Application of Monte Carlo methods in finance

Fred Espen Benth

Centre of Mathematics for Applications (CMA)
University of Oslo, Norway
Collaborators: Lars Oswald Dahl, Martin Groth and Paul Kettler

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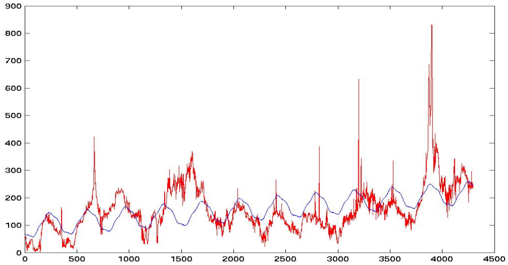
Overview of the lectures

1. Stochastic processes in finance
2. Financial derivatives
   - Options
   - Arbitrage and hedging
   - Pricing of financial derivatives
3. Pricing using Monte Carlo
   - Simulation of expectations
   - Quasi-MC as variance reduction

Stochastic processes in finance

- Starting point in finance: Asset prices
- Traded prices in the market place:
    - Stocks, oil, gas, electricity, metals, coffee,...



Bachelier 1900

- Future asset prices are uncertain
- Bachelier 1900: Theorie de la Speculation
  - Modelled the uncertain price dynamics of stocks on the French "Bourse"
  - Used probability theory to price options on stocks
  - Options traded on the exchange those days......
- Regnault 1853:
  - The square root law for stock price variations:

$$S(t+s) - S(t) \sim \sigma\sqrt{s}$$

- Bachelier proposed *Brownian motion* with drift

$$S(t) = S(0) + \mu t + \sigma B(t)$$

- Definition of Brownian motion $B(t)$
  1. $B(t+s) - B(t)$ is statistically independent of $B(t)$
  2. $B(t+s) - B(t)$ is stationary, e.g., its distribution depends only on s
  3. $B(t+s) - B(t)$ is normally distributed with zero mean and variance s.

- Property of Bachelier's model (recall Regnault's square root law)

$$\mathrm{Var}[S(t+s)-S(t)] = \sigma^2 \mathrm{E}[(B(t+s)-B(t))^2] = \sigma^2 s$$

- Price differences are independent and normally distributed
- Bad property: May get negative prices....

Samuelson 1965

- Samuelson: Make price dynamics positive by exponentiation
- Geometric Brownian motion (GBM)

$$S(t) = S(0)\exp(\mu t + \sigma B(t))$$

- What are the statistical implications of GBM?

GBM and logreturns

- Key factor in investments: The return!

$$\frac{S(t)-S(t-1)}{S(t-1)}$$

- Relative profit/loss from buying today, and selling tomorrow
  - Stated in percent, usually
- Key factor for the GBM model: The logreturn!

$$R(t)=\ln\left(\frac{S(t)}{S(t-1)}\right)=\mu+\sigma\left(B(t)-B(t-1)\right)$$

- Logreturns are independent and normally distributed
  - Mean μ
  - Standard deviation σ

- If the returns are small

$$\ln\left(\frac{S(t)}{S(t-1)}\right) \approx \frac{S(t)-S(t-1)}{S(t-1)}$$

- Use Taylor expansion: $\ln(1+x) \approx x$
- Returns are the key in practice
- Logreturns mathematically convenient

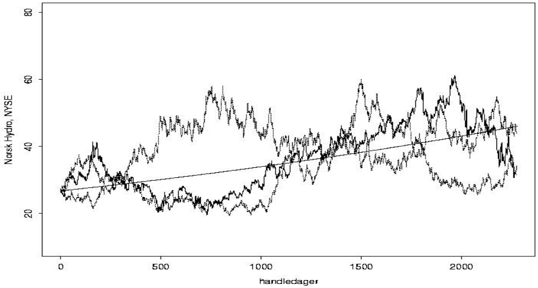
Fitting a GBM to data

- Transform price data to logreturns

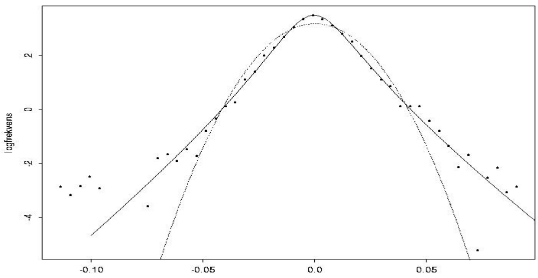
$$S_0, S_1, \ldots, S_n \Rightarrow R_1 = \ln(S_1/S_0), \ldots, R_n = \ln(S_n/S_{n-1})$$

- Use maximum likelihood to estimate μ and σ^2 from the logereturn data
- Example: Norsk Hydro at NYSE
  - Daily closing prices from Jan 1, 1990, to Dec 31, 1998
  - 2274 price data
  - $\widehat{\mu} = 2.75\%$, $\widehat{\sigma} = 32.8\%$ (annualized)

- Daily closing prices of Norsk Hydro at NYSE
- Two simulated GBM's (using Monte Carlo....will come back to this...)



- Is the normality hypothesis valid for logreturns?
- Empirical distribution of NH together with fitted normal
  - Note the logarithmic y-axis (frequencies)
  - Heavy tails
  - Less probability in the center than assigned by normal

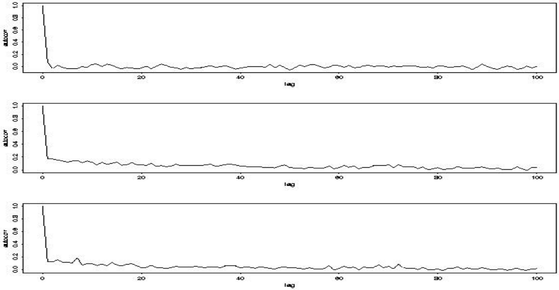


- Is the independence hypothesis valid for the logreturns?
- The autocorrelation function (ACF) for the logreturns

$$\rho(k) = \text{Cov}\left(R(t+k), R(t)\right)$$

- For GBM, we find that $\rho(k) = 0$ theoretically
- Example where ACF for $R(t), R^2(t)$ and $|R(t)|$ are calculated empirically for NH
- Note that GBM should have zero ACF for all these cases

- $R(t)$ has zero ACF
- The ACF for $R^2(t)$ and $|R(t)|$ display long range dependency



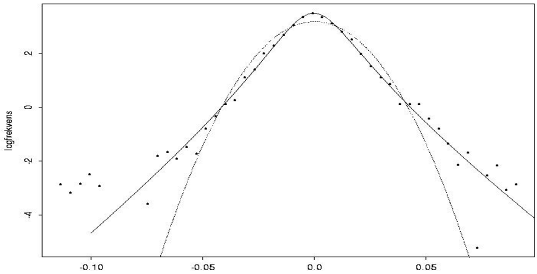
A small digression

- Suppose logreturns (or returns) have non-zero correlation
- Violates the market efficiency hypothesis of Samuelson (and others)
  - The prices today contain all available information
- Non-zero correlation means we can predict whether returns will be positive or negative
- Leads to possibilities to earn money riskless over time
- The traders will catch this, and speculation will rule out such possibilities quickly
  - Known as *arbitrage*
- The market is efficient

Alternative model for the logreturns

- Mandelbrot 70ties:
  - Logreturns of cotton and wool prices modelled using stable Pareto distributions
- A recent successful model by Barndorff-Nielsen:
  - The normal inverse Gaussian (NIG) distribution
  - Used to model logreturns of stocks, currency and power prices
  - ....and even temperature variations
- Original use of the NIG: Sand grain size distribution
  - Empirical studies on the beaches of Denmark.....
  - See video `http://home.imf.au.dk/oebn/blownsand.mpg`

- Fitting the NIG distribution to NH logreturns
- Heavy tails are explained
- Close to perfect fit in the center



The NIG distribution

- 4 parameters
  - μ: location
  - δ: scale
  - α: steepness, or tail heaviness
  - β: skewness
- Explicit density function

$$f_{\mathrm{NIG}}(x) = c \times \exp(\beta(x-\mu)) \frac{K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\sqrt{\delta^2 + (x-\mu)^2}}$$

- K_1 is the modified Bessel function of third kind and index 1 (!)

$$K_1(x) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}x(z+z^{-1})\right)\, dz$$

- Maximum likelihood estimation of NIG for NH logreturns

$$\widehat{\alpha} = 56, \widehat{\beta} = 2.6, \widehat{\mu} = 0.001, \widehat{\delta} = 0.015$$

- The order of estimates are typical for stock prices

Stochastic process yielding NIG logreturns?

- Let $L(t)$ be a Levy process
  1. $L(t+s)-L(t)$ is statistically independent of $L(t)$
  2. $L(t+s)-L(t)$ is stationary, e.g., its distribution depends only on s
- Specify $L(1)$ to be NIG-distributed
  - From property 2 we then have $L(t+1)-L(t)$ being NIG
- Define stochastic process for the price as

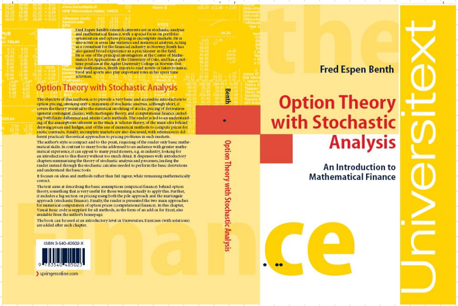
$$S(t)=S(0)\exp(L(t))$$

- Exponenential NIG-Levy process

Summary so far

- 3 basic models for asset price dynamics
  - Bachelier model
  - Samuelson model
  - exponential NIG model
- The Samuelson model is the starting point for option pricing
  - The theory of Black, Scholes and Merton
- Modern finance deals with incomplete markets
  - The NIG model provides a typical example

Commercial break & Lottery



Financial derivatives

What are financial derivatives?

- Assets that can be traded, where the value depends on another asset
- The standard example: Call option on stock

The owner of a call option has the right, but not the obligation, to buy one share of a stock at an agreed price and time

- The stock is specified in the contract (e.g. NH)
- Exercise time T and strike price K likewise

- Example: Call option on Microsoft (NYSE), with exercise time Feb 16 and $K = 30$
- Today's (Jan 23) MSFT price: USD 30.74
  - The price of this option was USD 1.20 (Jan 23)
- Question: If you own the call option, what do you do if
  - the price of MSFT is USD 32 on Feb 16?
  - the price of MSFT is USD 30 on Feb 16?

- Payment from call option is $\max(S(T) - K, 0)$
- The price of a call on Microsoft should be a function of the stock price
  - After all, the payment at exercise is a function of the stock price

$$C(t) = f(t, S(t))?$$

- This is the starting point of the Black, Scholes and Merton theory

- The option market is huge
  - Put, barrier, compound, Asian options, etc.....
  - European and American options
- Options written on stocks, power, commodities
- Options on temperature indices
- Options on CO2 allowances

- The biggest derivatives market is the *forward* market
- A forward/futures contract is an agreement to trade an underlying asset at a fixed time and price
- Example: Forward contract on electricity
  - Delivery of 1MWh electricity over the month of March
  - Nordpool price: 32Euro per MWh
  - Spot price Jan 23: 26.09 Euro per MWh
- Note that the *forward price* is agreed today, but paid on delivery
- Forwards are the main traded assets in power markets
- Options are frequently written on forwards

- Forwards are (in some sense) predictions of future spot prices
  - However, you pay an extra fee for the certainty
- Forward price should be a function of today's spot price

$$F(t, T) = f(t, T, S(t))?$$

- Let us use the Black, Scholes and Merton theory to derive the forward price

Black, Scholes and Merton (BSM) theory

- Assume a market consisting of a spot, forward and a bank
- Spot price $S(t)$
- Forward price denoted $F(t,T)$
  - Delivery of spot at time $T \geq t$
- Bank's interest rate r (same for borrowing and investing)
- Assume the following position: You have sold a forward contract
  - The counter-party will require the spot at time T
  - and pay you $F(t,T)$
- What is your risk?
  - You need to provide the spot at time T!

Hedging strategy for the forward

- Buy the spot today at the price $S(t)$
- Finance the purchase with a loan
- Save the spot untill delivery at time T
- Receive the payment at delivery, $F(t,T)$
- Your position at time t (today):
  - Bank loan $S(t)$ minus purchase of spot $S(t)$ equal 0

- Your position at time T (delivery time)
  - You receive $F(t,T)$ for delivering the spot
  - You need to pay the loan, $S(t)\exp(r(T-t))$

$$F(t,T) - S(t)e^{r(T-t)}$$

- If $F(t,T) > S(t)e^{r(T-t)}$, you have earned a riskless profit
- In a big market, you can then scale up your position, and earn an arbitrary high profit
- If $F(t,T) < S(t)e^{r(T-t)}$, buy forwards instead of selling
  - and turn the position in the spot around
- In both cases, there are opportunities to earn a riskless profit

The no-arbitrage price

- To earn money without taking any risk is called an arbitrage
  - More precisely: there is arbitrage
  - if your position costs zero to buy,
  - it ends with a non-negative value,
  - but have a positive probability of ending strictly positive
- In any liquid market, arbitrage possibilities are ruled out quickly by competition

- Conclusion: The arbitrage-free price is

$$F(t,T) = S(t)e^{r(T-t)}$$

- The hedging strategy is to buy spot at time t, financed by a bank loan
  - An investment replicating the forward completely

The forward price as an expectation

- Suppose $S(t)$ is a GBM
- Direct calculations show

$$S(T) = S(t)\exp\left(\mu(T-t) + \sigma(B(T)-B(t))\right)$$

$$\mathrm{E}\left[S(T)\,|\,S(t)\right] = S(t)\exp\left((\mu + 0.5\sigma^2)(T-t)\right)$$

- The forward price has r instead of $\mu + 0.5\sigma^2$
- A trick from stochastic analysis: Change probability!
  - Everything said so far is supposing a probability space $(\Omega, \mathcal{F}, P)$

- Girsanov's Theorem: There exists a probability Q equivalent to P such that

$$W(t) = \frac{\mu + 0.5\sigma^2 - r}{\sigma} t + B(t)$$

  is a Brownian motion
- Simple algebra then yields

$$\mathrm{E}_Q\left[S(T) \,|\, S(t)\right] = S(t)\exp(r(T-t)) = F(t,T)$$

- Note that the conditional expectation gives the price for general processes $S(t)$
  - First hint why MC is useful....

A little summary

- A forward is a derivative contract of the spot
- We have
  1. There exists a hedging strategy, replicating the forward
  2. By arbitrage, we can derive the forward price from the hedge
  3. There exists a probability Q such that the forward price is expressed in terms of a conditional expectation
- These are exactly the three steps used by BSM in pricing call options!

Call options

- Black, Scholes and Merton developed the arbitrage theory for valuing call options in 1972-3
- Highlight is the Black & Scholes pricing formula
- So-called "closed-form" formula for the price of call options
- Gives also the hedging strategy/replicating portfolio
- Was implemented on pocket calculators almost immediately after its publication
  - Publication was difficult, however!

The rise....

- Scholes and Merton were awarded the Nobel Prize in Economics in 1997

  "For a new method to determine the value of derivatives"

....and fall....

- In 1998....

  "Long-Term Capital Management (LTCM) was a hedge fund founded in 1994 by John Meriwether. On its board of directors were Myron Scholes and Robert C. Merton , who shared the 1997 Nobel Memorial Prize in Economics. Initially enormously successful with annualized returns of over 40% in its first years, in 1998 it lost 4.6 billion USD in less than four months and became the most prominent example of the risk potential in the hedge fund industry. The fund folded in early 2000."
- Quoted from Wikipedia

The derivation of BSM

- First, some discussion of the GBM model:
- Using Ito's Formula to find the differential $dS(t)$
  - Second-order Taylor expansion in $B(t)$, using $dB(t)^2 = dt$

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t) + \frac{1}{2}\sigma^2 S(t)\,dt$$

- For simplicity, we write the differential as

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dB(t)$$

- May interpret $dS(t)$ as $S(t + \Delta t) - S(t)$

- Suppose that there exists a replicating strategy (a hedge) consisting of borrowing/investing in the bank and buying/selling the underlying asset

$$V(t) = a(t)S(t) + b(t)D(t), dD(t) = rD(t)\,dt$$

- a and b are stochastic processes only depending on $S(s)$ for $s \leq t$
- $V(t)$ is supposed to be self-financing
  - No witdrawal and injection of money in the hedge
  - Change of positions (a, b) must come from transfer of funds from bank to stock, or vice versa

$$dV(t) = a(t)\,dS(t) + b(t)\,dD(t), dD(t) = rD(t)\,dt$$

- Denote by $P(t)$ the price, and suppose that there is a function $u(t,x)$ such that

$$P(t) = u(t, S(t))$$

- Applying Ito's Formula again

$$dP(t) = \left(u_t + \alpha u_x S(t) + \frac{1}{2}\sigma^2 u_{xx} S^2(t) \right) dt + \sigma u_x S(t)\, dB(t)$$

- To avoid arbitrage, $V(t) = P(t)$

$$dV(t) = (a(t)\alpha S(t) + rb(t)D(t))\, dt + \sigma a(t)S(t)\, dB(t)$$

- Comparing the dt and $dB(t)$ terms give

$$a(t) = u_x(t, S(t)), \quad b(t) = (u(t, S(t)) - a(t)S(t))/D(t)$$

and the PDE

$$u_t + rxu_x + \frac{1}{2}\sigma^2 x^2 u_{xx} = ru, \quad (t, x) \in [0, T) \times (0, \infty)$$

- Terminal condition $u(T, x) = \max(x - K, 0)$
- Boundary condition $u(t, 0) = 0$.
- Conclusion: Solve the PDE, and you have the price (and hedge)!

The Black & Scholes Formula

- Solution of the PDE: *The Black & Scholes formula*

$$P(t) = S(t)\mathrm{N}(d_1) - K\mathrm{e}^{-r(T-t)}\mathrm{N}(d_2)$$

where

$$d_1 = \frac{\ln(S(t)/K) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- $\mathrm{N}(\cdot)$ is the cumulative normal distribution function

$$\mathrm{N}(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,dy$$

How did they do it?????

They went down to the physics department and asked......!

A simple outline of derivation....

- Do a change of variables

$$v(\tau, y) = e^{ay+b\tau} u(T - c\tau, e^y)$$

- Direct differentiation

$$v_\tau = \frac{1}{2} v_{yy}\,, \quad v(0, y) = e^{ay} \max(e^y - K, 0)$$

- Fundamental solution of the heat equation known to be

$$\phi(\tau, z) = \frac{1}{\sqrt{2\pi\tau}} e^{-z^2/2\tau}$$

- Solution v becomes

$$v(\tau, y) = \int_{-\infty}^{\infty} e^{az} \max(e^z - K, 0)\phi(\tau, y - z)\, dz$$

- Transforming back to u gives the B&S formula
- Note: $\phi(\tau, z)$ is the density of a Brownian motion at time τ
- Another expression of v

$$v(\tau, y) = \mathrm{E}\left[e^{aB(\tau)} \max\left(e^{B(\tau)} - K, 0\right) \mid B(0) = y\right]$$

A Feynman-Kac solution of the PDE

- Tracing backwards the expectation representation of v

$$u(t,x) = \mathrm{e}^{-r(T-t)}\mathrm{E}\left[\max\left(\widetilde{S}(T) - K, 0\right) \mid \widetilde{S}(t) = x\right]$$

where

$$d\widetilde{S}(t) = r\widetilde{S}(t)\,dt + \sigma\widetilde{S}\,dW(t)$$

- W is a Brownian motion
- Note that $\widetilde{S}$ is NOT S!

The option price as an expectation

- Principle in economics: The value today of a future cash flow is the discounted future cash-flow
- In language of options:

$$\widetilde{P}(t) = \mathrm{e}^{-r(T-t)}\mathrm{E}\left[\max(S(T)-K,0)\,|\,S(t)\right]$$

- A direct calculation gives $\widetilde{P}(t) \neq P(t)$!
- Thus, $\widetilde{P}(t)$ is an arbitrage price

- Recall dynamics of S

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dB(t)$$

- Let us use Girsanov again: There exists a Q equivalent to P so that

$$W(t) := \frac{\alpha - r}{\sigma} t + B(t)$$

is a Brownian motion
- A similar direct calculation shows

$$P(t) = \mathrm{e}^{-r(T-t)} \mathrm{E}_Q \left[\max(S(T) - K, 0) \,|\, S(t)\right]$$

Risk-neutral probability

- Or equivalent martingale measure
- The discounted asset price $S(t)$ is a martingale under Q

$$S(t) = e^{-r(T-t)} E_Q [S(T) \mid S(t)]$$

- Using Girsanov representation, Q-dynamics is

$$dS(t) = rS(t)\, dt + \sigma S(t)\, dW(t)$$

General derivatives pricing formula

- Suppose X is a random variable which depends on $S(t)$ for $0 \leq t \leq T$
- X is a derivative, describing the random payment resulting from different scenarios of S
- Price is

$$P(t) = e^{-r(T-t)} E_Q [X \mid S(t)]$$

Conclusions so far...

- Price of a derivative expressible in terms of the expected present value of the payoff
- The expectation with respect to Q
  - Which in practical terms mean that α is substituted with r in the dynamics of $S(t)$
- The hedge position is $a(t) = u_x(t, S(t))$
- We are ready for some Monte Carlo

Pricing using Monte Carlo

Pricing of Basket options

- Call option written on several assets
- Examples:
  - Option on a portfolio
  - Option on the average of several asset prices
  - Spread options
- These options do not have a closed-form formula for the price

- Starting point: An d-dimensional GBM

$$dS_i(t) = rS_i(t)\,dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}\,dW_j(t)$$

- Each asset a GBM
  - with volatility given by (squared vol)

$$\sum_{j=1}^{d}\sigma_{ij}^2$$

  - and covariance

$$\sum_{j=1}^{d}\sigma_{i_1 j}\sigma_{i_2 j}$$

- Dynamics directly under the risk-neutral probability Q

- Example: Option on a basket of two assets
- Exercise time T and strike price K

$$\max\left(S_1(T) + S_2(T) - K, 0\right)$$

- Example: Spread option

$$\max\left(S_1(T) - S_2(T) - K, 0\right)$$

- Relevant in power markets
  - For $K = 0$, Margrabe's Formula
- General basket option

$$\max\left(\sum_{i=1}^{d} a_i S_i(T) - K, 0\right)$$

Pricing the basket

- Recall expectation operator for the price

$$P = e^{-rT} E_Q \left[\max \left(\sum_{i=1}^{d} a_i S_i(T) - K, 0 \right) \right]$$

- Explicit solution of $S_i(T)$ (using Ito's formula...)

$$S_i(T) = S_i(0) \exp \left((r - 0.5 \sum_{j=1}^{d} \sigma_{ij}^2)\, T + \sum_{j=1}^{d} \sigma_{ij} W_j(T) \right)$$

- Note that in distribution

$$W_j(T) = \sqrt{T}\, X_j\,, \quad X_j \text{ iid } \mathrm{N}(0,1)$$

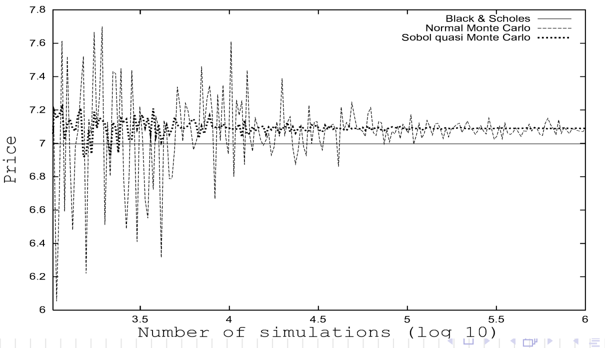
- In distribution

$$S_i(T) = S_i(0) \exp\left((r - 0.5 \sum_{j=1}^{d} \sigma_{ij}^2)\, T + \sum_{j=1}^{d} \sigma_{ij} \sqrt{T}\, X_j \right)$$

- Simulation of the price consists of simulation of d iid standard normal variables X_j.

Example

- Basket of two options
- MC vs. approximative Black & Scholes price



- Note the large variations of the price
- Time consuming to compute accurately
- In practice: Banks need prices “on-line”
- Variance-reduction is thus not for “academic fun”
    - ...but crucial for the market
- One method: Quasi-Monte Carlo

Quasi-Monte Carlo

- Draw uniform numbers so that $[0,1]^d$ gets filled optimaly
- Create a sequence of sample points $u_1, \ldots, u_N$ such that

$$D_N = \sup_{B \subset [0,1]^d} \left| \frac{\#\{u_n \in B\}}{N} - \lambda(B) \right|$$

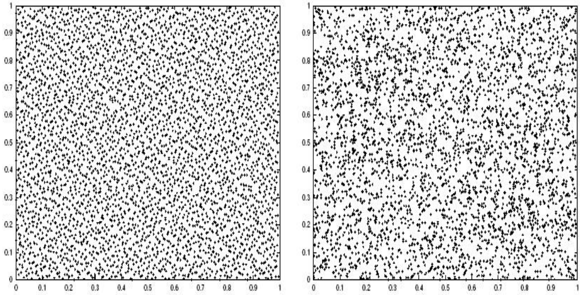
  gets "small"
- B are boxes of the form $\prod_{i=1}^{d} [0, a_i)$
- D_N is the *discrepancy* of the sequence
- It measures the "closeness" to the uniform distribution

- A sequence is called a *low discrepancy sequence* if

$$D_N \leq C \frac{(\ln N)^d}{N}$$

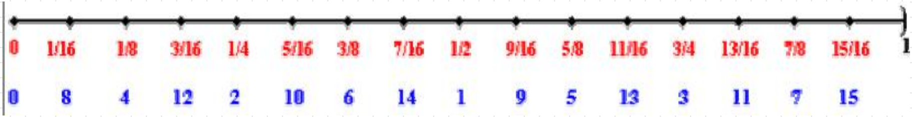
- Many such sequences exist
  - Sobol, Halton, van der Corput, de Faure etc....

Example: Halton vs. random in $d = 2$



Some low-discrepancy sequences

- The van der Corput sequence for $d = 1$ with *base 2*
- Example of the first 16 numbers
  - Sequential halfing of intervals



- Definition for a general base b
- van der Corput u_n: Find the b-ary representation of n

$$n = \sum_{k=0}^{L-1} d_k(n) b^k$$

- Next, calculate

$$u_n = \sum_{k=0}^{L-1} d_k(n) b^{-k-1}$$

- Example: Generation of u_3 with base $b = 2$

$$3 = 1 \times 2^0 + 1 \times 2^1 , \Rightarrow d_0(3) = 1 , \quad d_1(3) = 1$$

$$u_3 = 1 \times 2^{-1} + 1 \times 2^{-2} = \frac{3}{4}$$

The Halton sequence

- The Halton sequence is the d-dimensional version of the van der Corput sequence
- Choose d different bases $b_1, \ldots, b_d$
- Generate $u_n = (u_n^1, \ldots, u_n^d)$
  - u_n^i generated from van der Corput with base b_i
- Condition: the bases must be coprime integers
  - Their greates common divisor is 1

The Koksma-Hlawka bound

- Consider the integral

$$I = \int_{[0,1]^d} f(u)\, du$$

- Suppose that $\{u_n\}_{n=1}^N$ is a d-dimensional low discrepancy sequence

$$I_N = \frac{1}{N} \sum_{n=1}^{N} f(u_n)$$

- The Koksma-Hlawka bound

$$|I - I_N| \leq V_f D_N \leq C \frac{(\ln N)^d}{N}$$

- V_f dependent on the variation of f

What does this have to do with finance?

- Recall the pricing expectation

$$P = e^{-rT} E_Q \left[\max \left(\sum_{i=1}^{d} a_i S_i(T) - K, 0 \right) \right]$$

with

$$S_i(T) = S_i(0) \exp \left((r - 0.5 \sum_{j=1}^{d} \sigma_{ij}^2)\, T + \sum_{j=1}^{d} \sigma_{ij} \sqrt{T}\, X_j \right)$$

- As an integral

$$P = \mathrm{e}^{-rT} \int_{R^d} \phi(T, x_1, x_2, \ldots, x_d) \frac{\exp(-\frac{1}{2}\sum_{j=1}^{d} x_j^2)}{(2\pi)^{d/2}} \, dx_1 \cdots dx_d$$

- Change of variables: $y = \mathrm{N}(x)$
  - Recall $\mathrm{N}(x)$ is the cumulative standard normal distribution function

$$P = \mathrm{e}^{-rT} \int_{[0,1]^d} \phi(T, \mathrm{N}^{-1}(y_1), \mathrm{N}^{-1}(y_2), \ldots, \mathrm{N}^{-1}(y_d)) \, dy_1 \cdots dy_d$$

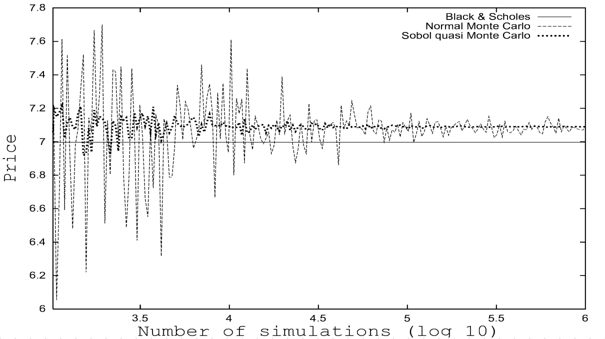
- Suitable for QMC simulation

- Remark: the change of variables corresponds to

$$X = \mathrm{N}^{-1}(U)$$

- X normally distributed, U uniform

- Example: A basket option with two assets
- Simulation of the price using a 2D Halton sequence



The exponential NIG model

- Pricing of options when asset follows an exponential NIG model
- Consider a call option

$$P(0) = e^{-rT} E_Q \left[\max \left(S(0) e^{L(T)} - K, 0 \right) \right]$$

- $L(T)$ is an NIG random variable

Questions....

1. What is Q ?
2. $L(T)$ is NIG under P, but what is it under Q?
3. How to calculate the price ?

Question 1: Q?

- NIG-Levy process gives rise to an *incomplete market*
- Incomplete market:
  - Option can not be replicated
- Theory of B&S: the option price is the price of replication
- So why is the price still an expectation operator under some Q?

- There exists a lot of super and sub-hedging strategies
- Superhedge
  - Investment in underlying and bank
  - Self-financing
  - Worth *more than* option at exercise time
- Subhedge: *less than*
- Introduce

$$P_{\text{super}} = \inf\{P \mid P \text{ is the price of a superhedge}\}$$
$$P_{\text{sub}} = \sup\{P \mid P \text{ is the price of a subhedge}\}$$

- Let P be so that

$$P \in (P_{\text{sub}}, P_{\text{super}})$$

- It can be shown that P is an *arbitrage-free* price
  - Not possible to make arbitrage if option has this price
- Further, for every P there exists a Q so that

$$P = e^{-rT} E_Q \left[\max \left(S(0) e^{L(T)} - K, 0 \right) \right]$$

- But what is Q?????
- Q is a probability with two properties
    1. Q is equivalent to P
    2. $\exp(-rt)S(t)$ has expectation $S(0)$ (i.e. a martingale)
- There are many such Q's
- One choice: Esscher transformation

Question 2: L under Q?

- Esscher transform: Structure preserving
- $L(t)$ will be NIG-Levy under Q
- Radon-Nikodym derivative:

$$\frac{dQ}{dP} = \exp\left(\theta L(T) - \phi(\theta)T\right)$$

- ϕ is the log-moment generating function of $L(1)$
- θ chosen so that $\exp(-rT)S(T)$ has expectation $S(0)$

- A little calculation:

$$
\begin{aligned}
S(0) &= \mathrm{E}_Q\left[\mathrm{e}^{-rT} S(T)\right] \\
&= \mathrm{e}^{-rT} S(0) \mathrm{E}\left[\mathrm{e}^{L(T)} \mathrm{e}^{\theta L(T)}\right] \mathrm{e}^{-\phi(\theta) T} \\
&= \mathrm{e}^{-rT} S(0) \mathrm{e}^{\phi(1+\theta)T - \phi(\theta)T}
\end{aligned}
$$

- Choose θ so that

$$
r = \phi(1+\theta) - \phi(\theta)
$$

- Suppose $L(T) \sim \text{NIG}(\alpha, \beta, \mu, \delta)$
- Then, under Q given from Esscher

$$L(T) \sim \text{NIG}(\alpha, \widehat{\beta}, \mu, \delta)$$

$$\widehat{\beta} = -\frac{1}{2} + \text{sgn}\beta\sqrt{\frac{\alpha^2(\mu - r)^2}{\delta^2 + (\mu - r)^2} - \frac{(\mu - r)^2}{4\delta^2}}$$

- We need to calculate the expecation of the payoff from an exponential NIG

Question 3: Calculation of price

- We must simulate

$$P = e^{-rT} E \left[\max \left(S(0) e^{X} - K, 0 \right) \right]$$

  for $X \sim \mathrm{NIG}(\alpha, \beta, \delta, \mu)$
- Note: removed the hat of β for notational laziness
- Do this by (quasi) Monte Carlo

MC simulation of NIG

- Algorithm:
  1. Sample Z from $\mathrm{IG}(\delta^2, \alpha^2 - \beta^2)$
  2. Sample Y from $\mathrm{N}(0, 1)$
  3. Return $X = \mu + \beta Z + \sqrt{Z}\, Y$
- The sampling of Z goes in several steps:

- Sampling of Z:
  1. Sample V being the square of a $\mathrm{N}(0,1)$ variable
  2. Define
  $$W = \xi + \frac{\xi^2 V}{2\delta^2} - \frac{\xi}{2\delta^2}\sqrt{4\xi\delta^2 V + \xi^2 V^2}$$
  with $\xi = \delta/\sqrt{\alpha^2 - \beta^2}$
  3. Let
  $$Z = W \cdot \mathbf{1}_{U_1 < \xi/\xi + W} + \frac{\xi^2}{W} \cdot \mathbf{1}_{U_1 \geq \xi/\xi + W}$$

- Sampling of V requires one U_2, uniform

$$V = (\mathrm{N}^{-1}(U_2))^2$$

- Sampling of Z thus requires two uniforms, U_1 and U_2
- Sampling of Y requires uniform U_3
- For appropriately defined function q

$$X = \mu + \beta q(U_1, U_2) + \sqrt{q(U_1, U_2)}\,\mathrm{N}^{-1}(U_3)$$

- NIG is simulated from 3 uniforms

- Rewriting of price expectation

$$P = e^{-rT}\mathrm{E}\left[\max\left(S(0)e^{X} - K, 0\right)\right]$$
$$= \mathrm{E}\left[f(U_1, U_2, U_3)\right]$$

- Feasible for quasi-MC simulation

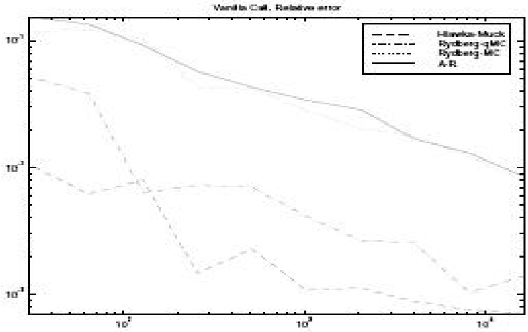
Example

- NIG parameters being "typical" for stocks

$$\mu = 0.004\,, \quad \beta = -15.2\,, \quad \alpha = 136\,, \quad \delta = 0.0295$$

- $S(0) = K = 100$, $r = 3.75\%$
- Exercise time being 1 month
- Prices simulated using MC and QMC

- Relative error as a function of number of points simulated
- “Correct” price obtained from long MC simulation



Conclusions

- Modelled financial price series with stochastic processes
  - BM, GBM and exponential NIG
- Priced options using arbitrage theory
- Prices are given as expectations
- Monte Carlo simulation of basket options
  - GBM models
  - Using quasi-MC as variance reduction technique
- MC and exponential NIG

Not mentioned in these lectures...

- Calculations of the Greeks
- Quantification of risk in portfolios
  - Value-at-Risk
- Optimization of portfolios
  - What is the best allocation of money....
  - ...given a criterion for balancing risk and return

- Valuation of swing options
  - Options where the owner has multiple rights
  - Combination of control and option theory
- Real options
  - Valuation of investments
  - Example: What is the value of having the option to build a gas pipeline to the UK?
  - Option theory
- All above may be solved using MC simulation

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