}\Omega + \int_\Omega p(J - \overline{J})\mathrm{d}\Omega - \Pi_{\mathrm{ext}}$$

- $p$ is a Lagrange multiplier that constrains $J$ to its independent representation, denoted $\overline{J}$. $p$ may be identified as the Cauchy mean or hydrostatic stress

$$\sigma_{ij}^{\mathrm{dil}} = p\delta_{ij}$$

- For computations we let $\overline{J}$ be related to $\theta$ through

$$\overline{J} = 1 + \theta \Rightarrow \theta = 0 \quad \mathrm{in} \quad \mathcal{C}_0$$

# Linearized discrete form of the variational equations

- If we approximate the volume change $\theta$ and the pressure $p$ by interpolation functions in reference coordinates $\mathbf{X}$

$$\theta = \sum_{b=1}^{n_\theta} L_b(\mathbf{X})\tilde{\theta}_b = \mathbf{L}\tilde{\boldsymbol{\theta}} \quad \text{and} \quad p = \sum_{b=1}^{n_p} M_b(\mathbf{X})\tilde{p}_b = \mathbf{M}\tilde{\mathbf{p}}$$

the linearized discrete form of the variational equation reads

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\theta} & \mathbf{K}_{up} \\ \mathbf{K}_{\theta u} & \mathbf{K}_{\theta\theta} & \mathbf{K}_{\theta p} \\ \mathbf{K}_{pu} & \mathbf{K}_{p\theta} & \mathbf{0} \end{bmatrix} \left\{ \begin{array}{c} d\tilde{\mathbf{u}} \\ d\tilde{\boldsymbol{\theta}} \\ d\tilde{\mathbf{p}} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{R}_u \\ \mathbf{R}_\theta \\ \mathbf{R}_p \end{array} \right\}$$

- Residuals are expressed as sums over elements as

$$\begin{aligned} \mathbf{R}_u &= \mathbf{f} - \sum_e \int_{\Omega_e} \mathbf{B}^T \widehat{\boldsymbol{\sigma}} \bar{J} \mathrm{d}\Omega \\ \mathbf{R}_\theta &= \sum_e \int_{\Omega_e} \mathbf{L}^T (\bar{p} - p) \mathrm{d}\Omega \\ \mathbf{R}_p &= \sum_e \int_{\Omega_e} \mathbf{M}^T (J - \bar{J}) \mathrm{d}\Omega \end{aligned} \quad \text{where} \quad \begin{aligned} \widehat{\boldsymbol{\sigma}} &= \bar{\boldsymbol{\sigma}} + \mathbf{m}(\hat{p} - \bar{p}) \\ \bar{p} &= \tfrac{1}{3}\mathbf{m}^T \bar{\boldsymbol{\sigma}} \\ \hat{p} &= (J/\bar{J})p \\ \mathbf{m}^T &= [1, 1, 1, 0, 0, 0] \end{aligned}$$

# Discontinuous $\theta - p$ approximations

- Approximations for $\theta$ and $p$ are identical ($\mathbf{L} = \mathbf{M}$) and assumed to be discontinuous between contiguous elements

$$\Rightarrow \tilde{\boldsymbol{\theta}} \text{ and } \tilde{\mathbf{p}} \text{ are condensed out on the element level}$$

- Direct solution $\Rightarrow \mathbf{R}^e_\theta$ and $\mathbf{R}^e_p$ vanish and the linearized form is reduced to

$$\bar{\mathbf{K}}_{uu} d\tilde{\mathbf{u}} = \mathbf{R}_u$$

  where

$$\bar{\mathbf{K}}_{uu} = \mathbf{K}_{uu} + \mathbf{K}_{up}\mathbf{K}^{-1}_{\theta p}\mathbf{K}_{\theta\theta}\mathbf{K}^{-1}_{p\theta}\mathbf{K}_{pu} - \mathbf{K}_{u\theta}\mathbf{K}^{-1}_{\theta p}\mathbf{K}_{pu} - \mathbf{K}_{up}\mathbf{K}^{-1}_{\theta p}\mathbf{K}_{\theta u}$$

- An efficient procedure to compute the reduced tangent may be found in[3]

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[3] Zienkiewics, O.C. and Taylor, R.L. The Finite Element Method for Solid and Structural Mechanics (6th ed, Elsevier, 2005)

# $Q_p/P_{p-1}$ and $Q_p/Q_{p-1}$ mixed formulations

- Implemented and studied two different constraint approximations based on the three-field variational form:

$Q_p/P_{p-1}$: **u** continuous of order $p$ with $\mathcal{C}^{p-1}$ continuity on "patches", $\theta$ and $p$ discontinuous of order $p-1$

**Note:** $\theta$ and $p$ are expanded in individual Lagrange elements whereas for Splines $\theta$ and $p$ expanded in individual knot-spans $\Rightarrow$ as the polynomial order increases the pressure space for Splines increases compared to Lagrange

$Q_p/Q_{p-1}$: **u** continuous of order $p$ with $\mathcal{C}^{p-1}$ continuity on "patches", $\theta$ and $p$ also continuous, but of order $p-1$ with $\mathcal{C}^{p-2}$ continuity on "patches"

# Babuška–Brezzi condition – Volumetric locking

- To avoid volumetric locking the Babuška–Brezzi condition must be satisfied

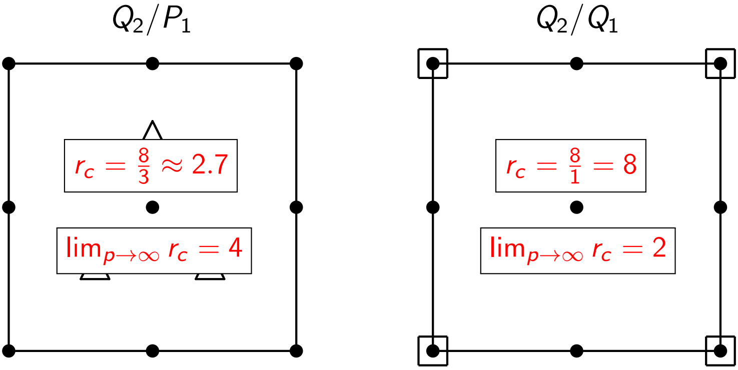
$$n_u \geq n_\theta = n_p$$

  where $n_u$, $n_\theta$ and $n_p$ denote the number of unknown displacement $\tilde{\mathbf{u}}$, volume $\tilde{\boldsymbol{\theta}}$, and pressure parameters $\tilde{\mathbf{p}}$
- In order to predict the propensity of volumetric locking, we define the constraint ratio

$$r_c = \frac{n_u}{n_p} = \frac{n_u}{n_\theta}$$

- The ideal value of the ratio $r_c$ would then be the ratio between number of equilibrium equations ($= n_{sd}$), divided by number of incompressibility conditions (=1)
  $\Rightarrow r_c = n_{sd} \Rightarrow$ the ideal ratio would be $r_c = 2$ in 2D
- If $r_c < n_{sd}$ volumetric locking may be anticipated

# $Q_2/P_1$ and $Q_2/Q_1$ mixed Lagrange elements

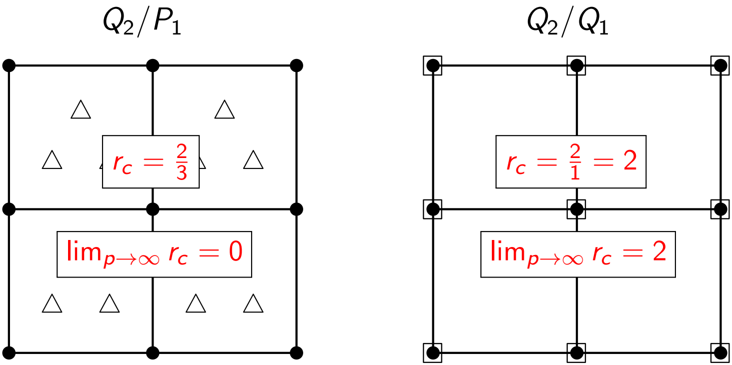



● = Displacement node

□ = Pressure/volume change node

△ = Internal pressure/volume change node

# $Q_2/P_1$ and $Q_2/Q_1$ mixed Spline elements




$\bullet$ = Displacement node

$\square$ = Pressure/volume change node

$\triangle$ = Internal pressure/volume change node

# Implementational issues

- *IFEM* : Object-oriented toolbox for isogeometric FE analysis
  - Problem-independent computational core
  - 2D and 3D continuum formulations, Kirchoff–Love thin plate and shell formulations
  - Lagrange, Spectral, Splines and NURBS basis functions
- Basis function evaluation (Splines/NURBS) based on GoTools
  `http://www.sintef.no/Projectweb/Geometry-Toolkits/GoTools`
- Element-level linear algebra: Use machine-optimized **BLAS**
  - For higher-order elements, the element matrices become large
  - Important to express the nonlinear FE formulation on *matrix* form (Voigt notation) — not *tensor* form
- Linear equation solvers
  - SuperLU (direct methods) `http://crd.lbl.gov/~xiaoye/SuperLU`
  - PETSc (iterative methods) `http://www.mcs.anl.gov/petsc`
  - Parallelization in progress (based on MPI message passing)

# Implementing the weak form (reference config.)

- Using tensor notation:

$$k_{mn}^{ab} = \int_{\Omega_0} N_{a,i} F_{mj} C_{ijkl} F_{nk} N_{b,l} \mathrm{d}\Omega$$

- Gives 8 nested loops, within the integration point loop!!

```
for (a = 1; a <= N.size(); a++)
  for (b = 1; b <= N.size(); b++)
    for (m = 1; m <= nsd; m++)
      for (n = 1; n <= nsd; n++)
      {
        double km = 0.0;
        for (i = 1; i <= nsd; i++)
          for (j = 1; j <= nsd; j++)
            for (k = 1; k <= nsd; k++)
              for (l = 1; l <= nsd; l++)
                km += dNdX(a,i)*F(m,j)*D(i,j,k,l)*F(n,k)*dNdX(b,l);

        EM(nsd*(a-1)+m,nsd*(b-1)+n) += km*detJW;
      }
```

- 11400 DOFs $\Rightarrow$ 48s CPU time for one element assembly step (89% of total simulation time) T = 2665s

- Using Voigt notation:

$$\mathbf{k} = \int_{\Omega_0} \mathbf{B}^T \mathbf{D}_T \mathbf{B} \mathrm{d}\Omega$$

- Implemented by two calls to BLAS-subroutine `DGEMM`

- 11400 DOFs $\Rightarrow$ 19s CPU time for one element assembly step (82% of total simulation time) T = 1176s

# Numerical examples

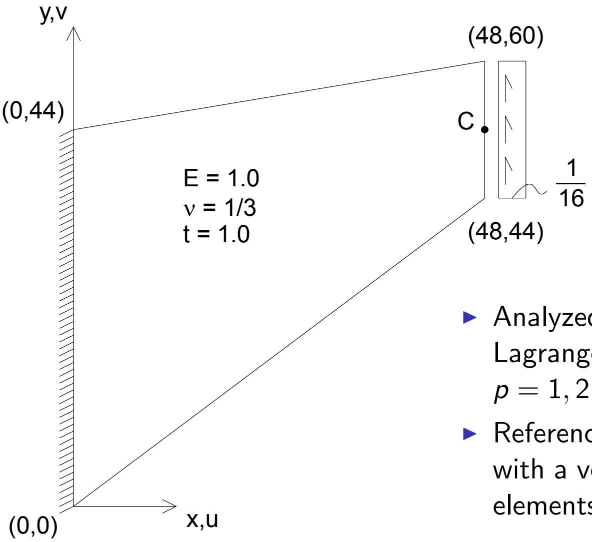
# Numerical examples

- The performance of the three-field mixed forms:
  - $Q_p/P_{p-1}$ : discontinuous $p$ and $\theta$, and
  - $Q_p/Q_{p-1}$ : continuous $p$ and $\theta$

  is numerically assessed and compared to the one-field $Q_p$ displacement formulation with Splines and Lagrange basis functions

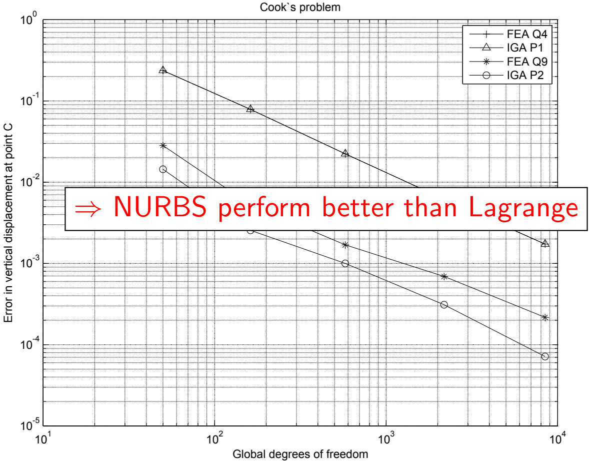
- The accuracy and the convergence characteristics are assessed in the finite deformation regime for elastic and elasto-plastic materials

# Cook's problem: Linear plane stress problem



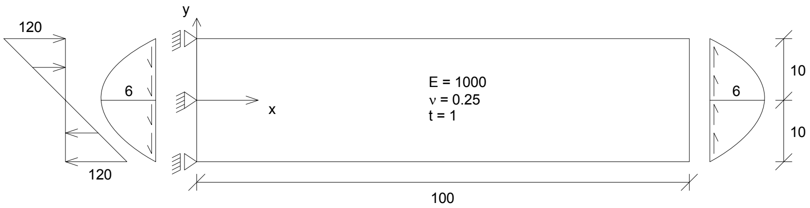

- Analyzed with NURBS and Lagrange $Q_p$ elements, $p = 1, 2$
- Reference solution obtained with a very fine mesh of LST elements (ndof $= 41586$)

# Cook's problem: Error in vertical tip displacement, $v_C$



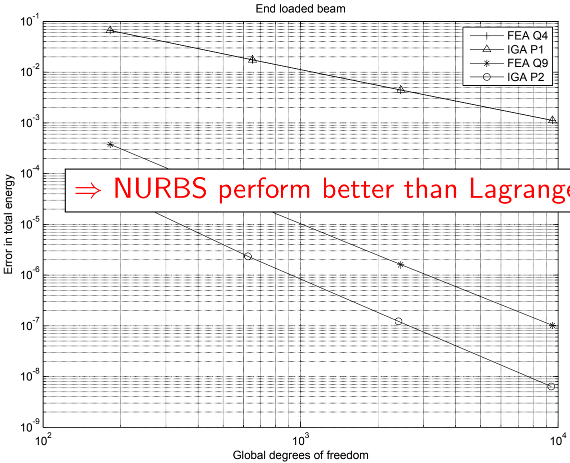

- As expected NURBS and Lagrange elements of polynomial order $p = 1$ coincide
- NURBS converge at the same rate but is more accurate than Lagrange elements of polynomial order $p = 2$

# End loaded cantilever beam: Linear plane stress problem




- Exact solution may be obtained for this particular problem
- Analyzed with NURBS and Lagrange $Q_p$ elements, $p = 1, 2$
- Note: In order to be compatible with the exact solution, a parabolic transverse traction field acting downward and a normal traction field equivalent to the transverse shear force and the moment, respectively, must be applied to the supported end

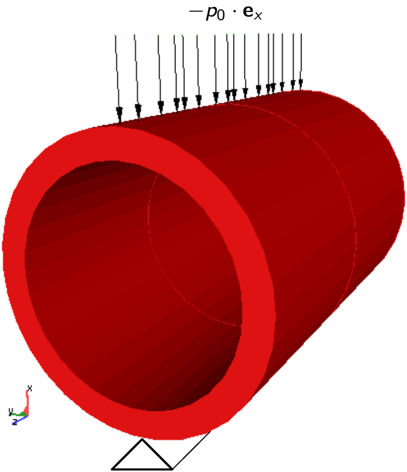
# End loaded cantilever beam: Error in potential energy




$\Rightarrow$ NURBS perform better than Lagrange !

- As expected NURBS and Lagrange elements of polynomial order $p = 1$ coincide
- NURBS converge at the same rate but is more accurate than Lagrange elements of polynomial order $p = 2$

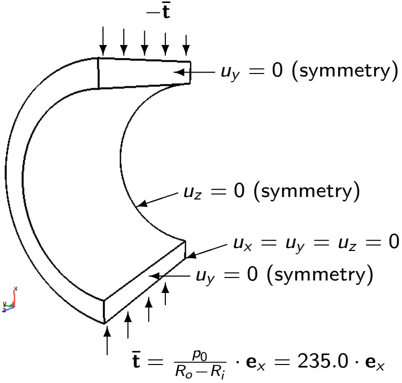
# Compression of a Thick Cylinder




| | | |
|---|---|---|
| Length | : | $L = 30.0$ |
| Outer radius | : | $R_o = 10.0$ |
| Inner radius | : | $R_i = 8.0$ |
| Young's modulus | : | $E = 16800$ |
| Poisson's ratio | : | $\nu = 0.4$ |
| Load intensity | : | $p_0 = 470.0$ |

Note: Quadratic NURBS describe exact geometry in $\mathcal{C}_0$ !

# Compression of a Thick Cylinder



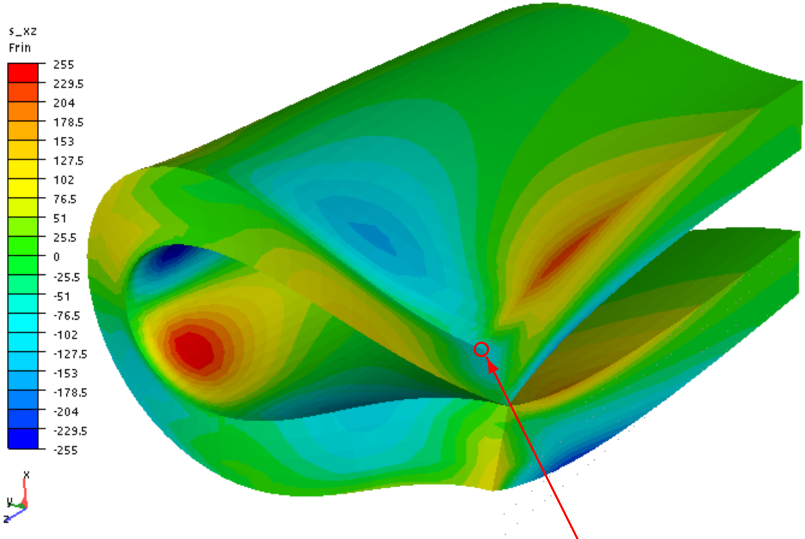

- Due to symmetry, only one quarter is modelled
- One element over the thickness, varying in length and circumferential direction
- NURBS and Lagrange $Q_p$ elements, $p = 2, 3, 4$
- Compressible neo-Hookean material model

$$\Psi(J, \mathbf{b}) = \frac{1}{2}\mu(\mathrm{tr}\mathbf{b} - 3) - \mu \ln J + \frac{1}{2}\lambda(\ln J)^2$$

- Load is here applied as two oppositely directed tangential tractions

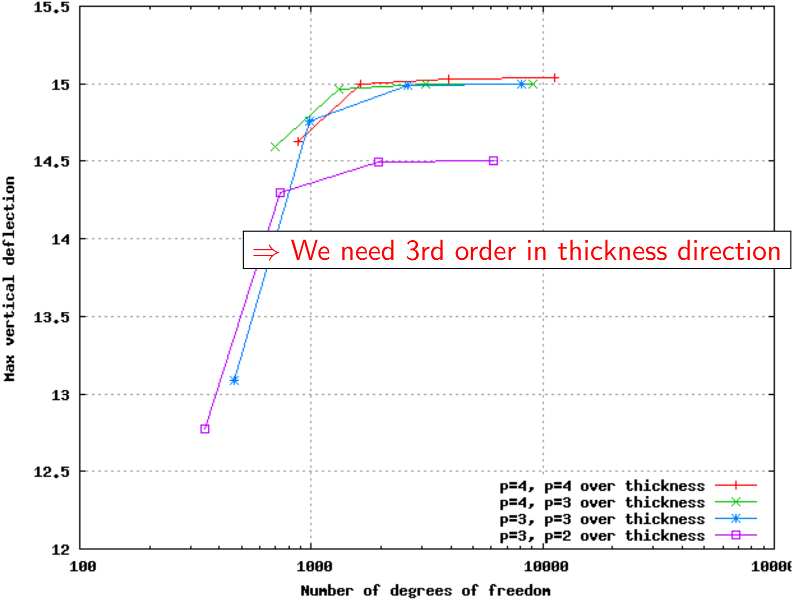
# Compression of a Thick Cylinder

Deformed configuration with the Cauchy stress $\sigma_{xz}$



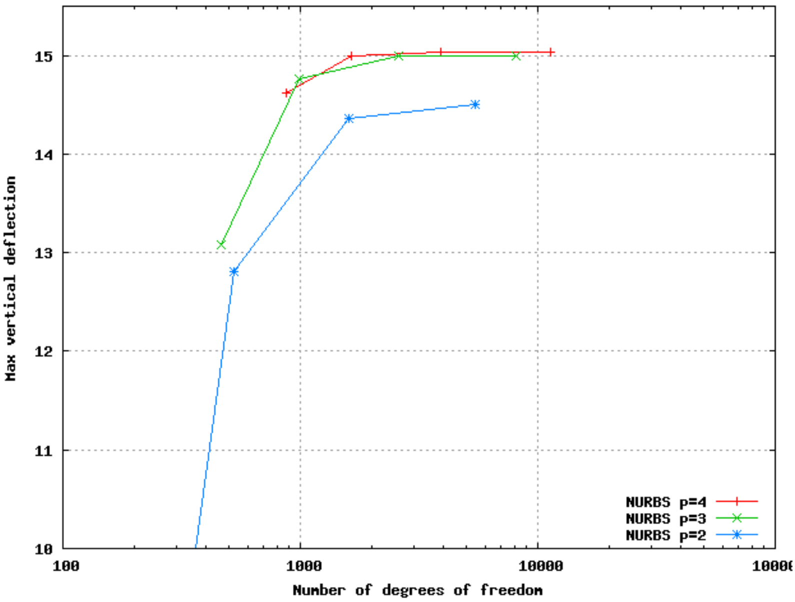

Let's look at the max vertical displacement, at this point

# Considering basis order through thickness (NURBS)

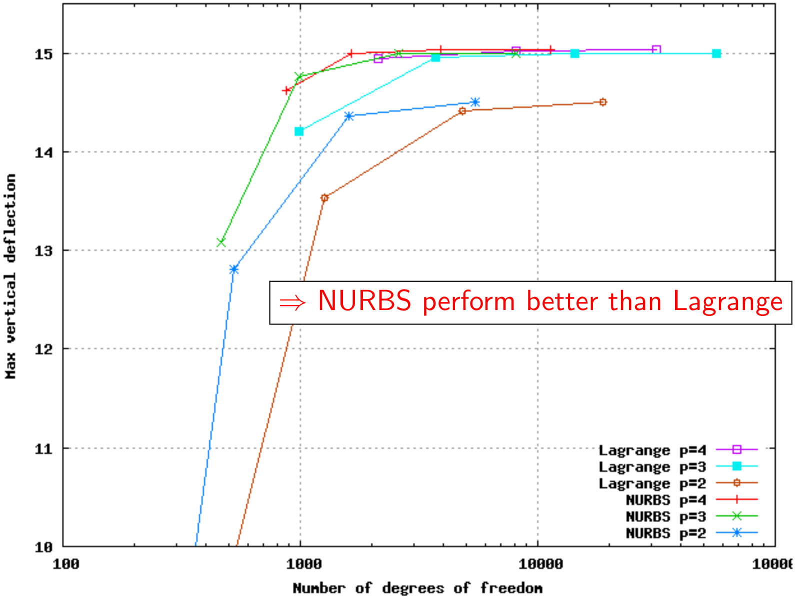



⇒ We need 3rd order in thickness direction

# Convergence for NURBS

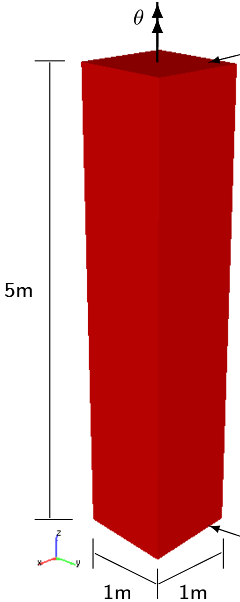


# Convergence for NURBS and Lagrange




⇒ NURBS perform better than Lagrange

# Torsion of a square column




$$u_x = \sin\theta(x - x_0) + \cos\theta(y - y_0) - (x - x_0)$$
$$u_y = \cos\theta(x - x_0) - \sin\theta(y - y_0) - (y - y_0)$$
$$u_z = 0$$

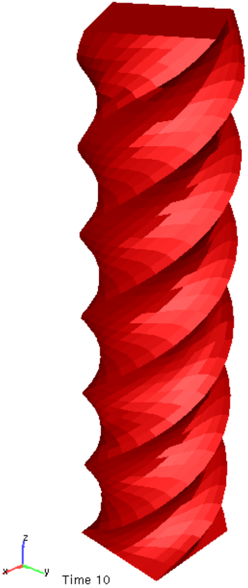
Modified neo-Hookean material model:

$$\Psi(J, \bar{I}) = \tfrac{1}{4}\kappa(J^2 - 1 - 2\ln J) + \tfrac{1}{2}\mu(\bar{I} - 3)$$

| | | | | |
|---|---|---|---|---|
| Bulk modulus | : | $\kappa$ | $=$ | $400942.0\ N/m^2$ |
| Shear modulus | : | $\mu$ | $=$ | $80.1938\ N/m^2$ |
| $\Rightarrow$ Poisson's ratio | : | $\nu$ | $=$ | $0.4999$ |

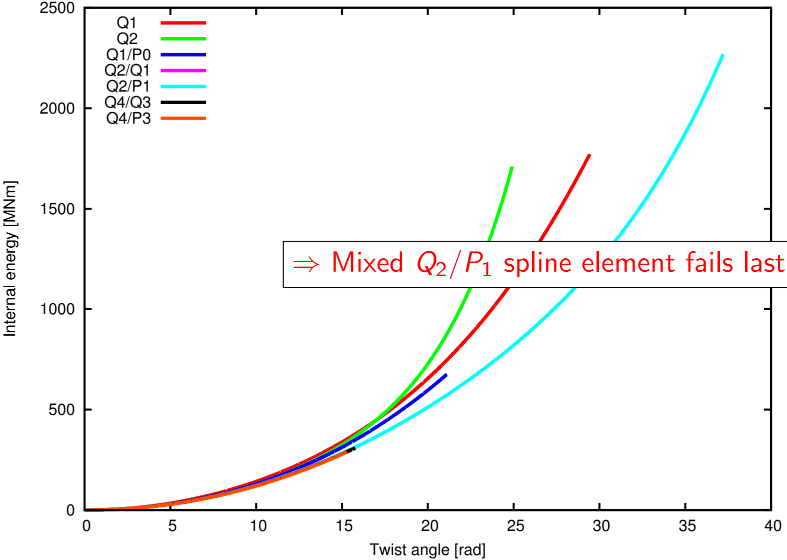
$$u_x = u_y = u_z = 0$$

# Torsion of a square column: $Q_2/P_1$ ($\theta = 10.0\,\mathrm{rad} \approx 573^o$)




- $5 \times 5 \times 17$ control points $\Rightarrow$ 1125 DOFs
- Analyzed with both $Q_p$, $Q_p/P_{p-1}$ and $Q_p/Q_{p-1}$ elements ($p = 1, 2, 3, 4$)
- Increment size: $d\theta = 0.01$

# Stored elastic strain energy for spline elements




$\Rightarrow$ Mixed $Q_2/P_1$ spline element fails last

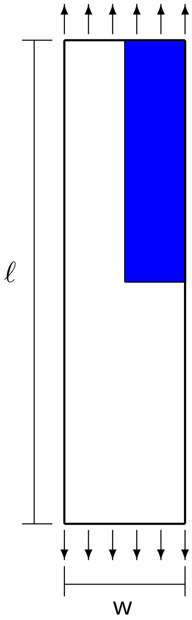
# Torsion of a square column: Failure angle ($\theta_f$)

| Approx. | Grid | Formulation | $\theta_f$ [rad] |
|---|---|---|---|
| Lagrange/ Splines | $4 \times 4 \times 16$ | Q1<br>Q1/P0 | 29.45<br>21.12 |
| Lagrange | $2 \times 2 \times 8$ | Q2<br>Q2/P1 | 10.39<br>19.22 |
| Splines | $3 \times 3 \times 15$ | Q2<br>Q2/P1 | 24.93<br>37.22 |
| Splines | $2 \times 2 \times 14$ | Q3<br>Q3/P2 | 23.86<br>27.36 |
| Lagrange | $1 \times 1 \times 4$ | Q4<br>Q4/P3 | 0.12<br>14.96 |
| Splines | $1 \times 1 \times 13$ | Q4<br>Q4/P3 | 0.27<br>15.30 |

Number of degrees of freedom = 1125 for all grids !

Prescribed incremental size $d\theta = 0.01$ [rad] for all analyses !

# Necking of elastic-plastic tension strip:




- Geometry:

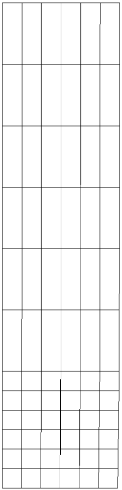
| | | | |
|---|---|---|---|
| Length | : | $\ell$ = | 53.334 |
| Width | : | $w$ = | 12.826 |
| Center width | : | $w_c$ = | $0.982w$ |

- Finite strain elastic-plastic model with a $J_2$ yield function; uniaxial yield stress given by:

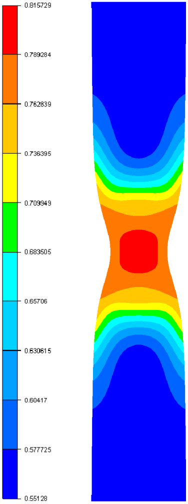
$$\sigma_y = \sigma_\infty + (\sigma_0 - \sigma_\infty)\exp(-\beta e_p) + \sqrt{\frac{2}{3}}he_p$$

| | | | |
|---|---|---|---|
| Bulk modulus | : | $\kappa$ = | 164.206 |
| Shear modulus | : | $\mu$ = | 80.1938 |
| Initial yield stress | : | $\sigma_0$ = | 0.45 |
| Residual yield stress | : | $\sigma_\infty$ = | 0.715 |
| Isotropic hardening | : | $h$ = | 0.12924 |
| Saturation exponent | : | $\beta$ = | 16.93 |
| Effective plastic strain | : | $e_p$ | |

# Necking of elastic-plastic tension strip:

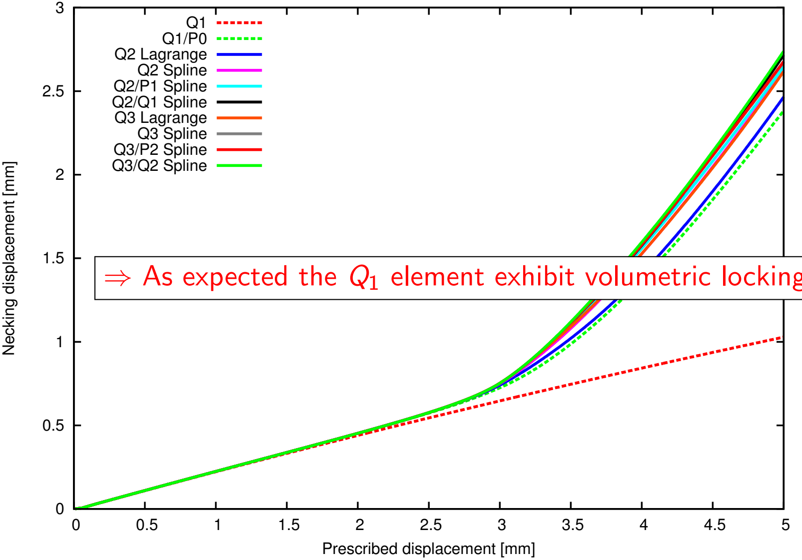


a) Mesh for one quadrant of the strip with 6 knot spans in the width and 12 in length $\Rightarrow 7 \times 13$ grid points (basis functions)



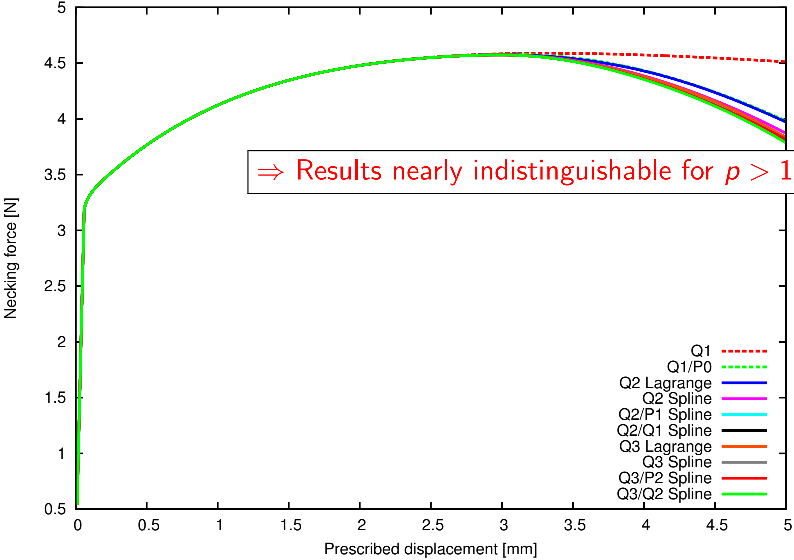

b) Mises stress distribution at final configuration obtained with the $Q_3/P_2$ spline element with a $49 \times 97$ grid

# Necking of elastic-plastic tension strip : $7 \times 13$ grid points



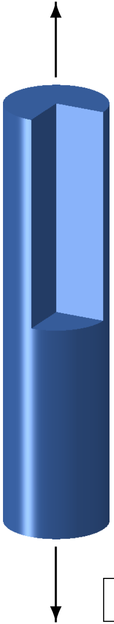

$\Rightarrow$ As expected the $Q_1$ element exhibit volumetric locking !

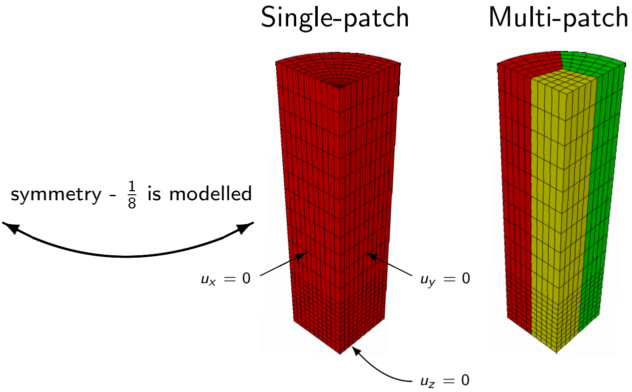
# Necking of elastic-plastic tension strip : 7 $\times$ 13 grid points




$\Rightarrow$ Results nearly indistinguishable for $p > 1$ !

# Necking of elastic-plastic cylinder






Cylinder length : $\ell = 53.334$

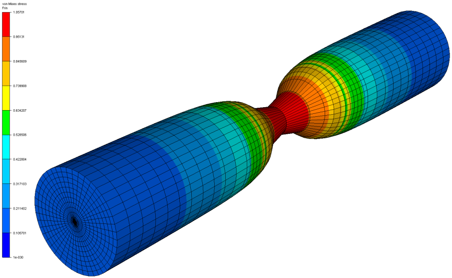
Cylinder diameter : $d = 12.826$

Center diameter : $d_c = 0.982d$

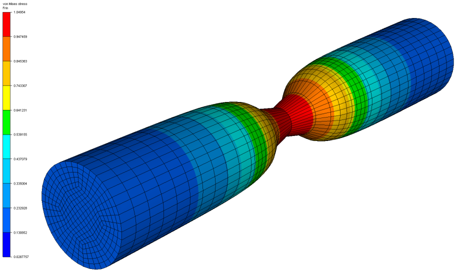
Note: Quadratic NURBS describe exact geometry in $\mathcal{C}_0$ !

# Necking of elastic-plastic cylinder

Single-patch analysis

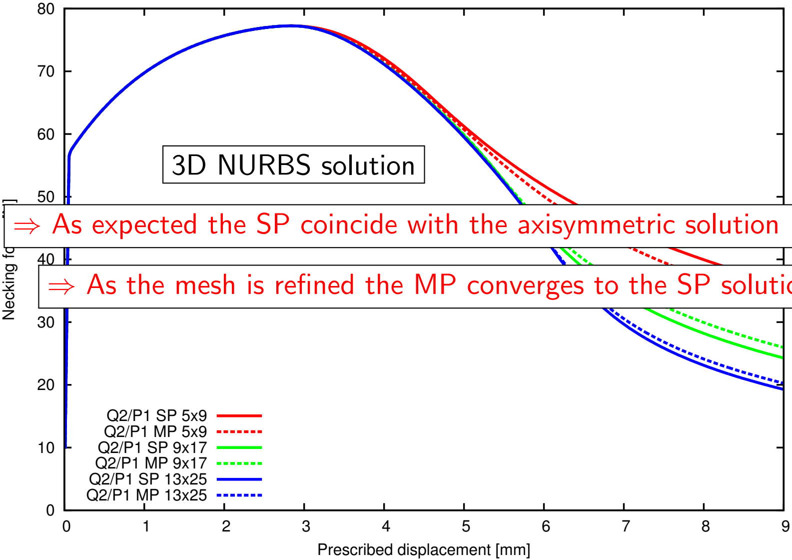



Multi-patch analysis




- Mises stress distribution at final configuration
- Three dimensional analysis
- Approximation: $Q_2$ NURBS element
- Discretization: $13 \times 25$ control points

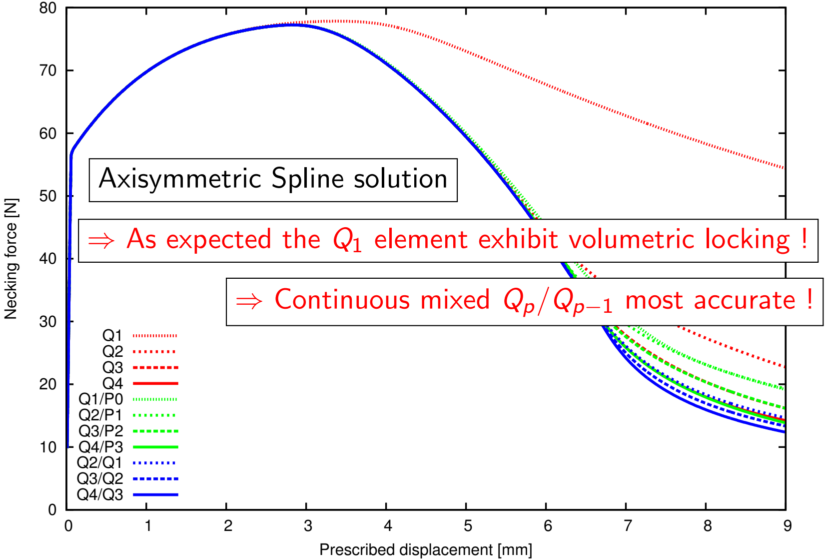
# Necking of elastic-plastic cylinder : $Q_2/P_1$ NURBS element



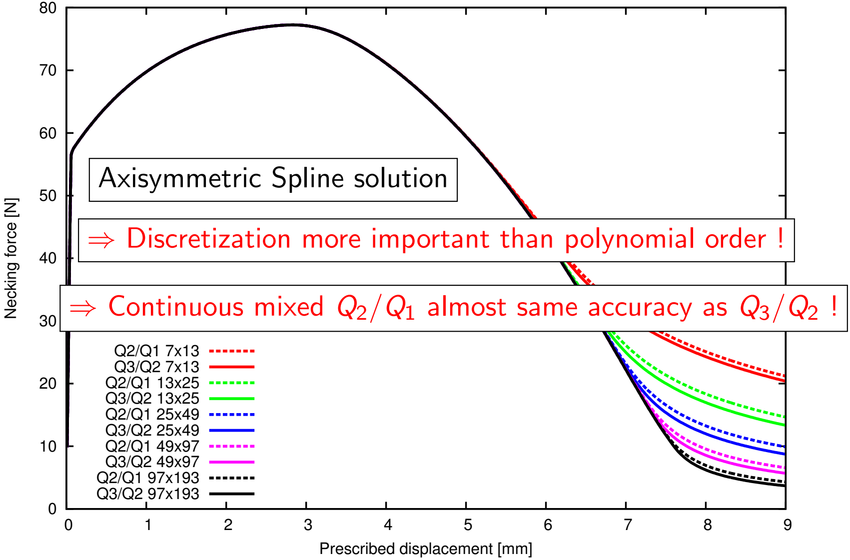

$\Rightarrow$ As expected the SP coincide with the axisymmetric solution !

$\Rightarrow$ As the mesh is refined the MP converges to the SP solution !

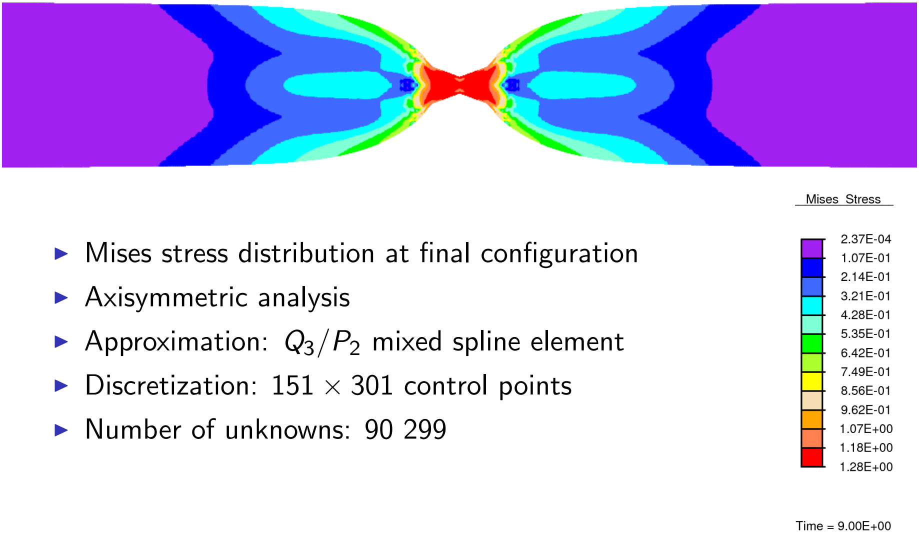
# Necking of elastic-plastic cylinder : 13 $\times$ 25 grid points



# Necking of elastic-plastic cylinder : $Q_p/Q_{p-1}; p = 2, 3$



# Necking of elastic-plastic cylinder




- Mises stress distribution at final configuration
- Axisymmetric analysis
- Approximation: $Q_3/P_2$ mixed spline element
- Discretization: $151 \times 301$ control points
- Number of unknowns: 90 299

# Concluding remarks:

- By means of numerical examples, the performance of isogeometric Splines and NURBS elements have been assessed on problems involving *compressible* and *nearly incompressible* hyperelasticity and finite multiplicative plasticity
- A remarkable ability in capturing strain localization phenomena has been verified for both *plane strain*, *axisymmetric* and *three-dimensional* isogeometric solid elements
- While *continuous* mixed $Q_p/Q_{p-1}$ is preferable for both nearly incompressible elastic materials and finite strain plasticity
- *Discontinuous* mixed $Q_p/P_{p-1}$ is far more efficient in terms of computer resources
- Further work:
  - A more detailed study of the 3D necking problem with *adaptively refined* meshes in the necking zone
  - Isogeometric finite element analysis of *thin-walled* structures

*Thank you for your attention!*