

# ISO GUM, Uncertainty Quantification, and Philosophy of Statistics

Gunnar Taraldsen  
Acoustics Research Centre  
NTNU and SINTEF  
January 21th 2015



---

## **GUIDE 98-3**

---

### **Uncertainty of measurement —**

Part 3:

### **Guide to the expression of uncertainty in measurement (GUM:1995)**

*Incertitude de mesure —*

*Partie 3: Guide pour l'expression de l'incertitude de  
mesure (GUM:1995)*

# Abstract and history

In 1978, recognizing the lack of international consensus on the expression of uncertainty in measurement, the world's highest authority in metrology, the Comité International des Poids et Mesures (CIPM), requested the Bureau International des Poids et Mesures (BIPM) to address the problem in conjunction with the national standards laboratories and to make a recommendation.

As a result the International Organization for Standardization (ISO) published the first version of the Guide to the expression of Uncertainty of Measurements (GUM) in 1993.

This session will discuss uncertainty quantification in relation to the ISO GUM, which states:

**"Just as the nearly universal use of the International System of Units (SI) has brought coherence to all scientific and technological measurements, a worldwide consensus on the evaluation and expression of uncertainty in measurement would permit the significance of a vast spectrum of measurement results in science, engineering, commerce, industry, and regulation to be readily understood and properly interpreted. In this era of the global marketplace, it is imperative that the method for evaluating and expressing uncertainty be uniform throughout the world so that measurements performed in different countries can be easily compared."**

Additionally, we will discuss the relation between uncertainty quantification and the philosophical foundations of statistics.

# What is a measurement?

- Measurement of temperature outside of the hotell.
- Measurements of height to obtain a terrain model.
- .....
- If  $X$  is the result of a measurement, then  $Y=f(X)$  is a measurement.
- Any function applied on a measurement is itself a measurement.
- The function can be simple as exemplified by  $Y=1/X$ , or it can be more complicated.
- The mapping from the initial data of a system of conservation equations into the solution of the equations is a more complicated example.

# The ISO GUMs main message.

## Always report:

- **Estimate**
- **Standard uncertainty**
- **Expanded uncertainty**
- **Credibility level**

# ISO and GUM [1, British foreword]

The International Organization for Standardization (ISO) requires that the 1993 edition of the Guide to the expression of uncertainty in measurement (GUM) be referenced when writing standards concerning the expression of uncertainty in measurement. The purpose of such guidance is:

- (1) to promote full information on how uncertainty statements are arrived at;
- (2) to provide a basis for the international comparison of measurement results.

# ISO GUM introduction [1]

Just as the nearly universal use of the **International System of Units (SI)** has brought coherence to all scientific and technological measurements, a worldwide consensus on the evaluation and expression of uncertainty in measurement would permit the significance of a vast spectrum of measurement results in science, engineering, commerce, industry, and regulation to be readily understood and properly interpreted. In this era of the global marketplace, it is imperative that the method for evaluating and expressing uncertainty be uniform throughout the world so that measurements performed in different countries can be easily compared.



# ISO GUM introduction [1]

- The ideal method for evaluating and expressing the uncertainty of the result of a measurement should be:
  - **universal**: the method should be applicable to all kinds of measurements and to all types of input data used in measurements; The actual quantity used to express uncertainty should be:
  - **internally consistent**: it should be directly derivable from the components that contribute to it, as well as independent of how these components are grouped and of the decomposition of the components into subcomponents;
  - **transferable**: it should be possible to use directly the uncertainty evaluated for one result as a component in evaluating the uncertainty of another measurement in which the first result is used.

# The 8 step GUM procedure [1,8.1-8]

1. Define the measurand  $Y = f(X_1, X_2, \dots, X_N)$
  2. Determine input quantity ( $x_i$ )
  3. Evaluate the standard uncertainty  $u(x_i)$
  4. Evaluate the covariances
  5. Calculate  $y = f(x_1, x_2, \dots, x_N)$
  6. Evaluate the combined standard uncertainty  $u_c(y)$
  7. Evaluate the expanded uncertainty  $U$
  8. Report  $y, u_c(y), U(y)$  with level
- I would prefer  $u(y)$ . The  $x_i$ 's and the  $y$  are conceptually similar.

# Interpretation of the ISO GUM [1]

## **standard uncertainty**

uncertainty of the result of a measurement expressed as a standard deviation

## **type A evaluation (of uncertainty)**

method of evaluation of uncertainty by the statistical analysis of series of observations

## **type B evaluation (of uncertainty)**

method of evaluation of uncertainty by means other than the statistical analysis of series of observations

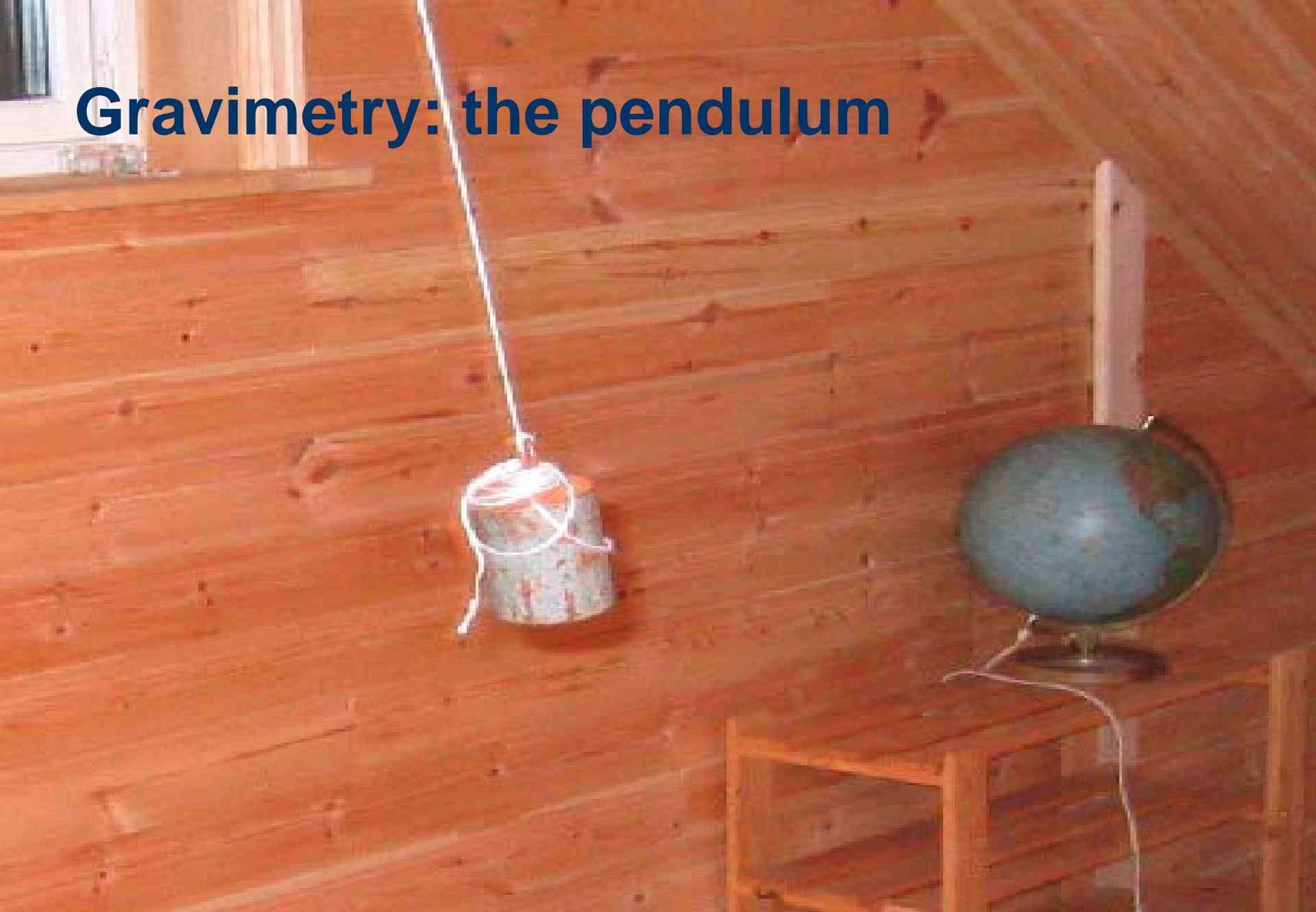
# Taraldsen: On the blackboard...

- Noise mapping example: Terrain, sources, meteorology, ...
- MC evaluation of uncertainty (GUM supplement)
- Elementary evaluation of uncertainty
- Bayes, Frequentist, Fiducial statistical philosophies
- Improper priors
- Mapping of probability distributions.
- Uncertainty on spheres, circles, manifolds: Variance???
- Confidence distributions: What is uncertainty?
- Linear system and error: Not always least squares solution!
- Savage book on decision theory: Estimation of error is meaningless!
- Mishra: Example of a different kind of mapping of a probability measure into a new probability measure. Foundation?
- MC evaluation of statistics versus drawing from distribution.
- Experimental design ... and MC error reduction and MLMC.

# The end

- But you may glance through the rest if you like 😊

# Gravimetry: the pendulum



# The simple pendulum

- The acceleration  $g$  of gravity is given by

$$g = \left( \frac{2\pi}{T} \right)^2 l$$

where  $T$  is the period and  $l$  is the length of the pendulum.

- Measurement of  $g$  can hence be done by measurement of  $T$  and  $l$ .
- There are several possible estimation strategies, and some of them will be investigated.

# Measurement of $l$

- The length was measured with manually with a measuring tape, and the result was

$$l = 366.7(1.0)\text{cm}$$

where the number in parentheses is the numerical value of the expanded uncertainty  $U(l)$  referred to the corresponding last digits of the quoted result. This defines an interval estimated to have a level of confidence of 95 percent. The standard uncertainty is estimated to be  $u(l) = 0.5\text{cm}$

- The uncertainty has been determined by a Type B evaluation.



# Measurement of T

- The period was measured manually 10 times, and the result for an average of 10 periods was  $T = 3.8\text{s} +$

$$\Delta T = [33, 41, 53, 41, 34, 52, 55, 34, 28, 34]ms$$

- Assuming a normal distribution, and application of the Student t distribution gives

$$T = 3.8405(69)s$$

- The standard error is  $u(T) = 3.1 \text{ m s}$  corresponding to a coverage factor  $k = 2.26$  and 9 degrees of freedom.

# Measurement of $g$

- The acceleration of gravity is

$$g = 9.8153(352)m/s^2$$

# The ISO GUMs main message

- **Estimate**
- **Standard uncertainty**
- **Expanded uncertainty**
- **Credibility level**

# References

It may be beneficial to consider (download your own private version from British standard <http://www.bsonline.bsi-global.com>)

[A] ISO, Guide to the expression of uncertainty in measurement (GUM)

[B] ISO 3534, 1985, Statistics - Vocabulary and symbols

[C] ISO 5725, 1994, Accuracy (trueness and precision) of measurement methods and results

[D] INCE (2005). Managing uncertainties in noise measurements and predictions : a new challenge for acousticians. Uncertainty Noise Symposium, LeMans, INCE.

# References

- [1] J.O. Berger. **Statistical Decision Theory and Bayesian Analysis**. Springer, 1985.
- [2] W. Bich, M.G. Cox, and P.M. Harris. **Evolution of the Guide to the Expression of Uncertainty in Measurement**. *Metrologia*, 43:161–166, 2006.
- [3] B. Blanquart. **Basic of uncertainties for acousticians**. Symposium Le Mans, June 2005.
- [4] M. Born. **Physics in my generation**. Springer, 1969.
- [5] K. Brinkmann, R. Higginson, and L. Nielsen. **Treatment of measurement uncertainties in international and european standards on acoustics**. In *Managing Uncertainty in Noise Measurement and Prediction - Symposium Le Mans (France)*. INCE EUROPE, 2005.
- [6] G. Casella and R.L. Berger. **Statistical inference**. Duxbury, 1990.

# References

- [7] D.R. Cox and D.V. Hinkley. Theoretical statistics. Chapman-Hall (2000), 1974.
- [8] S.V. Crowder and S.L. Kupferman. Use of Welch-Satterthwaite Approximation in Calibration of Voltage Standards. *Journal of Quality Technology*, 36:38–52, 2004.
- [9] R.J. Douglas, A.G. Steele, B.M. Wood, and K.D. Hill. A useful reflection. *Metrologia*, 42:35–39, 2005.
- [10] M. Evans and T. Swartz. Approximating integrals via Monte Carlo and deterministic methods. Oxford, 2000.
- [11] L.J. Gleser. Assessing Uncertainty in Measurement. *Statistical Science*, 13:277–290, 1998.
- [12] ISO. Guide to the expression of uncertainty of measurements. International Organisation for Standardisation, 1995.
- [13] ISO 3534. Statistics - vocabulary and symbols. 1993.

# References

- [14] ISO 5725. Accuracy (trueness and precision) of measurement methods and results. 1994.
- [15] H.K. Iyer, C.M.J. Wang, and T. Mathew. Models and Confidence Intervals for True Values in Interlaboratory Trials. *Journal of the American Statistical Association*, 99:1060–1071, 2004.
- [16] R. Kacker and A. Jones. On use of Bayesian statistics to make the Guide to the Expression of Uncertainty in Measurement consistent. *Metrologia*, 40:235–248, 2003.
- [17] R.N. Kacker. Bayesian alternative to the ISO-GUMs use of the Welch-Satterthwaite formula. *Metrologia*, 43:1–11, 2006.
- [18] E.L. Lehmann. Testing statistical hypotheses. Springer (1997), 1986.
- [19] E.L. Lehmann and G. Casella. Theory of point estimation. Springer (1998), 1983.

# References

- [20] I. Lira. **Evaluating the Measurement Uncertainty: Fundamentals and practical guidance.** Institute of Physics Publishing, Bristol, 2002.
- **[21] I. Lira. Resolution revisited. Metrologia, 43:14–17, 2006.**
- [22] B. Ripley. **Stochastic simulation.** Wiley, 1987.
- [23] M.J. Schervish. **Theory of Statistics.** Springer, 1995.
- [24] R. Willink. **A procedure for the evaluation of measurement uncertainty based on moments. Metrologia, 42:329–343, 2005.**
- **[25] R. Willink. Principles of probability and statistics for metrology. Metrologia, 43:211–219, 2006.**
- [26] R. Willink and I. Lira. **A united interpretation of different uncertainty intervals. Measurement, 38:61–66, 2005.**



# A simple example [4,Blanquart, B. 2005a]

An International-Ince Symposium

[www.uncertainty-noise.org](http://www.uncertainty-noise.org)

OBJECTIVES | PAPERS | SEARCH | EXHIBITION | COMMITTEES & SPONSORS | CONFERENCE VENUE |

## Managing Uncertainties in Noise Measurements and Prediction *A new challenge for acousticians*

Symposium **Le Mans, France**

27 - 29 June 2005

ENSIM



**PAPERS**

- SEARCH
- *Listed by* SESSIONS
- *Listed by* AUTHORS
- *Listed by* TITLES
- *Book of abstracts* 

To get additional information, please contact :

**CIDB :**

Website : <http://www.cidb.org>  
Phone : +33 (0)1 47 64 64 64

**INCE Europe :**

Website : <http://www.inceeurope.org>  
Phone : +44 (0)151 638 0181

 This symposium is organized with cooperation of  
the Directorate-General Environment of the European Commission

# A simple example [4,Blanquart, B. 2005a]

- An operator has obtained 7 values of temperature in a water vessel, in repeatability conditions. The specifications of the thermometer are: “accuracy : +/- 0,5 °C”

- 32,5 °C      32,8 °C      32,1 °C
- 33,1 °C      32,7 °C      32,9 °C
- 32,3 °C

$$x_c = \bar{x}_{random} + \underbrace{C_{representativity}}_{???} + C_{bias} + C_{displaying}$$

# A simple example [4,Blanquart, B. 2005a]

$$u^2(x_c) = u^2(\bar{x}_{random}) + u^2(C_{representativity}) + u^2(C_{bias}) + u^2(C_{displaying})$$

$$u^2(x_c) = \frac{0,35^2}{7} + \left(\frac{??}{??}\right)^2 + \left(\frac{0,5}{\sqrt{3}}\right)^2 + \left(\frac{0,05}{\sqrt{3}}\right)^2$$

# A simple example [4,Blanquart, B. 2005a]

- Expression of the result

$$u(x_c) = 0,32$$

$$U(x_c) = 2 \times 0,32 = 0,64$$

- Temperature:  $32,63 \text{ }^\circ\text{C} \pm 0,64 \text{ }^\circ\text{C}$  (k=2) (95%)
  - unknown contribution of the homogeneity of the temperature in the vessel
  - major component: thermometer accuracy

# A simple example [Blanquart, B. 2005a]: Interpretation

- The standard uncertainty  $u(y)$  is the ISO GUM notation for the square-root of the estimated variance of the estimator corresponding to  $y$ . This is an estimate of the standard deviation.
- The quantity  $y \pm k u(y)$  defines an interval estimated to have a given level of confidence - typically 95%. The coverage factor  $k$  is usually in the range 2 to 3. The quantity  $U(y) = k u(y)$  is the expanded uncertainty.
- The following concepts are hence essential:
  - **Variance and standard deviation.**
  - **Estimator.**
  - **Estimate.**
  - **Level of confidence.**

# Abstract of resolution paper

- A procedure is presented to evaluate the expanded uncertainty of a quantity about which discretized measurement data are available. The method is based on conventional statistics and depends on the value of the experimental variance and the resolution.
- The suggested procedure is compared with a recently suggested procedure based on Bayesian statistics.
- The ISO Guide to the expression of Uncertainty of Measurements (GUM) is discussed briefly. It is argued that both conventional and Bayesian statistics give a consistent interpretation of the GUM procedure, and the two approaches supplement each other.
- Conventional statistics estimates the uncertainty of the measurement procedure, while Bayesian statistics gives the uncertainty of the measurand.

# Micrometer example

- Let 7.489, 7.503, 7.433, 7.549, 7.526, 7.396, 7.543, 7.509, 7.504, 7.383 be the result in mm of the measurement of some length  $\mu$  with a micrometer. [Lira, I: 2006]

- Model

$$X_i = \mu + \sigma Z_i$$

- where

$$Z_i \sim N(0, 1)$$

are independent random variables.

- The idealization is that the measurand  $\mu$  is a property of the object, and  $\sigma$  is a property of the experiment as a whole.

# Micrometer example

A coverage level  $p = 95\%$  gives the coverage factor  $k_p = t_p(n - 1) = 2.2622$ . The estimate for  $\mu$  is  $\hat{\mu} = \bar{x} = 7.4835$  mm, the standard error is  $u(\mu) = s_x / \sqrt{n} = 0.0187$  mm, and the expanded error is  $U(\mu) = k_p u(\mu) = 0.0423$  mm.



## Micrometer example

The length is determined to be 7.483(42) mm, where the number in parentheses is the numerical value of the expanded uncertainty referred to the corresponding last digits of the quoted result. The coverage level is 95% and the coverage factor is 2.26.

# Calliper example

Let 7.5 , 7.5 , 7.4 , 7.5 , 7.5 , 7.4 , 7.5 , 7.5 , 7.5 , 7.4 be the result in mm of a series of measurements of the same length  $\mu$  with a caliper [21]. Assume that this can be modelled by independent random variables  $Y_i \sim N(\mu, \sigma_Y^2)$ .

## Calliper example

The length is determined to be 7.47(3) mm, where the number in parentheses is the numerical value of the expanded uncertainty referred to the corresponding last digits of the quoted result. The coverage level is 95% and the coverage factor is 2.26.

## Calliper and micrometer examples

The result  $\mu = 7.47(3)$  mm of the calliper measurement is more accurate than the result  $\mu = 7.48(4)$  mm of the micrometer measurement. The result is counterintuitive, but sometimes counterintuitive results are correct. In this particular case it is however the result of a too crude model.

## Calliper and micrometer examples

The GUM, and common sense, demands that every relevant source of uncertainty shall be included in the model. The uncertainty due to instrument resolution has not been taken into account in the model. The calliper has a resolution  $d = 100 \mu\text{m}$  and the micrometer has a resolution  $d = 1 \mu\text{m}$ .

## Calliper and micrometer examples

A seemingly reasonable model for the calliper measurement is given by  $Y_i = \mu + \sigma Z_i + dU_i$ , where  $Z_i \sim N(0, 1)$  and  $U_i \sim U(0, 1)$ . The uniform variable has  $\text{Var}(dU_i) = d^2/12$ . The standard error  $s_y/\sqrt{10} = 15.3 \mu\text{m}$ , the standard error  $d/\sqrt{12} = 28.9 \mu\text{m}$ , and  $u(\mu)^2 = s_y^2/10 + d^2/12$  give a combined standard error  $u(\mu) = 32.7 \mu\text{m}$ .

## Calliper and micrometer examples

The micrometer measurement gives  $s_x/\sqrt{10} = 18.7 \mu\text{m}$ ,  $d/\sqrt{12} = 0.3 \mu\text{m}$ , and a combined standard error  $u(\mu) = 18.7 \mu\text{m}$ . In this case the instrument resolution is of no concern, and the micrometer result is more accurate than the calliper result.

## Calliper and micrometer examples

There are several weak points in the argument given above. The model  $Y_i = \mu + \sigma Z_i + dU_i$  fails to give realizations with values which are integer multiples of  $d$ . The  $S_Y^2/n$  is an unbiased estimator of the variance of the estimator  $\bar{Y}$ , so the above calculation of a combined standard error is dubious: *The effect of the resolution is already included in  $S_Y^2/n$ .*



# Calliper example

The length is determined to be 7.46(3) mm, where the number in parentheses is the numerical value of the Bayesian expanded uncertainty referred to in the corresponding last digit of the quoted result. The credibility level is 95% and the coverage factor is 2.0.

# Micrometer example

The length is determined to be 7.48(4) mm, where the number in parentheses is the numerical value of the Bayesian expanded uncertainty referred to in the corresponding last digits of the quoted result. The credibility level is 95% and the coverage factor is 2.3.

# The ISO GUMs main message

- **Estimate**
- **Standard uncertainty**
- **Expanded uncertainty**
- **Credibility level**