STREAMLINE-BASED HISTORY MATCHING: A REVIEW

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Abstract. In recent years, several methods for streamline-based history matching have been developed. These methods have proved to be efficient for three reasons: First, streamlines delineate flow patterns and can therefore be used to define reduced inverse models. Second, streamline methods provide fast forward simulation. Third, streamline-based sensitivities can be evaluated directly based on one flow simulation for different reservoir responses. We here give a literature review of streamline-based sensitivities and streamline methods used for history matching.

Man skal ej læse for at sluge,
men for at se, hvad man kan bruge.[1]
Henrik Ibsen, Peer Gynt (1876)

1. Introduction

A reservoir model typically consists of a differential equation of the type

$$F\big(\mathbf{y}, \mathbf{x}, t, \mathbf{p}, \frac{\partial^n \mathbf{y}}{\partial \mathbf{x}^n}, \frac{\partial \mathbf{y}}{\partial t}\big) = 0, \tag{1}$$

equipped with appropriate initial and boundary conditions. Here $\mathbf{y}$ denotes responses of the system, $\mathbf{x}$ the spatial coordinates, t time, and $\mathbf{p}$ the set of rock and fluid parameters. The forward problem consists of solving (1) to compute $\mathbf{y}(\mathbf{x}, t)$ for a given set of parameters $\mathbf{p}$. The inverse problem consists of finding a set of reservoir parameters $\mathbf{m} \subseteq \mathbf{p}(\mathbf{x})$ such that the calculated responses $\mathbf{d}^{\text{cal}} \subseteq \mathbf{y}(\mathbf{x}, t)$ match a set of observations $\mathbf{d}^{\text{obs}}$ from the actual system. For the purpose of the inverse problem we will denote the forward model, based on the actual numerical grid-implementation of (1), by $\mathbf{d} = g(\mathbf{m})$.

In a realization of a reservoir model, the two main rock parameters, rock porosity ϕ and the absolute permeability $\mathbf{K}$, are defined as piecewise constants over a grid. These parameters describe the void volume fraction of the rock and the ability of the rock to transmit a single fluid and are therefore the parameter that often have the largest influence on the fluid flow in a reservoir. Permeability and porosity have considerable spatial variability (especially permeability) and are typically strongly correlated. Unfortunately, they are difficult to measure: direct measurements are only available at a few spatial locations (e.g., from core samples) and one therefore generally has to rely on geostatistical algorithms for generating plausible realizations that can be adjusted using indirect measurements and inverse estimation methods. There are also a large number of other parameters that are not necessarily directly related to the spatial grid. Examples include fluid parameters (e.g., viscosities and densities), rock-fluid parameters (end-point relative permeabilities, residual saturations), well indices, water aquifer size, fault multipliers, and permeability multipliers ($\mathbf{K}_v/\mathbf{K}_h$). In this paper we will mainly present methods for adjusting permeability (or porosity) based on fluid production data observed in wells.

The data available about a reservoir are often classified as two types depending on their association with fluid movement. Static data or *a priori* (prior) data, come from core analysis, well logs, seismics, outcrops, and so on. Dynamic data or *a posteriori* (posterior) data, primarily come from production history, e.g., rate, fractional flow (water-cut), well pressure, well testing, tracer testing, and so on. Common for all dynamic data is that they originate from dynamic processes in the reservoir. Therefore time-lapse seismics can also indirectly be

[1] One should not read for the sake of reading, but rather, to seek what may be useful.

considered as dynamic data. In this review, the term 'history matching' will be used for the process of integrating data to match dynamic observations of the reservoir in the past.

History matching has traditionally been a manual and time-consuming task for the reservoir engineer, consisting of iteratively modifying the reservoir description and running flow simulations for evaluating the resulting reservoir responses. The most common approaches for automated estimation of reservoir parameters, e.g., permeability and porosity, are based on minimization of an objective function (sometimes called a misfit function), which typically has the following form

$$\mathcal{O} = \sum_{j=1}^{N_d} w_j \big(d_j^{\text{obs}} - d_j^{\text{cal}}\big)^2. \tag{2}$$

Here the scalars w_j have been introduced to weight the influence of the individual observations. Algorithms used for minimizing the objective function can be classified as two types: gradient and nongradient methods. Gradient methods use the gradient of the objective function, where the gradient is defined as $\nabla\mathcal{O} = \partial\mathcal{O}/\partial\mathbf{m}$. This gradient is in turn given by the sensitivity matrix, which is the gradient of the calculated responses $\mathbf{d}^{\text{cal}} = g(\mathbf{m})$ with respect to the parameters $\mathbf{m}$,

$$\mathbf{G} = \frac{\partial \mathbf{d}^{\text{cal}}}{\partial \mathbf{m}}, \qquad G_{ji} = \frac{\partial d_j^{\text{cal}}}{\partial m_i}.$$

The sensitivity coefficients G_{ji} measure how a perturbation in the parameter vector effects the responses of the system. Efficient computation of these quantities is a crucial point when developing an efficient parameter-estimation method. Commonly used gradient algorithms include Gauss–Newton, quasi-Newton, steepest descent, conjugate gradients, and Levenberg–Marquardt, see e.g., [10, 125]. Gradient methods converge relatively fast, but may easily fail if the objective function is nonsmooth, in which case the solution may get stuck in a local minimum. Nongradient methods, as the name says, do not use gradients to minimize the objective function. Common algorithms of this group include simulated annealing, genetic algorithms, neighborhood algorithms, etc. [118, 125]. These methods are fairly simple to implement, are always able to reach a global minimum, but may have relatively slow convergence and thus require a large number of forward simulations, which are usually the most computationally expensive part of a history-matching algorithm.

History-matching is usually an ill-posed problem, for which a unique solution seldom exists. Indeed, the number of data points $\mathbf{d}$ to be matched is typically much lower than the number of parameters $\mathbf{m}$ to be modified. Further, there may be redundancy in the information represented in the data. The inverse problem is therefore usually strongly under-determined, so a lot of possible reservoir parameters $\mathbf{m}$ can potentially match the data $\mathbf{d}$. Moreover, there are strong nonlinearities, model errors, and numerical errors involved in the forward model. In addition, there are uncertainties associated with the measured data. Thus, constraints are required to guide the descent towards the inverse solution and make it more stable. In practice this is often done by adding regularization terms to the objective function, e.g., by constraining to prior geological information. Moreover, the non-uniqueness and all the errors involved make uncertainty assessments important.

History-matching methods can be divided into deterministic and stochastic methods. A deterministic method can be described as a function $\mathbf{m} = f(\mathbf{m}_p, \mathbf{d}, \ldots)$ that takes a single prior reservoir model $\mathbf{m}_p$ to a single updated reservoir model $\mathbf{m}$ that accounts for the production data. In other words, deterministic methods intend to obtain an inverse/backward solution $\mathbf{m} = g^{-1}(\mathbf{d})$ for the deterministic forward model $\mathbf{d} = g(\mathbf{m})$. As discussed above, this is a very hard problem that requires some kind of constraining.

Stochastic methods are often referred to as geostatistical methods, and are sometimes (as we will see) coupled in some sense with deterministic methods. Geostatistical methods describe the reservoir model, more or less formally, by a probability distribution $f(\mathbf{m})$, for which a realization is denoted by $[\mathbf{m}] \sim f(\mathbf{m})$. Rather than having an analytic representation, probability distributions are often represented by an ensemble of realizations said to be sampled from or span the probability distribution. The initial probability distribution is often referred

to as the prior distribution, and should incorporate the static data in a geostatistical manner. The spatial covariance structure of the reservoir properties can be incorporated via a variogram or covariance function. This is referred to as two-point statistics (variogram based), but multi-point statistics are needed to describe complex structures like fractures and channels [25, 129]. Conditioning on the dynamic data can be done by specifying a likelihood model $f(\mathbf{d}|\mathbf{m})$ that relates the reservoir parameters $\mathbf{m}$ of interest to the dynamic data $\mathbf{d}$ through the forward model plus observation error. The prior model and forward model fully specify the posterior model $f(\mathbf{m}|\mathbf{d})$ via Bayes' rule

$$f(\mathbf{m}|\mathbf{d}) = \frac{f(\mathbf{d}|\mathbf{m})f(\mathbf{m})}{f(\mathbf{d})} \propto f(\mathbf{d}|\mathbf{m})f(\mathbf{m}). \tag{3}$$

Although the posterior distribution is generally only known up to a constant, it is possible to sample from the distribution to obtain realizations denoted by $[\mathbf{m}|\mathbf{d}] \sim f(\mathbf{m}|\mathbf{d})$. Methods intended to sample from the posterior distribution, by some kind of simulation-based interference, are referred to as Monte Carlo methods. Commonly used Monte Carlo methods are Markov-chain Monte Carlo (McMC) [44, 119], ensemble Kalman filter (EnKF) [50, 51], particle filter [47], randomized maximum likelihood [87, 107], SIR-algorithm [70], pilot-point (PP) methods [93, 115], and sequential self-calibration (SSC) [28, 62]. The main advantages of geostatistical methods are that small-scale geological variability is incorporated and that uncertainty can be assessed from the realizations. However, history matching of multiple realizations is often very computationally expensive. Therefore, the number of realizations are often kept low or reduced, which may cause poor uncertainty estimates. The selection of a subset of realizations to be pursued for further uncertainty assessments is referred to as ranking, and is usually based on some criteria intended to preserve the information sought to the maximum extent [108]. Comparative studies of geostatistical history-matching methods are reported in the literature [see e.g. 14, 56, 96, 151].

This paper is meant as a review of history-matching methods that are based on a streamline formulation. Streamline simulation has experienced a revival in recent years and has proved to be an effective tools for fast reservoir simulation. Streamline simulators are most efficiently applied to injection-dominated cases and cases where the fluid flow is governed by heterogeneities in the rock properties, well positions and rates, fluid mobilities, etc. Streamlines are well suited for history-matching of reservoir properties to fit dynamic data due to three main reasons: (i) streamline methods are relatively fast compared with traditional finite-difference methods for forward simulation, and (ii) by nature streamline methods give precise information about the geometries of the flow pattern and can be used to define reduced models, for instance injector-producer pairs. (iii) streamline-based sensitivities can be evaluated directly from analytic expressions after a single flow simulation. In this paper we will focus on the latter two points. To this end, we start out by giving a review of streamline-based sensitivities in Section 4, after having introduced the model equations most commonly used in streamline-based history matching in Section 2 and given a quick introduction to streamline simulation in Section 3.

Streamline sensitivities are defined as analytical integrals along streamlines and can be computed very efficiently based on a single flow simulation. First, time-of-flight sensitivities [67, 135, 143] with respect to common reservoir parameters are presented. The time-of-flight sensitivities are the basic building blocks for obtaining streamline-based sensitivities for different dynamic data. Further, we describe streamline-based sensitivities for arrival-time, time-shift, saturation, tracer concentration, fractional flow, and gas-oil ratio. Moreover, we briefly discuss streamline-derived sensitivities for time-lapse amplitudes [136] and sensitivities for pressure interference tests [69, 92]. Finally, we briefly describe sensitivities with respect to the parameters of the gradual deformation method [117].

Section 5 constitutes the main part of this paper and in this section we give a survey of history-matching methods based on streamlines. Rather than discussing methods (or papers) where streamline simulation has been applied merely to provide fast forward simulation, we focus on different uses of streamline methods to modify the geological/reservoir simulation models throughout the history-matching process. However, we will neither go into great

details about different formulations of the history-matching problem as an inverse problem nor will we discuss methods for solving the corresponding inverse problems in great detail. The reference list of this review is quite extensive and will hopefully guide the reader to sources for further reading. In picking references we have tried to cite papers where a complete and/or mature presentation is given, which means that we are not always referring to the first occurrence of an idea. Further, we have prioritized, whenever possible, referring to peer-reviewed papers. Short reviews of streamline-based history matching are also given in [40, 126, 127]. A preliminary version of this paper has also been published as a technical report [95].

The surveyed methods for history matching based upon streamlines can roughly be divided into four different categories:

- *The Assisted History Matching* (AHM) approach was introduced by Emanuel and Milliken [49] and is outlined in Section 5.1. This method defines sub-regions associated with wells in which subsequent targeted changes of grid parameters can be performed manually (or semi-automatic) by a reservoir engineer.
- *Travel-Time Inversion* (TTI) methods were introduced by Vasco, Datta-Gupta and coworkers [135] based upon an analogy with seismic ray inversion. Streamlines are used to estimate sensitivity coefficients analytically, thereby speeding up the optimization on the grid-cell level. The first approach is a two-step approach with a travel-time matching followed by an amplitude matching. Later, so-called *Generalized Travel-Time Inversion* (GTTI) has been introduced to combine travel-time matching and amplitude matching while keeping the desirable convergence properties of travel-time inversion [67]. For the GTTI method time-shifts for the production curves, minimizing the misfit, are jointly propagated to necessary modifications in the reservoir parameters. The methods in this category are described in Section 5.2.
- Methods for matching *streamline effective properties* (SLEP) were first introduced by Wang and Kovscek [141] and have later been extended by others. The key idea of these methods is to relate the mismatch between observed and calculated production data to a mismatch in effective properties along streamlines or streamline bundles, and adjust the effective properties to obtain a satisfactory match. Then the perturbations in effective properties are propagated to individual grid cells (by direct mapping or by a geostatistical algorithm). These methods are described in Section 5.3.
- The final category consists of geostatistical history-matching methods that take advantage of streamline-defined regions or streamline-derived sensitivities. Methods discussed herein include Markov chain Monte Carlo [98], ensemble Kalman filter [9, 45], sequential self-calibration [143], and the gradual deformation method [58, 59]. The methods in this category are described in Section 5.4.

Moreover, in Section 6 we describe some methods for streamline-based ranking of geostatistical realizations of reservoir models [75, 142]. Finally, Section 7 contains a discussion and comparison of some of the methods introduced earlier in the paper.

2. Simplified Flow Models

Almost all the history-matching methods to be surveyed later in the paper are based on simplified flow models. For completeness, we will therefore introduce these models in some detail and specify the accompanying simplifying assumptions.

The fundamental equation describing flow in a porous media is the continuity equation which states that the mass is conserved for phase α

$$\frac{\partial}{\partial t}\left(\phi\rho_\alpha S_\alpha\right) + \nabla\cdot\left(\rho_\alpha \mathbf{u}_\alpha\right) = q_\alpha.$$

Here ϕ denotes porosity, ρ_α is density, S_α is saturation, $\mathbf{u}_\alpha$ is the phase velocity, and q_α models fluid sources and sinks. The saturations are volume fractions and must therefore add up to unity, i.e., $\sum_\alpha S_\alpha = 1$. For the phase velocity, we use the semi-empirical Darcy's law,

which relates the phase velocity to the gradient of the phase pressure p_α,

$$\mathbf{u}_\alpha = -\lambda_\alpha \mathbf{K}\big(\nabla p_\alpha - \rho_\alpha \mathbf{g}\big), \tag{4}$$

where λ is the relative fluid mobility, $\mathbf{K}$ is the tensor of absolute permeability, and $\mathbf{g}$ is the gravity vector (pointing downwards). The relative mobility is defined as $\lambda_\alpha = k_{r\alpha}/\mu_\alpha$, where $k_{r\alpha}$ is the relative permeability, generally a function of the concentrations/saturations of other present phases, and μ_α is the viscosity of phase α.

For two-phase flow of oil and water ($\alpha = o, w$) this gives three equations and a set of constitutive relations, for which it is common to choose a pressure and the water saturation as the primary unknowns. By manipulating continuity equations and Darcy's law, one can derive the so-called fractional formulation consisting of an equation for the pressure and an equation describing fluid transport, which is referred to as the saturation equation. The pressure equation has more or less elliptic characteristics depending on the compressibility of the rock and fluids, and the saturation equation is more or less hyperbolic, depending on capillary pressures. Most of the methods discussed later assume incompressible and immiscible flow. Using these assumptions and introducing the total velocity $\mathbf{u} = \mathbf{u}_o + \mathbf{u}_w$ and a so-called global pressure p (see e.g., [2]) as primary unknowns, the coupled system can be written as,

$$\nabla \cdot \mathbf{u} = q, \qquad \mathbf{u} = -\mathbf{K}\Big[\lambda_t \nabla p - (\lambda_w \rho_w + \lambda_o \rho_o)\mathbf{g}\Big], \tag{5}$$

$$\phi \frac{\partial S_w}{\partial t} + \nabla \cdot \Big[f_w\big(\mathbf{u} + \mathbf{K}\lambda_o \nabla p_{\text{cow}} + \mathbf{K}\lambda_o \mathbf{g}\Delta\rho\big)\Big] = \frac{q_w}{\rho_w}. \tag{6}$$

Here we have introduced the total mobility $\lambda_t = \lambda_w + \lambda_o$, the fractional-flow function of water $f_w = \lambda_w/\lambda_t$, the capillary pressure $p_{\text{cow}} = p_o - p_w$, the density difference $\Delta\rho = \rho_w - \rho_o$, and the total contribution from the wells $q = q_w/\rho_w + q_o/\rho_o$. The two equations are coupled since the mobilities λ_α depend on the water saturation. We will refer to (5) and (6) as the pressure- and the transport equation, respectively.

The majority of the history-matching methods also assume negligible gravity and capillary forces, i.e., that the terms involving $\mathbf{g}$ and p_{cow} vanish, and we can define $p = p_w = p_o$. Further, for incompressible flow $\nabla \cdot \mathbf{u} = 0$ away from the wells, so $\nabla \cdot (f_w \mathbf{u}) = \mathbf{u} \cdot \nabla f_w$. Moreover, when discussing oil-water systems, we drop the subscript 'w'. Hence, the system is considerably simplified

$$\nabla \cdot \mathbf{u} = q, \qquad \mathbf{u} = -\mathbf{K}\lambda_t \nabla p, \tag{7}$$

$$\phi \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla f = \tilde{q}. \tag{8}$$

Unless stated otherwise, this will therefore be our flow model in the following sections and the dynamic data observed will typically be the fractional flow (or water cut) in wells.

For streamline methods, the coupled system (7)–(8) is solved using a sequential splitting: First, the current saturation field is used to evaluate the mobilities $\lambda_t(S)$ in (7), and the equation is solved for the pressure and velocity. Then the velocity field $\mathbf{u}$ is held fixed for a given time step while the saturation is advanced forward in time according to (8). How this is done, will be explained in the next section. After the saturation has been advanced forward in time, the new values are used to update the mobilities in (7), an so on.

In the above model, the permeability has been assumed to be a tensor. Within streamline-based history matching, permeability usually is considered as isotropic and can therefore be described by a scalar function. Using anisotropic permeability will make the inversion problem much more under-determined. Moreover, the dynamic well data to be matched are often noisy and spatially convoluted, and therefore contain limited spatial and small-scale information. To adjust the permeabilities in other directions, multipliers or correlations are usually applied, for which the involved parameters may also be history-matched. However, some of the derivations presented later in this paper may in principle apply directly to a diagonal or full permeability tensors as well.

3. Streamline methods

For the sake of completeness we give a very brief introduction to streamline simulation. For a given velocity field $\mathbf{u}$, a streamline is a line that is everywhere tangential to $\mathbf{u}$, that is,

$$\frac{d\mathbf{x}}{dr} = \frac{\mathbf{u}}{|\mathbf{u}|}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

Rather than using the arc length r to parameterize streamlines, it is common to introduce the so-called time-of-flight τ, which takes into account the reduced volume available for flow, i.e., the porosity ϕ. Time-of-flight is defined by the following integral

$$\tau(R) = \int_0^R \frac{\phi(\mathbf{x}(r))}{|\mathbf{u}(\mathbf{x}(r))|}\, dr = \int_0^R s(\mathbf{x}(r))\, dr, \tag{9}$$

where τ expresses the time it takes a passive particle to travel a distance R along a streamline (in the interstitial velocity field $\mathbf{v} = \mathbf{u}/\phi$). The function $s(\mathbf{x})$ is often referred to as the slowness function. Alternatively, by the fundamental theorem of calculus and the directional derivative, τ can be expressed by the following differential equation [41]

$$\frac{\phi}{|\mathbf{u}|} = \frac{d\tau}{dr} = \frac{\mathbf{u}}{|\mathbf{u}|}\cdot\nabla\tau \quad \Rightarrow \quad \mathbf{u}\cdot\nabla\tau = \phi, \tag{10}$$

which we, in lack of a better name, will refer to as the time-of-flight equation. We will denote the time-of-flight increment over grid cell i by $\Delta\tau_i$. Hence, the time-of-flight at the well can be written as the sum of the traversal times for all the N_c grid cells intersected by the streamline; $\tau = \sum_{i=1}^{N_c} \Delta\tau_i$. (Regarding subscripts, we will henceforth use indices i, j, k and ℓ to denote grid cells, times, wells and streamlines, respectively. Moreover, we use the expression 'grid cell', rather than 'grid block', when a method in theory can be applicable to more general grid cells than non-degenerated quadrilateral or hexahedral grid blocks.)

Streamlines and time-of-flight can be used to define an alternative curvilinear and flow-based coordinate system in three dimensions. To this end, we introduce the bi-streamfunctions ψ and χ [18], for which $\mathbf{u} = \nabla\psi \times \nabla\chi$. In the streamline coordinates (τ, ψ, χ), the gradient operator is expressed as

$$\nabla_{(\tau,\psi,\chi)} = (\nabla\tau)\frac{\partial}{\partial\tau} + (\nabla\psi)\frac{\partial}{\partial\psi} + (\nabla\chi)\frac{\partial}{\partial\chi}. \tag{11}$$

Moreover, a streamline Ψ is defined by the intersection of a constant value for ψ and a constant value for χ. Because $\mathbf{u}$ is orthogonal to $\nabla\psi$ and $\nabla\chi$, it follows that

$$\mathbf{u}\cdot\nabla_{(\tau,\psi,\chi)} = (\mathbf{u}\cdot\nabla\tau)\frac{\partial}{\partial\tau} = \phi\frac{\partial}{\partial\tau}. \tag{12}$$

Therefore the coordinate transformation $(x, y, z) \to (\tau, \psi, \chi)$ will reduce the three-dimensional transport equation

$$\phi\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla f(S) = 0.$$

to a family of one-dimensional transport equations along each streamline [41, 85],

$$\frac{\partial S}{\partial t} + \frac{\partial f(S)}{\partial\tau} = 0. \tag{13}$$

In other words, there is no exchange of the quantity S between streamlines and each streamline can be viewed as an isolated flow system.

For each streamline a constant volumetric flux q_ℓ is associated. Quantities like the total water rate, tracer concentration, fractional flow and gas-oil ratio (GOR) at a well can be obtained by in some sense summing the quantities of the contributing streamlines. For instance the fractional flow and total rate at a producer at time t are given by [17]

$$f(t) = \frac{1}{q}\sum_{\ell=1}^{N_{\mathrm{sl}}} q_\ell f_\ell(t), \qquad q = \sum_{\ell=1}^{N_{\mathrm{sl}}} q_\ell, \tag{14}$$

where N_{sl} is the number of streamlines connected to the well, q_ℓ is the total flux assigned to streamline ℓ, and $f_\ell(t)$ is the fractional flow associated with streamline ℓ at time t.

In modern streamline methods one does not need to represent the path of a streamline explicitly in three-dimensional space to perform the mapping back and forth between physical space and streamlines. Instead, the parameterization, i.e., the integral in (9), can be computed numerically on a cell-by-cell basis. The one-dimensional time-of-flight grid is obtained by tracing a streamline forward and/or backward towards a sink/source (wells) [17]. Most commonly used is a semi-analytical tracing algorithm introduced by Pollock [112], which uses analytical expressions of the streamline paths inside each cell based on the assumption that the velocity field is piecewise linear locally. Although Pollock's method is only valid for regular grids, it is often used also for highly skewed and irregular grids. Other approaches for tracing on unstructured grids and the associated accuracy are discussed in [38, 66, 79, 102, 103, 113].

As mentioned above, (13) is solved numerically forward in time on a sequence of steady-state approximations for the velocity field, just as done for an IMPES formulation in a finite-difference simulator. In general, the streamline trajectories will change for unsteady flow cases, for instance for non-unit mobility ratios (especially favorable) or because of changes in the well configuration (e.g., infill drilling or temporal rates/pressure constraints). For unsteady flow the changes in streamline trajectories are accounted for by regenerating the streamlines periodically through pressure/velocity updates. The saturations/concentrations are then mapped back and forth between the pressure grid and the streamlines for each update (described below). The efficiency of streamline simulation compared to conventional finite-difference simulators is traditionally primarily caused by the ability of taking longer pressure steps within the sequential splitting formulation [86].

The streamline formulation can also be applied to describe flow including more physical effects than those described in the simple two-phase model (7)–(8). A similar decomposition of the 3D transport equation can also be performed for compressible flow [36], in which case the one-dimensional transport equation will have a source term on the right-hand side. Further, the decomposition has been extended to compositional flow with compressibility effects [109]. Moreover, it is possible to include gravity and capillary forces by operator splitting, as discussed in [22, 60, 61, 86]. However, gravity and capillary forces may enforce fluxes traverse to the direction defined by the total velocities, and therefore separate sets of streamlines have to be used for the gravity and capillary steps. Extensions to fractured reservoir flow have also been reported [6, 46, 88], also resulting in source terms in the 1D transport equations.

3.1. **Linear Transport.** For the special case of piston-like displacement, which will be a key assumption in several of the history-matching methods discussed below, the flux function is linear. Similarly, for the neutral advection of a passive tracer, (13) reads

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial \tau} = 0. \tag{15}$$

The injector has a concentration history $C_0(t)$, which gives a time-dependent boundary-value problem for (15). The response at the producer reads [41],

$$C(t) = C_0(t-\tau) = C_0\Big(t - \int_\Psi s(\mathbf{x})\, dr\Big), \tag{16}$$

which is easily verified by inserting the expression into (15) and the fact that the solution is unique [73]. For the special case of continuous and constant injection (which is equivalent to piston-like displacement), the solution is particularly simple

$$C(t) = \begin{cases} 0, & t < \tau, \\ C_0, & t > \tau. \end{cases}$$

Dispersion is not accounted for in (15) or (16), but this can also be accounted for [see e.g. 76]. Further, for instance for a partitioning tracer being partially absorbed into the oil phase the travel time along a streamline will be increased in the presence of oil saturation. This can be

accounted for by increasing the slowness $s(\mathbf{x})$ by the partitioning properties of the tracer [76]

$$s(\mathbf{x}) = \frac{\phi(\mathbf{x})}{|\mathbf{u}(\mathbf{x})|}(S_w + P_o S_o). \tag{17}$$

Here, S_o and S_w are the oil and water saturations, respectively, and P_o is the partitioning coefficient of the tracer defined as the ratio of tracer concentration in the oil phase to that in the water phase. Hence, P_0 will take on unity for a neutral tracer.

3.2. **Buckley–Leverett Displacement.** A common assumption in many history-matching methods is to assume a so-called Buckley–Leverett profile along each streamline. That is, one considers the one-dimensional transport equation in (13) and assumes constant initial saturation S_0 along each streamline and a constant injection state S_i. Mathematically, this corresponds to a so-called Riemann problem with initial data

$$S(0,\tau) = \begin{cases} S_i & \text{for } \tau < 0, \\ S_0 & \text{for } \tau \geq 0. \end{cases}$$

Since both the one-dimensional transport equation and the initial data are scale-invariant or self-similar—that is, invariant under the map $\tau \to k\tau$ and $t \to kt$—the solution should also have that property, i.e., $S(t,\tau) = S(\tau/t)$. More specifically, for Riemann initial data the solution of the one-dimensional transport equation is given by the analytic Buckley–Leverett solution (Riemann solution) [73]

$$S(t,\tau) = \begin{cases} S_i & \text{for } \tau \leq t\widetilde{f}'(S_i), \\ (\widetilde{f}')^{-1}(\frac{\tau}{t}) & \text{for } t\widetilde{f}'(S_i) \leq \tau \leq t\widetilde{f}'(S_0), \\ S_0 & \text{for } \tau \geq t\widetilde{f}'(S_0). \end{cases} \tag{18}$$

Here $\widetilde{f}$ denotes the upper concave envelope of f if $S_i > S_0$, and the lower convex envelope of f if $S_i < S_0$. The front saturation $\tilde{S}$ can be determined by solving the equation [73]

$$f'(\tilde{S}) = \frac{f(\tilde{S}) - f(S_0)}{\tilde{S} - S_0},$$

and the injection front will arrive at the well at time $\tau/f'(\tilde{S})$.

3.3. **General Displacement.** For cases where the injection problem is not a simple Riemann problem, the one-dimensional transport equation (13) must generally be solved numerically. The standard approach for solving the scalar problems numerically along streamlines is to use a finite-difference or finite volume method. The simplest such scheme is the first-order upwind scheme,

$$S_i^{j+1} = S_i^j - \frac{\Delta t}{\Delta \tau}\Big[f(S_i^j) - f(S_{i-1}^j)\Big].$$

For explicit schemes a so-called CFL condition has to be fulfilled with respect to time step size to keep the numerical solution stable. The CFL condition usually puts a severe restriction on the time-step size compared to what is required with respect to accuracy. To make the critical time-step size less restrictive, the time-of-flight grid is often mapped to a more regular grid for these schemes. For multi-phase and compositional flow with strong nonlinear couplings in the system of flow equations, the sharpness and the accuracy of the propagation speeds need to be accurately represented, which may require more accurate schemes for solving along streamlines. We here confine ourself with just listing a few recent related works [90, 99, 111, 128, 131].

Alternatively, one may use an implicit scheme of the form

$$S_i^{j+1} = S_i^j - \frac{\Delta t}{\Delta \tau}\Big[f(S_i^{j+1}) - f(S_{i-1}^{j+1})\Big].$$

to escape the stability restrictions. For each time step, a system of N_c equations, where N_c is the number of unknowns along the streamline, has to be solved. The implicit scheme has larger numerical diffusion than its explict counterpart, and the amount of diffusion (or smearing of sharp fronts) increases with the time step. It is therefore customary to use implicit

time steps that are smaller than the pressure steps. In other words, there are two different time steps involved in streamline simulation: the pressure step of the sequential splitting, and the local time step used in the transport solve.

For two-phase (and other scalar problems), a much better approach is to use front tracking [73] as discussed in [19–21]. This grid-independent method is based on an entirely different approach than finite-difference schemes and is unconditionally stable and devoid of numerical diffusion. Instead of discretizing the problem spatially, the initial data S_0 and the flux function f are approximated by piecewise constant and piecewise linear functions, respectively. The resulting approximated problem consists of a set of Riemann problems that can be solved exact analytically forward in time given an exact Riemann-solution for the given equation. During the forward solves the different Riemann solutions will interact and create new Riemann problems, and so on. The solution of each Riemann problem is given by the Buckley–Leverett construction discussed in the previous section, which for a linear flux function simplifies to a step-function with discontinuities propagating along space-time rays, see [73] for more details. Extensions to miscible and 3-phase flow for front tracking are addressed in [83, 84].

3.4. **Mapping Between Pressure Grid and Streamlines.** A crucial step in streamline methods is the mapping of saturations from the pressure grid to streamlines, and vice versa. Mapping from pressure grid to streamlines is usually performed by simply picking up the piecewise-constant saturation values from the grid cells that are intersected by the streamline. Alternatively, higher accuracy is obtained if one first makes a piecewise linear reconstruction on the pressure grid before mapping to streamlines, as suggested by Mallison et al. [100]. Contrary, mapping the saturations from streamlines to grid cells is done by

$$S_i = \frac{\sum_\ell S_{\ell,i} V_{\ell,i}}{\sum_\ell V_{\ell,i}}, \tag{19}$$

where $V_{\ell,i} = q_\ell \Delta\tau_{\ell,i}$ is the pore volume associated with streamline ℓ over grid cell i [17]. Unfortunately, this mapping may potentially introduce large errors in the mass balance of the reservoir. Commercial streamline solvers therefore use some kind of correction to counteract the lack of mass balance. One such simple approach is discussed by Stenerud et al. [122], who suggest to adjust the time-of-flight locally to preserve the local pore volume for both the mappings. Doing so reduces significantly the number of streamlines required to obtain accurate production curves, which is more important within history matching than obtaining high local accuracy in space. Mallison et al. [100] suggest another, and entirely different, approach based on a geostatistical kriging mapping in which streamlines are no longer seen as fluid carriers but rather as an unstructured, flow-based grid for computing fluid transport.

Finally, we mention that spatial errors and convergence in streamline simulation have been studied by Jimenez et al. [79]. For further details on streamline simulation, we refer to the upcoming textbook by Datta-Gupta and King [40].

4. Streamline-Based Computation of Sensitivities

One of the benefits of applying streamlines for history-matching is the possibility of fast evaluation of reservoir-response sensitivities. These sensitivities can be evaluated analytically after a single forward simulation. Using streamlines to compute sensitivities is thus an optimal approach. We will here review the current literature on streamline-based sensitivity computations, starting by discussing the computational cost and applicability of streamline-based relative to traditional methods for computing sensitivities.

4.1. **Efficiency and Applicability.** Traditional methods for computing sensitivities of multiphase production data with respect to reservoir parameters can be divided into three categories: perturbation methods, gradient-simulator methods [8, 62], and adjoint or optimal control methods [30, 32, 94, 148, 149]. We will not go into much details about these methods, but we will briefly discuss the computational costs and compare them with the streamline-based approaches.

The perturbation method is the brute force approach for obtaining sensitivities. Each parameter is perturbed followed by a flow simulation to evaluate the resulting perturbations in the production responses. For N parameters this approach therefore requires $N+1$ simulations and is thus very expensive. Both the gradient-simulator method and the adjoint method require one forward simulation and one or more solutions of a system of linear equations of the same size as the discretized system for the flow equations (different right-hand sides). The gradient-simulator method requires the solution of a linear system to obtain sensitivities for the state variables with respect to a parameter of interest. The linear system is obtained by differentiating a discretized version of the flow equations with respect to the parameter of interest. For instance, consider a pressure system $\mathbf{Ap} = \mathbf{b}$. By differentiating this equation with respect to a parameter m we obtain

$$\frac{\partial \mathbf{A}}{\partial m}\mathbf{p} + \mathbf{A}\frac{\partial \mathbf{p}}{\partial m} = \frac{\partial \mathbf{b}}{\partial m},$$

which can be solved to obtain $\partial \mathbf{p}/\partial m$. Hence, a linear system has to be solved one time for each reservoir parameter of interest for each simulator step. The gradient-simulator method is usually not as expensive as the perturbation method because the sensitivities are not necessarily needed for all steps. For the adjoint method, one needs to solve an adjoint linear system for every gradient needed. Therefore, one solution is needed to obtain the gradient of an objective function, while the number of solutions to obtain the sensitivity matrix is equal to the number of data points to be integrated. The adjoint systems are solved backward in time from the end of the last pressure step, and this requires storage of the intermediate saturation and pressure information for the pressure steps. For a more thorough presentation of the adjoint method see for instance [148], where it is described in the appendix how the adjoint method can be used to obtain time-shift sensitivities, cf. Section 4.4. An alternative to the adjoint method for obtaining the *gradient* of an objective function is the stochastic gradient approach used in the SPSA algorithm [57, 120], which requires two evaluations of the objective function (two flow simulations) to obtain a realization of the gradient. This approach thus has similar computational complexity as the adjoint method, but can easily be implemented on top of any flow simulator because mainly evaluations of the objective function are required. The theoretical foundation for this approach is that at least for a quadratic objective function the expectation value of the stochastic gradient is the true gradient [57]. However, the stochastic nature of the gradient may slow down the convergence of the history matching as demonstrated in [57].

The streamline-based approaches described below are superior with respect to efficiency for obtaining the sensitivity matrix for large systems because they only require one forward simulation and a post-processing step which basically boils down to bookkeeping of analytic arithmetic computations.

Sensitivities describe how calculated reservoir responses will react to a small perturbation in the reservoir description. The sensitivities depend, in principle, on the way the reservoir responses are calculated, e.g., by a flow simulator, by an inverse seismics-to-saturation model, etc., and should account for errors made in the calculation. However, highly accurate sensitivities are seldom needed for applications in inverse modeling of petroleum reservoirs, since sensitivities are mostly used within an iterative inversion algorithm to determine in which direction one should perturb the solution in the next iteration. Moreover, large uncertainties in the reservoir description will in general mask errors made in the calculation of sensitivities. In practice, sensitivities obtained by one reservoir response simulator may perform well for another simulator too. For instance, streamline sensitivities may be obtained by using the velocity fields of a finite-difference simulator, even though streamlines are not used for the actual flow simulation, see e.g., [35]. This extends the applicability of fast streamline-based sensitivity calculations considerably.

4.2. **Time-of-Flight Sensitivities.** The sensitivity of the time-of-flight with respect to reservoir parameters is the basic building block used to obtain streamline-based sensitivities for reservoir responses. We will therefore start out by presenting two different approaches for deriving time-of-flight sensitivities.

The first approach is used in particular by Datta-Gupta and coworkers. By Darcy's law (7) for the total velocity $\mathbf{u}$ and the time-of-flight definition in (9), the time-of-flight is related to reservoir properties by

$$\tau = \int_\Psi \frac{\phi(\mathbf{x})}{\lambda_t K(\mathbf{x})|\nabla p|}\, dr = \int_\Psi \frac{\phi(\mathbf{x})A(\mathbf{x})}{q}\, dr = \int_\Psi s(\mathbf{x})\, dr. \tag{20}$$

Here ϕ is the porosity, K is the absolute permeability, p is the pressure, λ_t is the total mobility, A is the streamtube cross sectional area, and q is the total volumetric streamline rate.

The sensitivity of τ with respect to a reservoir parameter m can then be defined by [135]

$$\frac{\partial \tau}{\partial m} = \int_\Psi \frac{\partial s(\mathbf{x})}{\partial m(\mathbf{x})}\, dr, \tag{21}$$

where $\partial s(\mathbf{x})/\partial m(\mathbf{x})$ typically is given by, for instance

$$\frac{\partial s}{\partial K} = -\frac{\phi}{\lambda_t K^2 |\nabla p|} = -\frac{s}{K}, \tag{22}$$

$$\frac{\partial s}{\partial \phi} = \frac{1}{\lambda_t K |\nabla p|} = \frac{s}{\phi}, \tag{23}$$

$$\frac{\partial s}{\partial |\nabla p|} = -\frac{\phi}{\lambda_t K |\nabla p|^2} = -\frac{s}{|\nabla p|}, \tag{24}$$

$$\frac{\partial s}{\partial \lambda_t} = -\frac{\phi}{\lambda_t^2 K |\nabla p|} = -\frac{s}{\lambda_t}, \tag{25}$$

$$\frac{\partial s}{\partial q} = -\frac{\phi A}{q^2} = -\frac{s}{q}. \tag{26}$$

Similar expressions for various relative permeability parameters are described in Appendix A. We now assume that each reservoir parameter m_i is constant inside grid cell i. Then a time-of-flight sensitivity can be associated with each grid cell: The sensitivity with respect to permeability, for instance, is given by

$$\frac{\partial \tau}{\partial K_i} = \frac{\partial \Delta\tau_i}{\partial K_i} = \int_{\Psi_i} \frac{\partial s(\mathbf{x})}{\partial K_i}\, dr = \int_{\Psi_i} -\frac{s(\mathbf{x})}{K_i}\, dr = -\frac{\Delta\tau_i}{K_i}. \tag{27}$$

The sensitivities are calculated under the assumption that the streamlines do not shift as a result of a small perturbation in the reservoir properties. Further, it is assumed that the different reservoir properties are independent in the sense that a small perturbation in one property does not perturb any of the other properties. However, especially the pressure will generally depend on the permeability distribution, but this dependence is usually neglected.

Tracer partitioning can be accounted for by defining the slowness function $s(\mathbf{x})$ as in (17) [43, 76]. Illiassov and Datta-Gupta [76] also use this formulation to compute time-of-flight sensitivities with respect to saturations. Further, we remark that it may be possible to account for gravity and capillary pressure in the time-of-flight sensitivities by using the total Darcy velocity (5) accounting for these effects in the slowness function $s(\mathbf{x})$.

Wen et al. [143] present a more general approach to account for the pressure impact on the time-of-flight and the spatial correlation for permeability. These sensitivities were derived for application to the sequential self-calibration (SSC) method, which will be discussed in Section 5.4.3. For the SSC method, the sensitivities are associated with master points rather than grid cells, and therefore the spatial correlations between grid cells and the master points are important. We will here index master points by subscript d. The time-of-flight in each cell i is a function of the transmissibilities $\{T_{i,n}\}$ associated with the cell faces and the pressures $\{p_{i,n}\}$ in the cell and its surrounding neighbors. A straightforward differentiation, applying the chain rule along a streamline intersecting N_c grid cells gives

$$\frac{\partial \tau}{\partial K_d} = \sum_{i=1}^{N_c} \Big[\sum_n \frac{\partial \Delta\tau_i}{\partial T_{i,n}} \frac{\partial T_{i,n}}{\partial K_d} + \sum_n \frac{\partial \Delta\tau_i}{\partial p_{i,n}} \frac{\partial p_{i,n}}{\partial K_d} \Big]. \tag{28}$$

To obtain $\partial\Delta\tau_i/\partial T_{i,n}$ and $\partial\Delta\tau_i/\partial p_{i,n}$ analytically, Wen et al. [143] differentiate the expressions for $\Delta\tau_i$ used in the Pollock's tracing algorithm. Further, the pressure sensitivities $\partial p_{i,n}/\partial K_d$ are obtained by the gradient-simulator method described above [62, 143] and are not streamline based, and thus more expensive. Finally, using the harmonic average to calculate the transmissibility between two cells gives [62]

$$\partial T_{i,n}/\partial K_d = \frac{T_{i,n}^2}{2}\Big(\frac{\omega_{d,i}}{K_i} + \frac{\omega_{d,n}}{K_n}\Big), \tag{29}$$

where $\omega_{d,i}$ and $\omega_{d,n}$ are the kriging weights associated with master point d, cell i, and face n (adjacent cell). Hence, the spatial correlations of permeability perturbations are accounted for through the kriging weights in (29).

If one assumes that the pressure (the gradient) is independent of a small perturbation in the permeability, like assumed in (27), the sensitivities are reduced to

$$\frac{\partial\tau}{\partial K_d} = \sum_{i=1}^{N_c}\frac{\partial\Delta\tau_i}{\partial K_d} = \sum_{i=1}^{N_c}\frac{\partial\Delta\tau_i}{\partial K_i}\frac{\partial K_i}{\partial K_d} = -\sum_{i=1}^{N_c}\frac{\Delta\tau_i}{K_i}\frac{\partial K_i}{\partial K_d} = -\sum_{i=1}^{N_c}\frac{\Delta\tau_i}{K_i}\omega_{d,i}, \tag{30}$$

where $\omega_{d,i}$ is the kriging weight of master point d and cell i. For given kriging weights, this is a fully analytic approximation to the sensitivity coefficients that should apply directly to the same reservoir parameters as (21) does, given that appropriate covariance structures can be defined for the parameters.

The kriging weights involved in (29) and (30) can be obtained by solving an ordinary kriging system [62]. Further, the kriging weights only depend on the spatial locations of the master points and the locations being interpolated, so the kriging system only need to be solved once for a fixed set of master points, interpolation points, and covariance structure. The same weights may also be used to propagate the updated permeabilities of the master locations in the inverse problem, see Section 5.4.3.

If a master point coincides with a cell j and we assume that a perturbation of the permeability K_j only contribute to a perturbation of $\Delta\tau_j$, i.e.,

$$\omega_{j,i} = \begin{cases} 0, & \text{for } i \neq j, \\ 1, & \text{for } i = j, \end{cases} \tag{31}$$

then (30) reduces to (27).

In choosing between the two approach introduced above, we note that calculating the time-of-flight sensitivities by (21) is the fastest approach because the approach mainly boils down to bookkeeping of time-of-flights over each grid cell. Moreover, we remark that for the purpose of history matching, it often turns out in practice (see [67, 146]) to be sufficient to apply the cell-based approximations (27) and (30), which are the less computationally expensive (but also more approximate).

4.3. **Arrival-Time Sensitivities.** An arrival time measures the time it takes a quantity to propagate from one point in the reservoir to another, e.g., the time it takes from one starts injecting water in at an injector to the water front break through in a producer. In this subsection we will present an approach for computing the sensitivity $\partial t_j/\partial m_i$ of an arrival time t_j with respect to reservoir parameter m_i of grid cell i. This sensitivity is also sometimes referred to as a travel-time sensitivity [67] (analogy to ray-tracing in seismics). Consider a system of two-phase flow given in the time-of-flight coordinate along each streamline by the one-dimensional transport equation (13). If the streamlines are assumed to be invariant under the perturbation in reservoir parameters, the shift in the saturation at the outlet nodes is given by

$$\delta S = \frac{\partial S}{\partial t}\delta t + \frac{\partial S}{\partial \tau}\Big[\frac{\partial\tau}{\partial \mathbf{m}}\Big]^T\delta\mathbf{m}.$$

Let us consider the propagation of a fixed saturation, i.e., $\delta S \equiv 0$, or in other words

$$0 = \frac{\partial S}{\partial t}\delta t + \frac{\partial S}{\partial \tau}\Big[\frac{\partial\tau}{\partial \mathbf{m}}\Big]^T\delta\mathbf{m}.$$

If we now perturb $\mathbf{m}$ only in the ith component and solve for $\delta t/\delta m_i$, we obtain

$$\frac{\partial t}{\partial m_i} = -\frac{\partial S}{\partial \tau}\frac{\partial \tau}{\partial m_i}\cdot\Big(\frac{\partial S}{\partial t}\Big)^{-1} = \frac{1}{f'(S)}\frac{\partial \tau}{\partial m_i}. \tag{32}$$

Here time-of-flight derivative $\partial\tau/\partial m_i$ is computed analytically as described in Section 4.2. In deriving (32) we have tacitly assumed that the fixed saturation propagates with a constant wave-speed $f'(S)$ from its 'release' (at an injector) to its 'arrival' at a well. Whereas this is true for a piston-like displacement or for a neutral tracer flow, for which $f(S) = S$ and we obtain $\partial t/\partial m_i = \partial\tau/\partial m_i$ as expected, it will generally not be true for a general *nonlinear* displacement. For a pure Buckley–Leverett displacement, $f'(S)$ should be replaced by the derivative of the convex envelope of the flux $\widetilde{f}'(S)$, see Section 3.2. (Alternatively, this can be derived directly by direct differentiation of the self-similar Buckley–Leverett solution, for which $\widetilde{f}'(S) = \tau/t$.) In other words, df/dS is evaluated at the saturation of outlet node of the streamline for streamlines with breakthrough (i.e., having outlet saturation larger than the front saturation), and for the front saturation for streamlines without breakthrough. For other flow cases where the initial boundary-value problem along each streamline does not consist of a single Riemann problem, the accuracy of (32) depends on how well $f'(S)$ (or $\widetilde{f}'(S)$) approximates the *true* wave-speed of the fixed saturation during the time interval from release to arrival.

He et al. [67] also extend the arrival-time sensitivity to account for changing saturation distribution along streamlines due to changes in the streamline geometry, pressure updates, and mapping of saturations between streamlines as part of an operator splitting algorithm to account for gravity and/or capillary forces. The change in the saturation in the outlet cell will now also be a function of the initial saturation distribution $\mathbf{S}_0$ along the streamline (i.e., the saturation after the previous pressure update)

$$\delta S = \frac{\partial S}{\partial t}\delta t + \frac{\partial S}{\partial \tau}\Big[\frac{\partial \tau}{\partial \mathbf{m}}\Big]^T\delta\mathbf{m} + \Big[\frac{\partial S}{\partial \mathbf{S}_0}\Big]^T\delta\mathbf{S}_0.$$

If we now assume that the change in the water saturation in the outlet cell is primarily a function of the initial saturation $S_{0,j}$ in the same cell (which is true for a small time due to finite speed of propagation in hyperbolic equations), the last term becomes

$$\Big[\frac{\partial S}{\partial \mathbf{S}_0}\Big]^T\delta\mathbf{S}_0 = \frac{\partial S}{\partial S_{0,j}}\delta S_{0,j} = \frac{\partial S}{\partial S_{0,j}}\Big[\frac{\partial S_{0,j}}{\partial \mathbf{m}}\Big]^T\delta\mathbf{m}.$$

Hence, the overall sensitivity reads [67]

$$\frac{\partial t}{\partial m_i} = -\Big(\frac{\partial S}{\partial \tau}\frac{\partial \tau}{\partial m_i} + \frac{\partial S}{\partial S_{0,j}}\frac{\partial S_{0,j}}{\partial m_i}\Big)\cdot\Big(\frac{\partial S}{\partial t}\Big)^{-1} = \frac{1}{f'(S)}\frac{\partial \tau}{\partial m_i} + \frac{\partial t_j^0}{\partial m_i}.$$

where $\partial t_j^0/\partial m_i$ denotes the travel-time sensitivity at the beginning of the update (i.e., the sensitivity at the end of the previous time step). If operator splitting is applied in the transport solve, e.g., to account for matrix-fracture exchange for fractured systems and/or gravity, $f'(S)$ is evaluated after the corrector steps of the operator splitting [5].

Using the compressible conservation equation for water saturation [36], we can generalize the sensitivity calculations presented above to compressible flow. The sensitivity of arrival time of a water saturation with respect to a reservoir parameter m_i is given by [37]

$$\frac{\partial t}{\partial m_i} = \frac{\dfrac{\partial \tau}{\partial m_i}\dfrac{\partial}{\partial \tau}\Big(\dfrac{S_w}{B_w}\Big)}{\dfrac{\partial}{\partial \tau}\Big(\dfrac{f_w}{B_w}\Big) + \dfrac{f_w}{B_w}\dfrac{c}{\phi}}, \tag{33}$$

where B_w is the volume formation factor of water, and c represents the divergence of flux ($c = \nabla\cdot\vec{u}$) along the streamline, which can be estimated from the velocity field. Again, the time-of-flight derivative $\partial\tau/\partial m_i$ is computed analytically as described above. The rest of the derivatives can be computed by (backward) finite-differences along the streamlines. For incompressible flow, $c \equiv 0$ and B_w is constant, so (33) reduces to (32). Similarly, for gas-oil

ratio (GOR), using the compressible conservation equation for gas [36], we can obtain the arrival time sensitivities [37].

As the primary example of an arrival time, we use the arrival of a fixed fractional-flow (fixed saturation) in both the well and along the streamlines. A common arrival-time sensitivity for each producer is then obtained by a flux-weighted average

$$\frac{\partial t_j}{\partial m_i} = \frac{1}{q}\sum_{\ell=1}^{N_{\text{sl}}} q_\ell \frac{\partial t_{j,\ell}}{\partial m_i}, \qquad q = \sum_{\ell=1}^{N_{\text{sl}}} q_\ell. \tag{34}$$

Here, q_ℓ is the total flux of each streamline, and N_{sl} is the number of streamlines connected to the well.

Finally, we mention that Al-Huthali et al. [7] use (14), (21), (26), and (32) to derive arrival-time sensitivities with respect to injection and production rate. These sensitivities are not used for history matching, but rather for optimal waterflood management by rate control. To compute the arrival-time sensitivities for the producers, the authors only consider a fraction of the streamlines (fastest). Moreover, they also consider sensitivities for a group of producers with a common contributing injector.

4.4. **Time-Shift Sensitivities.** A time-shift is a measure for how much a simulated production response curve should be shifted in time to maximize the cross correlation with an observed production-response curve. The time-shift is described and used with the generalized travel-time inversion method described in Section 5.2.

Consider a small perturbation $\delta\mathbf{m}$ in the reservoir parameters with an accompanying shift δt in the computed production response. In each data point t_j there will be a corresponding shift δt_j, where

$$\delta t = \delta t_j = \Big[\frac{\partial t_j}{\partial \mathbf{m}}\Big]^T \delta\mathbf{m}, \quad j = 1, \ldots, N_{\text{d}}.$$

Since a perturbation δm_i will lead to the same time-shift in all data points, we sum over all data points and define the sensitivity of the travel time-shift with respect to parameter m_i as the average of the above equations

$$\frac{\partial t}{\partial m_i} = \frac{1}{N_d}\sum_{j=1}^{N_d} \frac{\partial t_j}{\partial m_i}. \tag{35}$$

By convention, one defines $\partial\Delta\widetilde{t}/\partial m_i = -\partial t/\partial m_i$. Now, the arrival-time sensitivities given above can be used to obtain travel-time shift sensitivities, e.g., for fractional flow, gas-oil ratio, or tracer concentration [37].

Practical experience indicates that more robust history matching is achieved by making the sensitivities dimensionless by applying log-sensitivities [67]:

$$\frac{\partial \log(|\Delta\widetilde{t}|)}{\partial \log m_i} = \frac{m_i}{\Delta\widetilde{t}}\frac{\partial \Delta\widetilde{t}}{\partial m_i}.$$

For the generalized travel-time inversion described in Section 5.2, it is therefore common to use logarithmic modifications for the reservoir parameters.

4.5. **Saturation Sensitivities.** By differentiating the expression used for the streamline-to-grid mapping of saturation (see (19)), the sensitivities of saturation with respect to a reservoir parameter m_i at a given time can be calculated by [130]

$$\frac{\partial S_i}{\partial m_i} = \sum_{\ell=1}^{N_{\text{sl},i}} \frac{\partial S_\ell}{\partial m_i}\beta_\ell = \sum_{\ell=1}^{N_{\text{sl},i}} \frac{\partial S_\ell}{\partial \tau_\ell}\cdot\frac{\partial \tau_\ell}{\partial m_i}\beta_\ell, \tag{36}$$

where β_ℓ is the weight assigned to streamline ℓ in the mapping, $\partial\tau_\ell/\partial m_i$ is the time-of-flight sensitivity specified above, and $\partial S/\partial\tau_\ell$ is the derivative of the 1D saturation solution along the streamline. Here we have assumed that β_ℓ is independent of a perturbation in m_i.

For Riemann initial data the solution is self-similar $S(t,\tau) = S(\tau/t)$ and analytically known, as described in Section 3.2. The sensitivity of the saturation in streamline ℓ at a particular (t,τ) with respect to the reservoir parameter m_i in grid cell i is then given analytically by

$$\frac{\partial S_\ell}{\partial m_i} = \frac{\partial S_\ell}{\partial \xi}\frac{\partial \xi}{\partial \tau}\frac{\partial \tau}{\partial m_i} = \frac{1}{t}\frac{\partial S_\ell}{\partial \xi}\frac{\partial \tau}{\partial m_i}, \quad \xi = \frac{\tau}{t}. \tag{37}$$

Hence, if the reservoir parameter m_i is not located upstream, the sensitivity will be zero. Further, this expression may be used as an approximation in (36).

4.6. **Production Data Sensitivities (Amplitude Sensitivities).** Differentiating (14), the sensitivity of fractional flow at a producer with respect to a perturbation in reservoir parameter m_i is obtained by [143, 147]

$$\frac{\partial f(t)}{\partial m_i} = \frac{1}{q}\sum_{\ell=1}^{N_{\rm sl}} q_\ell \frac{\partial f_\ell(t)}{\partial m_i}.$$

This expression also applies to tracer concentration for which $f(C) = C$, and a similar result applies to gas-oil ratios.

We now need to evaluate $\partial f_\ell(t)/\partial m_i$ for the connected streamlines. This can be done by the chain rule, using an expansion involving either time-of-flight sensitivities [135, 143, 147]

$$\frac{\partial f_\ell}{\partial m_i} = \frac{\partial f_\ell}{\partial \tau_\ell}\frac{\partial \tau_\ell}{\partial m_i}, \tag{38}$$

arrival-time sensitivities [45]

$$\frac{\partial f_\ell}{\partial m_i} = \frac{\partial f_\ell}{\partial t}\frac{\partial t}{\partial m_i}, \tag{39}$$

or the saturation sensitivities

$$\frac{\partial f_\ell}{\partial m_i} = \frac{df_\ell}{dS}\frac{\partial S}{\partial m_i}. \tag{40}$$

Expressions for the the time-of-flight, the arrival-time, and the saturation sensitivities for each streamline have been introduced in the previous subsections. The derivatives of the fractional flow ($\partial f_\ell/\partial\tau$, $\partial f_\ell/\partial t$, or df_ℓ/dS) can be obtained either analytically or by using finite-differences.

Another possibility for obtaining fractional-flow sensitivities directly is to apply the chain rule to the production response (at the well), which yields

$$\frac{\partial f}{\partial m_i} = \frac{\partial f}{\partial t_j}\frac{\partial t_j}{\partial m_i}.$$

Here $\partial t_j/\partial m_i$ is given by (34), and $\partial f/\partial t_j$ can be evaluated numerically from the production response curve at the given time. A smoothed approximation to the generally noisy production curve might then be needed.

Below we will present two approaches for determining the fractional-flow derivative along streamlines, starting out by tracer concentration (or piston-like displacement). Following Vasco and Datta-Gupta [132, 133], we start by observing that the transport of tracer concentration along a streamline can be described by (16). Assume an initial distribution of reservoir properties along a streamline Ψ_0. To compute sensitivities, we give the underlying parameters a small perturbation, reflected as a single perturbation in the slowness function

$$s(\mathbf{x}) = s_0(\mathbf{x}) + \delta s(\mathbf{x}) \tag{41}$$

and seek to find the corresponding change in tracer production δC. According to Vasco and Datta-Gupta [134][2], the perturbation in the streamline is of second order in δs. One can

[2]Vasco and Datta-Gupta [134] refer to King and Datta-Gupta [85]. Unfortunately, we have so far not been able to locate the proof in [85] of the fact that the change in streamlines is second order in δs.

therefore assume that the change in the streamlines is so small that the integral over the new streamline Ψ equals that over the old streamline Ψ_0, that is

$$\tau = \int_{\Psi} s(\mathbf{x})\,dr \approx \int_{\Psi_0} s_0(\mathbf{x})\,dr + \int_{\Psi_0} \delta(\mathbf{x})\,dr = \tau_0 + \delta\tau_0.$$

Using the notation from Section 3.1, we may hence write

$$\begin{aligned}\delta C_\ell(\mathbf{x}) &= C_{\ell,0}\big(t-\tau\big) - C_{\ell,0}\big(t-\tau_0\big)\\ &\approx C_{\ell,0}\big(t-\tau_0-\delta\tau_0\big) - C_{\ell,0}\big(t-\tau_0\big) \overset{\text{Taylor}}{\approx} -C'_{\ell,0}\big(t-\tau_0\big)\delta\tau_0\end{aligned}$$

as a first-order approximation. It now remains to determine $\delta\tau_0$, i.e., the integral of δs over Ψ_0. The variation in $s(\mathbf{x})$ due to variation in all properties is given as

$$\delta s(\mathbf{x}) = \frac{\partial s(\mathbf{x})}{\partial K}\delta K + \frac{\partial s(\mathbf{x})}{\partial \phi}\delta\phi + \frac{\partial s(\mathbf{x})}{\partial |\nabla p|}\delta|\nabla p| + \ldots, \tag{42}$$

where the expressions for the partial derivatives were given in Section 4.2. For instance, the sensitivity of the concentration due to changes in the reservoir parameter m_i at time t is given by

$$\frac{\partial C_\ell}{\partial m_i} = -C'_{\ell,0}\big(t-\tau_0\big)\int_{\Psi_{0i}} \frac{\partial s(\mathbf{x})}{\partial m_i}\,dr = \underbrace{-C'_{\ell,0}\big(t-\tau_0\big)}_{\approx \frac{\partial C_\ell}{\partial \tau_\ell}}\frac{\partial \tau_\ell}{\partial m_i}.$$

Notice that the same expression could have been obtained directly by differentiating (16) under the assumption of no shift in streamlines due to the perturbation in reservoir parameters.

Wen et al. [143] present another approach to analytic calculation of $\partial C_\ell(t)/\partial m_i$ for tracer flow. We start by assuming a tracer flow with a monotone flow profile, where the analytical solution is given by (see Section 3.1)

$$C_\ell(t) = \begin{cases} 1, & \text{if } \tau_\ell \le t,\\ 0, & \text{otherwise.}\end{cases}$$

Here τ_ℓ denotes the time-of-flight of streamline ℓ at the well. To be able to differentiate this discontinuous profile, the authors use an approximation in terms of an error function E_σ for some small parameter σ,

$$C_\ell(t) \approx 1 - E_\sigma\Big(\frac{\tau_\ell}{t} - 1\Big), \qquad t \le \tau_\ell,$$

and hence

$$\frac{\partial C_\ell(t)}{\partial m_i} = \frac{\partial C_\ell(t)}{\partial \tau_\ell}\frac{\partial \tau_\ell}{\partial m_i} = -\frac{1}{t}G_\sigma\Big(\frac{\tau_\ell}{t}\Big)\frac{\partial \tau_\ell}{\partial m_i}, \qquad G_\sigma(r) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(r-1)^2}{2\sigma^2}}.$$

The same approach can be extended to two-phase incompressible flow [135, 147] described by transport equation (13). The analytic Buckley–Leverett solution for Riemann initial data (see Section 3.2), consisting of a shock followed by a rarefaction wave, can be used to calculate analytic fractional-flow sensitivities by

$$\frac{\partial f_\ell}{\partial m_i} = \frac{df_\ell}{dS}\frac{\partial S}{\partial \tau}\frac{\partial \tau}{\partial m_i} = \frac{\tau}{t}\frac{\partial S}{\partial \tau}\frac{\partial \tau}{\partial m_i}$$

since $f'_\ell(S) = \tau/t$ for a self-similar profile.

As mentioned above, $\partial f_\ell/\partial t$ and $\partial f_\ell/\partial \tau$ can be evaluated by finite-differences along the streamlines, which is the most general approach. However, the fully analytic approximations may often be sufficiently accurate.

Finally, we remark that gravity and capillary forces can be accounted for in the fractional flow derivative by defining the fractional flow function from Darcy's law incorporating gravity and capillary forces; i.e., [39]

$$f_w(S) = \frac{\mathbf{q}_w\cdot\mathbf{n}}{\mathbf{q}_t\cdot\mathbf{n}} = \frac{\lambda_w + \frac{\lambda_w\lambda_o}{\mathbf{u}\cdot\mathbf{n}}K(\nabla p_{\text{cow}} + (\rho_w-\rho_o)\mathbf{g})\cdot\mathbf{n}}{\lambda_w+\lambda_o}. \tag{43}$$

Here $\mathbf{n}$ is the unit vector in the flow direction that easily can be estimated from the streamline geometry; i.e. by $\mathbf{n} = \mathbf{u}/|\mathbf{u}|$. Hence, gravity and capillary forces will only be fully accounted for if both (43) is applied and the forward simulator accounts for these effects.

In [150] an argument is given for the spatial additivity of production response sensitivities. We will here outline this argument by a small example. Let $g(\mathbf{m})$ be the production response at a well, and let there be N grid cells $\{1, 2, \ldots, N\}$, with corresponding grid parameters $\{m_1, m_2, \ldots, m_N\}$. The set of production-response sensitivities is then $\{\partial g/\partial m_1, \partial g/\partial m_2, \ldots, \partial g/\partial m_N\}$. By perturbing a subset of the reservoir parameters, e.g., $\{m_4, m_5, m_6, m_7\}$, we obtain the following differential for the resulting perturbation in the production response

$$\delta g = \frac{\partial g}{\partial m_4}\delta m_4 + \frac{\partial g}{\partial m_5}\delta m_5 + \frac{\partial g}{\partial m_6}\delta m_6 + \frac{\partial g}{\partial m_7}\delta m_7.$$

By assuming the same modification or perturbation $\delta m_c = \delta m_4 = \delta m_5 = \delta m_6 = \delta m_7$ in all cells, the differential is

$$\delta g = \underbrace{\left(\frac{\partial g}{\partial m_4} + \frac{\partial g}{\partial m_5} + \frac{\partial g}{\partial m_6} + \frac{\partial g}{\partial m_7}\right)}_{\frac{\partial g}{\partial m_c}} \delta m_c.$$

Hence, it is reasonable to approximate the sensitivities of the production response with respect to a coarse-cell reservoir parameter m_c by the sum of the sub-cell sensitivities. Further, it should be noted that small cells then in general will have smaller sensitivities than large cells.

4.7. **Miscellaneous.** In this last subsection we will briefly comment on a few other uses of streamlines to calculate various sensitivity coefficients that fall in neither of the above categories.

Vasco et al. [136] derive sensitivities for amplitudes from time-lapse seismic with respect to changes in reservoir parameters. We will not go into the details, but the key to obtaining the sensitivities is to relate perturbations in the amplitudes to perturbations in the upstream saturations along streamline trajectories. Further, the perturbations in the upstream saturations can be related to perturbations in the reservoir parameters via the perturbations in time-of-flight by (37).

Kulkarni et al. [92] derive streamline-based sensitivities for the arrival time of a 'pressure front' for use in pressure interference tests, see Section 5.2.3 for details about streamline-based integration of transient pressure data. The arrival time is related to the so-called diffusive time-of-flight by (51) in Section 5.2.3. Similar calculations as used for ordinary time-of-flight and arrival-time sensitivities can then be applied. In [69], sensitivities for the amplitude of the 'pressure front' of a pressure interference test are derived by simply differentiating (49) in Section 5.2.3 in the time domain (inverse Fourier transformed).

In [58, 59] sensitivities of fractional flow and well pressure with respect to the parameters of the gradual deformation method (GDM) are derived by direct differentiation of the discrete equations of a streamline method. Numerical values for the gradients are computed in a two-step procedure (corresponding to the two steps in the fractional step solution method):

(1) Given boundary conditions, the pressure and its gradient are computed on a 3D grid. The pressure gradients with respect to a parameter are obtained by the gradient-simulator method discussed in Section 4.1. This is therefore not a streamline-based approach, but the saturation derivatives involved in the derivation of the linear equation system are streamline based. The pressure gradients are calculated based on information from the previous pressure step.
(2) Then velocities are computed from Darcy's law and streamlines are traced from injectors to producers. The one-dimensional saturation/transport equation is solved along each streamline, and saturation gradients with respect to reservoir properties are computed. Finally, the streamline gradients are mapped back onto the 3D grid to obtain grid-block saturation gradients.

Part of the intermediate derivations are similar or identical to derivations presented above. The entire derivations are too technical to give a condensed presentation here. One should therefore instead read the paper in full [59].

We will just remark that the derivations presented involve a specific streamline simulator implementation involving expressions used in the Pollock's tracing algorithm, the Peaceman well model, and the standard first-order upwind finite-difference scheme. The calculations may therefore need to be adapted for other implementations. Currently, there are two competing streamline technologies: (i) the 3DSL technology of StreamSim, which uses finite differences along each streamline; and (ii) the FrontSim technology by Schlumberger, which also uses front-tracking along each streamline. The results from [59] are based upon the 3DSL-type streamline simulator, and may therefore not be applied directly to FrontSim-type streamline simulators. By using the front-tracking method, one avoids the mapping back and forth between an irregular and a regular discretization along streamlines. This simplifies the calculations of the sensitivities, since the terms arising from equations (22) and (25) in [59] are not needed. On the other hand, an equation for the saturation gradients cannot be obtained by simply differentiating the discretized saturation equation. Instead, one could try to use the approach of Vignes [139], in which saturation gradients are computed 'recursively' as part of the front-tracking algorithm.

As the authors point out in [59], including gravity in the computations should be straightforward. It may also be possible to generalize the computations to obtain sensitivities of other parameters than the gradual deformation parameters. However, for the gradual deformation method, only a few tens of parameters are usually employed [59], which keeps the number of linear solves for the gradient-simulator method down. The streamline-assisted gradual-deformation approach of Gautier et al. [59] is discusses in Section 5.4.4.

Caers [25] and Ravalec-Dupin and Fenwick [116] present analytic sensitivities for the particular gradual deformation approach presented in Section 5.3.4 in the case of Gaussian permeability field. i.e. sensitivities for streamline-effective permeability with respect to the gradual deformation parameters.

5. History-Matching Methods

In this section, which forms the core of the paper, we will review methods for streamline-based history matching. The methods will be sorted into four categories as outlined in the introduction: assisted history matching, (generalized) travel-time inversion, streamline-effective properties, and miscellaneous. The main emphasize will be put on travel-time inversion methods and methods using streamline-effective properties.

5.1. **The Assisted History Matching Approach.** Emanuel and Milliken [49] describe what they call an assisted history matching (AHM) approach, where streamline methods are used to assist in the matching of conventional finite-difference reservoir simulation models. The key idea in the AHM approach is to alter "geologic properties along the flow paths connecting a producing well and its flow source" [49]. A 3D streamline method is used to define these flow paths. Once the streamlines are computed, all streamlines are traversed and the grid cells are assigned to the producer at which the particular streamline terminates. This way, the AHM approach identifies bundles of streamlines where the reservoir engineer later must change the reservoir properties, either manually or by some algorithm, in the grid-cells containing the identified streamlines. Rock properties (permeabilities and porosities) are changed on a well-by-well basis through traditional multipliers, or the heterogeneity is changed through a renormalization based upon the Dykstra–Parsons coefficient for controlling the spatial heterogeneity. The manual work typically amounts to targeted adjustments of a few parameter for each well. Emanuel and Milliken [49] illustrate that by using AHM one is often able to reveal nonintuitive connections between grid cells and wells. In [104], the AHM approach is extended to placement of shale bodies between well pairs and the utility of the method is demonstrated for three field cases.

The streamline distribution is generated based on average well conditions over the production period. However, several streamline distributions may be used if there are significant

changes in the well configuration [49]. The AHM method changes properties along flow paths derived from the initial geological model and thus relies upon a well-constructed initial model. The history match is obtained through relatively minor local changes of the initial model under the assumption of invariant streamlines. Thus, the AHM approach is different from automated approaches that come in the form of a computer algorithm for minimizing a mismatch functional. In particular, since the AHM approach uses no gradient-based minimizing technique, the method does not provide any means for sensitivity computations.

The AHM method has been applied with success to a number of real fields [13, 29, 97, 101]. In particular, Cheng et al. [33] present two field cases for which both AHM and the generalized travel-time inversion to be introduced in Section 5.2 are applied.

5.2. **(Generalized) Travel-Time Inversion.** In a series of papers Datta-Gupta, Vasco, and coworkers have developed methods for integrating dynamic data, using a combination of streamline methods and streamline-based sensitivities. The travel-time approach for matching production data is basically motivated by an analogy between seismic ray inversion and streamlines, which will be outlined briefly below. This initial approach consists of travel-time matching at each well of breakthrough time, a distinct peak in tracer concentration, etc., followed by an amplitude matching [135]. An approach built on the same principles is proposed for incorporating transient pressure data [92]. Later a so-called generalized travel-time inversion (GTTI) approach was introduced [67], which can be considered as a combination of travel-time matching and amplitude matching into one step. In contrast to traditional amplitude matching, both the travel-time matching and GTTI have quasilinear properties [34]. Therefore, (generalized) travel-time history matching proceeds rapidly even if the initial model is not close to the global minimum. The original travel time matching and the generalized approach are both deterministic algorithms, but a geostatistical version of GTTI is introduced in [138, 148].

5.2.1. *The Analogy with Seismic Ray Inversion.* For a neutral tracer, the transport is described by the time-of-flight equation, see (10):

$$\mathbf{v} \cdot \nabla\tau(\mathbf{x}) = 1 \quad \Leftrightarrow \quad \mathbf{u} \cdot \nabla\tau(\mathbf{x}) = \phi. \tag{44}$$

Here $\mathbf{v} = \mathbf{u}/\phi$ is the interstitial velocity. A key point in [135] is the observation that the time-of-flight equation has certain properties in common with the Eikonal equation describing (seismic) travel time tomography,

$$\nabla T(\mathbf{x}) \cdot \nabla T(\mathbf{x}) = 1/c(\mathbf{x})^2. \tag{45}$$

Here T is the travel time and c is the propagation speed. (A common form of the Eikonal equation is to write $|\nabla T| = 1/c$). The Eikonal equation allows for wave propagation in both directions along ∇T, whereas the time-of-flight equation only allows for particles traversing in the positive direction of $\nabla\tau$, i.e., toward increasing values of τ along the streamline. Equation (44) can be thought of as the square root of (45) with the positive sign.

5.2.2. *A Two-Step Travel-Time/Amplitude Matching Method.* Motivated by inversion methods for seismic travel times, a two-step inversion method for tracer and fractional-flow data is developed in [135]. In the first step, one chooses a certain characteristic feature of the production curves, e.g., time to breakthrough or a distinct peak. Then the observed and calculated responses are lined up in all wells such that the characteristic features coincide in time. during this stage the dominant features of the permeability field will be matched and the majority of the misfit reduced. In the second step, the 'amplitude' of the production responses are matched.

Cheng et al. [34] give a systematic investigation and comparison of the nonlinearity of travel-time matching, amplitude matching, and generalized travel-time inversion (described below). Their investigations demonstrate quasilinear properties for travel-time matching, while amplitude matching can be order of magnitudes more nonlinear. Travel-time matching generally has fewer local minima and is therefore more robust and has better convergence characteristics. Further, Cheng et al. [34] report that travel-time sensitivities are more uniformly distributed between the wells, in contrast to amplitude sensitivities that tend to be

localized near the wells. This contributes to the good robustness and convergence properties of the travel-time matching.

Apart from the two-step methodology, a central ingredient in the method is the use of an efficient method for computing analytic streamline-based sensitivities. The arrival time and fractional-flow sensitivities described in Sections 4.2, 4.3, and 4.6 are used for the travel-time and the amplitude matching, respectively.

The inverse problem is formulated as a minimization of the misfit function defined over a set of N_d observations d_j,

$$\sum_{j=1}^{N_d} \big(d_j - g_j(\mathbf{m})\big)^2,$$

where $\mathbf{m}$ denotes the reservoir parameters (permeability, porosity, etc) and $g_j(\mathbf{m})$ the forward model (streamline simulator). If we now use a Taylor series expansion of $g_j(\mathbf{m})$ about some initial model $\mathbf{m}_p$, we can linearize the residuals as follows

$$d_j - g_j(\mathbf{m}_p) = \delta d_j = \sum_i G_{ji}\,\delta m_i, \qquad G_{ji} = \frac{\partial g_j}{\partial m_i}.$$

Here $\{G_{ji}\}$ are sensitivity coefficients. Since the number of parameters usually is very large compared to the amount of data, the corresponding minimization problem is numerically unstable. The authors therefore add two regularizing terms and seek the modification $\delta\mathbf{m}$ that minimizes the following function [135]

$$\|\delta\mathbf{d} - \mathbf{G}\delta\mathbf{m}\|_2^2 + \beta_1\|\delta\mathbf{m}\|_2^2 + \beta_2\|\mathbf{L}\delta\mathbf{m}\|_2^2. \tag{46}$$

The first regularization term tends to keep the modifications made to the reservoir parameters small, while the second term tends to make the modifications smooth; see the discussion of ill-posedness and regularization in Section 1. The minimum of the regularized function in (46) is obtained as a least-squares solution to the following augmented linear system

$$\begin{pmatrix} \mathbf{G} \\ \beta_1\mathbf{I} \\ \beta_1\mathbf{L} \end{pmatrix} \delta\mathbf{m} = \begin{pmatrix} \delta\mathbf{d} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{47}$$

and may for instance be computed by an iterative sparse least-squares solver, for instance, LSQR [110].

The travel-time step is faster than the amplitude matching step [34]. In the travel-time inversion a single parameter is matched in each well, giving a total number of N_w parameters to be matched using (47). In the amplitude step, all N_d^k observations per well are matched, giving a total of $N_w \times N_d^k$ parameters to be matched in (47).

Finally, we mention that an overview of the framework under which the two-step approach, and some related approaches, have been developed is given by Vasco and Datta-Gupta [132].

5.2.3. *Inversion of Pressure Interference Tests.* Extensions to compressible flow and integration of dynamic pressure data from pressure interference tests are considered in [42, 92]. A pressure interference test is an important source of dynamic data. The pressure responses from injecting or producing wells are observed in surrounding distant wells. An advantage of pressure interference tests is that the transient pressure responses can be obtained more quickly than tracer and fractional-flow responses, so that the data integration can take place at an earlier stage.

As in the previous papers, the central idea is to draw upon the analogy between propagating waves and propagating fronts and apply the inversion algorithm to a propagating pressure front. The pressure front is obtained by studying high-frequency asymptotic solutions of the diffusivity equation

$$\mu c_t \phi(\mathbf{x}) \frac{\partial p}{\partial t} = \nabla\big(K(\mathbf{x})\nabla p\big). \tag{48}$$

Applying a Fourier transformation to (48) one obtains an equation in the frequency domain

$$(-i\omega)\frac{\phi(\mathbf{x})\mu c_t}{K(\mathbf{x})}\widetilde{p}(\mathbf{x},\omega) = \nabla^2\widetilde{p}(\mathbf{x},\omega) + \frac{\nabla K(\mathbf{x})}{K(\mathbf{x})}\cdot\nabla\widetilde{p}(\mathbf{x},\omega).$$

One can now define a phase function $\tau(\mathbf{x})$ and seek approximations in terms of a series of inverse powers in $\sqrt{-i\omega}$, where the 'pressure front' would correspond to the zeroth order term

$$\widetilde{p}(\mathbf{x},\omega) = A_0(\mathbf{x})e^{-\sqrt{-i\omega}\tau(\mathbf{x})}. \tag{49}$$

By inserting this term and equating the coefficients with highest order in $\sqrt{-i\omega}$, i.e., $(\sqrt{-i\omega})^2$, the following phase function for a propagating pressure front is obtained

$$\sqrt{\alpha(\mathbf{x})}|\nabla\tau(\mathbf{x})| = 1, \qquad \alpha(\mathbf{x}) = \frac{K(\mathbf{x})}{\phi(\mathbf{x})\mu c_t}.$$

Based on the similarity with the time-of-flight equation (see (44)), the authors define what they call the 'diffusive time-of-flight' by

$$\tau(\mathbf{x}) = \int_\Psi \sqrt{\frac{\phi(\mathbf{x})\mu c_t}{K(\mathbf{x})}}\,dr = \int_\Psi \frac{dr}{\sqrt{\alpha(\mathbf{x})}}. \tag{50}$$

The values of $\sqrt{\alpha(\mathbf{x})}$ at the cell faces are used to generate streamlines along which the pressure front will propagate. Hence, these streamlines will not coincide with the velocity-based streamlines, but will be similar. It is also showed that the arrival time of the 'pressure front' in a 3D medium is related to the diffusive time-of-flight by

$$t_{\max} = \frac{\tau^2(\mathbf{x})}{6}. \tag{51}$$

Using this association, the authors derive a travel-time inversion method for the transient pressure data by applying the sensitivities discussed in Section 4.7. Moreover, Kulkarni et al. [92] present a relation between the drainage radius of a well and the diffusive time-of-flight.

The derivation above is performed for a sharp pressure impulse (a propagating peak). In practice, the source function is more like the Heaviside function. Observing that the time derivative of the Heaviside function is an impulse function, the travel-time analysis should instead be carried out with respect to the time derivative of the pressure response at the well [92]. In [23] the relation between a Heaviside source and an impulse source is discussed in more detail, and a conversion factor is derived.

Datta-Gupta et al. [42] compares history matches obtained by travel-time matching of transient pressure, tracer, and fractional-flow data. Further, in [69] both travel-time and amplitude matching of transient pressure responses from pressure interference tests were performed in a similar manner as for the two-step method presented in Section 5.2.2.

5.2.4. *Generalized Travel-Time Inversion.* He et al. [67] introduce an alternative single-step version of the above two-step travel-time inversion method. Assume for simplicity that there are N_d^k observations $y(t_1^k),\dots,y(t_{N_d^k}^k)$ that are to be matched for well k. A traditional amplitude matching would try to minimize a misfit function of the type

$$J = \sum_{j=1}^{N_d^k}\Big(y^{\text{obs}}(t_j^k) - y^{\text{cal}}(t_j^k)\Big)^2,$$

for each well k. The GTTI method, on the other hand, proceeds by selecting an optimal time shift $\Delta\widetilde{t}_k$ in the observed data that minimizes the misfit function

$$J(\Delta\widetilde{t}_k) = \sum_{j=1}^{N_d^k}\Big(y^{\text{obs}}(t_j^k + \Delta\widetilde{t}_k) - y^{\text{cal}}(t_j^k)\Big)^2,$$

or alternatively maximizes the coefficient of determination

$$R^2(\Delta\widetilde{t}_k) = 1 - \frac{\sum_{j=1}^{N_d^k}\left[y^{\text{obs}}(t_j^k + \Delta\widetilde{t}_k) - y^{\text{cal}}(t_j^k)\right]^2}{\sum_{j=1}^{N_d^k}\left[y^{\text{obs}}(t_j^k) - \overline{y^{\text{obs}}(t_j^k)}\right]^2}.$$

In other words, we seek a time-shift $\Delta\widetilde{t}_k$ in each well that maximizes the correlation between the observed and calculated production curves. These time-shifts are then used to match the reservoir properties for all wells jointly; i.e., by setting $\delta\mathbf{d} = \mathbf{\Delta t} = \{\Delta\widetilde{t}_k\}$ in the minimization system in (47). As for the two-step inversion method, a central part of the algorithm is the calculation of analytic sensitivities as discussed in Section 4.4.

It can be shown that GTTI reduces to the traditional least-square amplitude matching as the match is getting close to the observed data [67]. Thus, GTTI combines travel-time and amplitude matching to some extent, while preserving most of the quasilinear properties of the travel-time matching [34].

Field case studies for GTTI are presented in e.g., [72, 114]. Further, in [33] the performance of GTTI and AHM (described in Section 5.1) is compared.

5.2.5. *Bayesian Generalized Travel-Time Inversion.* In [138, 148] a stochastic version of GTTI is developed, based on Bayesian statistics. Again, let $\mathbf{m}$ and $\mathbf{d}$ denote the reservoir parameters and observed data, respectively. By assuming a Gaussian prior distribution

$$[\mathbf{m}] \sim f(\mathbf{m}) = \mathcal{N}_N(\mathbf{m}_p, \mathbf{\Sigma}_m) = \text{const}_\text{A} \cdot \exp\Big[-\frac{1}{2}(\mathbf{m}-\mathbf{m}_p)^T\mathbf{\Sigma}_m^{-1}(\mathbf{m}-\mathbf{m}_p)\Big],$$

and a Gaussian likelihood model for the observations

$$\begin{aligned}[\mathbf{d}|\mathbf{m}] = \mathbf{g}(\mathbf{m}) + \mathbf{u} &\sim f(\mathbf{d}|\mathbf{m}) = \mathcal{N}_{N_d}(\mathbf{g}(\mathbf{m}), \mathbf{\Sigma}_d) \\ &= \text{const}_\text{B} \cdot \exp\Big[-\frac{1}{2}(\mathbf{d}-\mathbf{g}(\mathbf{m}))^T\mathbf{\Sigma}_d^{-1}(\mathbf{d}-\mathbf{g}(\mathbf{m}))\Big],\end{aligned} \tag{52}$$

the posterior distribution is, by Bayes' rule (3), given by

$$\begin{aligned}[\mathbf{m}|\mathbf{d}] \sim f(\mathbf{m}|\mathbf{d}) = \text{const}_\text{C} \cdot \exp\Big[-\frac{1}{2}\big((\mathbf{d}-\mathbf{g}(\mathbf{m}))^T\mathbf{\Sigma}_d^{-1}(\mathbf{d}-\mathbf{g}(\mathbf{m})) \\ + (\mathbf{m}-\mathbf{m}_p)^T\mathbf{\Sigma}_m^{-1}(\mathbf{m}-\mathbf{m}_p)\big)\Big].\end{aligned} \tag{53}$$

Here $\mathbf{m}_p$ is the prior mean for the reservoir parameters, $g(\mathbf{m})$ is a forward model, $[\mathbf{u}] \sim \mathcal{N}_{N_d}(\mathbf{0}, \mathbf{\Sigma}_d)$ represents the measurement errors, $\mathbf{\Sigma}_d$ is the covariance matrix for the measurement errors, and $\mathbf{\Sigma}_m$ is the (prior) covariance matrix for the reservoir parameters. If the forward model is represented by a linear relation $\mathbf{g}(\mathbf{m}) = \mathbf{A}\cdot\mathbf{m}$, the posterior distribution is Gaussian and can be determined analytically, see Appendix B. However, $\mathbf{g}(\mathbf{m})$ is generally nonlinear, so we only know the posterior distribution up to a constant. Still, it is possible to obtain an estimate for a $\mathbf{m}$ that maximizes the *a posterior* distribution given by

$$\arg\min_{\mathbf{m}} \ \big(\mathbf{d}-\mathbf{g}(\mathbf{m})\big)^T\mathbf{\Sigma}_d^{-1}\big(\mathbf{d}-\mathbf{g}(\mathbf{m})\big) + \big(\mathbf{m}-\mathbf{m}_p\big)^T\mathbf{\Sigma}_m^{-1}\big(\mathbf{m}-\mathbf{m}_p\big). \tag{54}$$

This maximum a posteriori (MAP) estimate gives in general a too smooth reservoir description ('regression toward the mean'). However, the large-scale structures of the reservoir may be discerned from the MAP estimate. The Gauss–Newton algorithm for the minimization of (54) leads to the following iterative scheme

$$\mathbf{m}^{l+1} = \mathbf{m}_p - \mathbf{\Sigma}_m\mathbf{G}_l^T\Big[\mathbf{\Sigma}_d + \mathbf{G}_l\mathbf{\Sigma}_m\mathbf{G}_l^T\Big]^{-1}\Big[\big(\mathbf{g}(\mathbf{m}^l)-\mathbf{d}\big) - \mathbf{G}_l\big(\mathbf{m}^l-\mathbf{m}_p\big)\Big], \tag{55}$$

where $\mathbf{G}$ is the sensitivity matrix for data misfit with respect to a perturbation in the reservoir parameters. Wu and Datta-Gupta [148] apply $\mathbf{g}(\mathbf{m}) - \mathbf{d} = \mathbf{\Delta\widetilde{t}} = \{\Delta\widetilde{t}_k\}$ to obtain a Bayesian version of GTTI, where $\Delta\widetilde{t}_k$ is the time-shift described above. If the data misfits are represented by the time-shifts $\mathbf{\Delta\widetilde{t}}$, then the inverse matrix on the right hand side of (55) has dimension $N_w \times N_w$, where N_w is the number of wells. In conventional methods the inverse matrix is usually of dimension $N_d \times N_d$, where N_d is the total number of observations. Hence, N_d is usually orders of magnitude larger than N_w, so the inversion of the matrix is therefore a minor issue.

Finally, we mention that conditional realizations can be obtained by a similar minimization problem, referred to as randomized maximum likelihood, see e.g., [96]

$$\arg\min_{\mathbf{m}} (\mathbf{d}_{\text{uc}} - \mathbf{g}(\mathbf{m}))^T \mathbf{\Sigma}_d^{-1} (\mathbf{d}_{\text{uc}} - \mathbf{g}(\mathbf{m})) + (\mathbf{m} - \mathbf{m}_{\text{uc}})^T \mathbf{\Sigma}_m^{-1} (\mathbf{m} - \mathbf{m}_{\text{uc}}),$$

in which the prior mean and the observed data have been replaced by unconditioned (uc) realizations $\mathbf{m}_{\text{uc}}$ and $\mathbf{d}_{\text{uc}}$, respectively: that is, random error have been added to the observed data.

Vega et al. [138] investigate the computational scalability of the deterministic and the Bayesian version of GTTI. They demonstrate that the deterministic version scales almost linearly with problem size and that the Bayesian version scales almost quadraticly (as expected). However, by reformulating the Bayesian method Vega et al. [138] were able to obtain an almost linear scaling for the computational cost and results with the same quality as for the deterministic approach, while preserving the statistical foundation of the Bayesian approach. Hence, conditioned realizations can also for this formulation be obtained by the randomized maximum likelihood approach.

To reformulate the Bayesian approach, Vega et al. [138] start by rewriting the minimization problem (54) to obtain an alternative minimization formulation. Further, they approximate the Hessian $\mathbf{H}$ by $\mathbf{J}^T\mathbf{J}$, where $\mathbf{J}$ is the Jacobian. This approximation is the same that is used in the Gauss–Newton algorithm, and is accurate near the solution or for quasilinear problems. The reformulated system reads

$$\begin{bmatrix} \mathbf{\Sigma}_d^{-1/2}\mathbf{G} \\ \mathbf{\Sigma}_m^{-1/2} \end{bmatrix} \delta\mathbf{m} = \begin{bmatrix} \mathbf{\Sigma}_d^{-1/2}[\mathbf{d} - \mathbf{g}(\mathbf{m})] \\ \mathbf{\Sigma}_m^{-1/2}[\mathbf{m}_p - \mathbf{m}] \end{bmatrix}. \tag{56}$$

This system is analogous to the deterministic formulation in (47). Here $\mathbf{\Sigma}_m^{-1/2}[\mathbf{m}_p - \mathbf{m}]$ plays an equivalent role as the regularization terms in (47), where the covariance matrix $\mathbf{\Sigma}_m$ imposes smoothing and the difference $\mathbf{m}_p - \mathbf{m}$ tends to keep the modifications small. Further, by using this formulation, we can thus solve iteratively for a least-square solution by, for instance, the LSQR algorithm. Here, $\mathbf{\Sigma}_m$ is generally a matrix of dimension $N \times N$, where N is the number of model parameters. Obtaining $\mathbf{\Sigma}_m^{-1/2}$ is therefore very computationally expensive if $\mathbf{\Sigma}_m^{1/2}$ is to be computed numerically. Vega et al. [138] therefore rely on a semi-analytical computation of $\mathbf{\Sigma}_m^{1/2}$. The key observation to this end is that the inverse of the covariance of the model parameters can be identified with the differential operator (the smoothing operator) in the deterministic approach. They therefore apply a computational stencil based on an approximation of the differential operator associated with $\mathbf{\Sigma}_m^{1/2}$, for which they assume an exponential covariance model [138]. The extension to other covariance models is presented in [137].

5.2.6. *Other Extensions.* Barman et al. [15] suggest a procedure for applying two-step inversion approach described above for fractured reservoirs. The idea is to use effective permeability representations for the inversion, obtained based on production based indicators (details not given), while the forward simulations are performed for fractured reservoir representations. Similarly, Al-Harbi et al. [5] also apply the Bayesian version of GTTI fractured reservoir models.

Kulkarni and Datta-Gupta [91] extend the two-step inversion approach to history-matching relative permeability curves. The sensitivities with respect to the parameters used in the relative permeability functions are specified in Appendix A. Further, appropriate regularization terms for (46) are specified in [91].

Illiassov and Datta-Gupta [76] present an extension of the two-step inversion to multiwell, multitracer partitioning interwell tracer tests, applied to a large oil field in Texas. Partitioning tracer sensitivities are described in Section 4 (in particular Section 4.2). They use the two-step inversion approach twice: first for matching permeability and then for matching oil saturation. This procedure is iterated if necessary. A similar work related to ground water transport is found in [43].

Although all the above methods have been based upon streamlines, they all amount more or less to matching the properties in all the grid cells in the reservoir model. To improve the convergence properties of the inversion process, a multiscale approach is proposed by Yoon et al. [150]. The central idea here is to use a hierarchy of coarsened grids to match the dynamic production data. The matching is first performed on the coarser scales, where the inversion problem is less under-determined, to reduce the ill-posedness of the problem. Moreover, the number of local minima is reduced on the coarser scales and this will speed up the iterative minimizing techniques. The inversion may be aborted before reaching the fine grid to prevent over-parameterization. Finally, the solution is downscaled to fine-grid realizations by sequential simulation, conditioning to well data. The multiscale idea was applied to the two-step inversion method, but the multiscale matching may be applied to other inversion approaches as well, e.g., as done by Stenerud and Lie [121] for matching streamline-effective properties using the Wang–Kovseck formulation to be discussed in the next section.

He et al. [68] propose a manual approach, using the relationship between the diffusive time-of-flight and the drainage radius outlined by Kulkarni et al. [92] to identify reservoir compartmentalizations and flow barriers during primary production. First, the drainage volumes and communications for the different wells are estimated by traditional decline-type-curve analysis of the primary production data. Second, starting from the geologic model, the drainage volumes are recalculated by the diffusive time-of-flight from a streamline-based flow simulation. Finally, reservoir compartmentalization and flow barriers are inferred by matching of the two estimates for the drainage volumes.

An approach for reconciling time-lapse amplitude changes using (47) with the time-lapse amplitude sensitivities discussed in Section 4.7 is proposed in [136].

As noticed in Section 4, any simulator can in theory be used to calculate streamline-based sensitivities, as long as intermediate velocity fields can be outputed during the simulation and streamlines can be traced on the cell geometry, see discussion in Section 4.1. Cheng et al. [35] demonstrate the applicability of a finite-difference simulator to the GTTI method.

In [124] and [122] the deterministic version of GTTI is combined with a very efficient multiscale-streamline simulator. A mixed multiscale finite-element pressure solver [1, 31] is combined with a transport solver based on streamlines and the unconditionally stable front-tracking method [73]. High efficiency of the forward simulator is obtained by selectively reusing the multiscale basis functions based on the spatial sensitivity distribution obtained from GTTI. In addition, a method for improved mass conservation in streamline simulation [122] is applied , which allows for a considerable reduction in the number of streamlines used in the forward simulations. For this combination of forward and backward methodology, a reservoir model with more than one million grid blocks was history-matched in less than twenty minutes on an ordinary desktop PC.

In [123], GTTI is applied to two-dimensional fully unstructured grids. A generalized smoothing operator for $\mathbf{L}$ in (47) is proposed for fully unstructured grids. Further, sensitivities on unstructured grids with varying grid-cell sizes are discussed. Because of spatial additivity (as discussed in Section 4), the sensitivities will scale with cell sizes. Consequently, larger modifications will be induced in large cells and small modifications in small cells, given the same conditions. The regularization involved in (47) can only to a small extent counteract this undesired effect. To remedy grid effects for cases with large variances in grid-cell sizes, Stenerud et al. [123] propose to rescale the sensitivities with the grid-cell volumes. The paper focuses on 2D numerical examples to investigate principal effects of performing GTTI on fully unstructured grids, and contains examples with large cell-size heterogeneities and faults with non-neighboring connections. The framework established in [123] is general and should therefore also apply in 3D, although some care should probably be taken when matching strongly layered structures.

5.3. **Methods Based on Streamline-Effective Properties.** For the methods within this category modifications to effective properties along either streamlines or bundles of streamlines are obtained to match dynamic production data. The modifications in effective properties

are then propagated back to the underlying simulation grid by either a deterministic or a geostatistical method. In the following M refers to a grid-cell property and m refers to a streamline-effective property. Similarly we use K and k for a grid and a streamline-effective permeability, respectively.

5.3.1. *Streamline-Effective Properties.* It is not necessary to actually compute the streamline-effective properties for all the methods described below, because it is often only the relative modifications to these properties that are needed. Nevertheless, we will here present and discuss a few approaches for obtaining streamline effective properties; in particular for permeability.

Obtaining effective properties along streamlines or streamline-bundles is simply an upscaling problem. For uni-directional one-phase flow in a medium with constant permeability values perpendicular to the flow direction (serial flow), it can be shown that the correct average is a harmonic average, see Appendix C. This makes sense because for the harmonic average is dominated by the lowest permeability value, which is the bottleneck for the flow. Further, it can also be shown that for uni-directional one-phase flow in a medium with constant permeability values along the flow direction (parallel flow), the correct average is an arithmetic average, see Appendix C. At a first glance, upscaling permeability along streamlines may seem like a 1D upscaling problem, but streamlines are not truly one-dimensional. The different cases of serial and parallel flow indicate that the correct streamline upscaling will depend on the underlying spatial structures in the permeability and thus not be a simple 1D upscaling problem. A one-dimensional streamline represents the flow in a 3D streamtube. Hence, even though streamlines are supposed to be aligned with the flow directions (total velocity), the actual flow may locally escape bottlenecks caused by low-permeable rock (traverse fluxes).

Unweighted arithmetic (A), geometric (G), and harmonic (H) averages are the three classical Pythagorean averages. For $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ with all elements positive, the following relation holds

$$\max(\mathbf{x}) \geq A(\mathbf{x}) \geq G(\mathbf{x}) \geq H(\mathbf{x}) \geq \min(\mathbf{x}),$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$. The geometric average is therefore also a reasonable candidate to calculate the effective permeabilities.

Based on this discussion, it seems reasonable to apply a harmonic or a geometric average to obtain effective permeabilities along streamlines (serial flow), and then an arithmetic or a geometric average of the streamline-effective permeabilities to obtain effective sensitivities for streamline bundles (parallel flow).

Wang and Kovscek [141], among others, suggest to represent the effective streamline permeability by the following weighted harmonic average

$$k_\ell^{\text{eff}} = \frac{\sum_{i=1}^{N_c^\ell} \Delta\tau_{\ell,i}}{\sum_{i=1}^{N_c^\ell} \dfrac{\Delta\tau_{\ell,i}}{K_i}}, \tag{57}$$

where N_c^ℓ is the number of grid cells intersected by streamline ℓ, $\Delta\tau_{\ell,i}$ is the time-of-flight increment of streamline ℓ across grid cell i, and K_i is the permeability in grid cell i. Hence, (57) can be considered a variant of (A-3).

Further, Ravalec-Dupin and Fenwick [116] suggest to use the following harmonic average for the effective permeability of a streamline bundle

$$k_\ell^{\text{eff}} = \frac{\sum_{\ell=1}^{N_{\text{sl}}} \sum_{i=1}^{N_c^\ell} q_\ell \Delta\tau_{\ell,i}}{\sum_{\ell=1}^{N_{\text{sl}}} \sum_{i=1}^{N_c^\ell} \dfrac{q_\ell \Delta\tau_{\ell,i}}{K_i}}. \tag{58}$$

Here q_ℓ is the volumetric flux assigned to streamline ℓ.

5.3.2. *The Wang–Kovscek Method.* The idea of using effective properties along streamlines for history matching was first introduced by Wang and Kovscek [141]. Their basic idea was to

relate the fractional-flow curve at a producer to the water breakthrough of individual streamlines. Then by adjusting the effective permeability along streamlines, one can determine the breakthrough time of each streamline that reproduces the reference producer fractional-flow curve. This is realized through the following simple algorithm

(1) Start with an initial permeability field
(2) Run a simulation and check match with observed data: fractional-flow, well rate and/or well pressure (drops).
(3) Obtain modifications in effective streamline permeability along each streamline to match the data, as discussed below.
(4) Propagate modifications in streamline permeabilities back to the grid.
(5) Iterate steps 2–4 until a satisfactory match is achieved.

The derivation of the original method assumes two-phase incompressible flow, piston-like displacement along each streamline, no capillary forces, and no gravity. Moreover, we here present the modifications obtained for the data associated with one single producer, but the extension to several producers is straightforward

The ordered streamlines are used to discretize the observed fractional flow in the wells. Since each individual streamline will contribute equal amounts to the total fractional flow for piston-like displacement, the fractional flow increases a fixed amount each time a streamline breaks through. Implicitly, we assume that the fractional-flow curve at the producer is monotone. When a mismatch between observed and calculated fractional flows arises, the streamlines responsible for the mismatch are identified by examining the breakthrough times and the effective permeabilities of the streamlines are adjusted.

The history match along streamlines is obtained as follows. Assuming equal flow rates, the streamlines are ordered with respect to their dimensionless breakthrough times. Let there be N streamlines, each having a length L_ℓ, an average porosity ϕ_ℓ, and an average cross-sectional area A_ℓ. Then the dimensionless breakthrough time of streamline ℓ is defined as

$$\widetilde{T}_\ell = \frac{\sum_{j=1}^{N_{\mathrm{sl}}} (A\phi L)_j \tilde{x}_j^\ell}{\sum_{j=1}^{N_{\mathrm{sl}}} (A\phi L)_j} = \sum_{j=1}^{N_{\mathrm{sl}}} \widetilde{V}_j \tilde{x}_j^\ell, \tag{59}$$

where $\tilde{x}_j^\ell$ is the position in dimensionless units of the displacing phase front in the jth streamline as streamline ℓ breaks through, and $\widetilde{V}_j$ is the ratio of the pore volume of streamline j over the total pore volume. Since the streamlines are considered as independent flow systems with piston-like displacement, the relative positions $\tilde{x}_j^\ell$ can be approximated using Dykstra–Parsons method for non-communicating layers, see e.g., [48, 121]. The expression for the relative position of the front in streamline j as streamline ℓ breaks through, is a function of the effective permeabilities in streamline ℓ and j, that is, $\tilde{x}_j^\ell = f(k_\ell, k_j)$. Now, since the sum in the numerator of (59) runs over all streamlines, the breakthrough time $\widetilde{T}_\ell$ is a function of the permeabilities of all streamlines connected to the producer. In vector notation this can be written $\widetilde{\mathbf{T}} = \mathbf{B}(\mathbf{k})$. Using a Taylor expansion, the mismatch in breakthrough times can be written:

$$\Delta\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}^{\mathrm{obs}} - \widetilde{\mathbf{T}}^{\mathrm{cal}} = \mathbf{B}(\mathbf{k}^{\mathrm{obs}}) - \mathbf{B}(\mathbf{k}^{\mathrm{cal}}) \approx \mathbf{A}(\mathbf{k}^{\mathrm{obs}} - \mathbf{k}^{\mathrm{cal}}) = \mathbf{A}\,\Delta\mathbf{k}. \tag{60}$$

Here the derivatives $A_{\ell,j} = \partial\widetilde{T}_\ell/\partial k_j$ are obtained by differentiating (59). The system is simplified by defining normalized parameters [141], which makes the system (60) strongly diagonally dominant for unit mobility ratio so that it approximately decouples. By neglecting off-diagonal entries, the relative modifications can then be obtained by

$$r_\ell^t = \frac{\Delta k_\ell^t}{k_\ell^{\mathrm{old}}} \approx \frac{\widetilde{T}_\ell^{\mathrm{cal}} - \widetilde{T}_\ell^{\mathrm{obs}}}{\widetilde{T}_\ell^{\mathrm{obs}}}. \tag{61}$$

This approximation is only valid for unit mobility ratios. For non-unit mobility ratios, the modifications must generally be obtained by solving the full system (60). Moreover, streamlines will generally evolve during a dynamic displacement (as discussed in the introduction), so an accurate inversion would require the use of several different streamline distributions in

time to match different segments of the fractional-flow curve. Altogether, this is computationally intensive. Rather than inverting the matrix $\mathbf{A}$, Wang and Kovscek [141] therefore suggest to use (61) for a representative set of streamlines as an approximation also in the non-unit mobility case.

Similarly, a match of pressure drop and total well rate is calculated by

$$r_\ell^{p,q} = \frac{\Delta k_\ell^{p,q}}{k_\ell^{\text{old}}} = \frac{\Delta p^{\text{cal}}\, q^{\text{obs}} - \Delta p^{\text{obs}}\, q^{\text{cal}}}{\Delta p^{\text{obs}}\, q^{\text{cal}}}, \tag{62}$$

which can be derived under the same assumptions as for the Dykstra–Parsons method from Darcy's law for an effective streamline permeability [121, 141]. The superscripts p and q indicate that the corresponding modification is due to mismatch in pressure drop and total well rate, respectively.

The two modifications (61) and (62) are then combined to define a total correction factor α_ℓ for each streamline, so that $k_{\text{new}} = \alpha_\ell k_{\text{old}}$. To this end, one should in general use a weighted geometric average

$$\alpha_\ell = \left[\left(1 + r_\ell^t\right)^{w_1} \cdot \left(1 + r_\ell^{p,q}\right)^{w_2}\right]^{1/(w_1+w_2)}.$$

In practice it turns out that equal weighting is acceptable [141].

Once the relative modifications to effective permeabilities are obtained for each streamline, they must be propagated back to the underlying grid. Then a forward simulation is run and the above process is repeated until the data are satisfactory matched.

To map the modification in effective permeability of a streamline back to grid-cell permeabilities, the simplest procedure would be to modify all grid cells along the streamline with the same amount as the modification in the effective property of the streamline that passes through it; that is, simply multiplying by α_ℓ so that $K_{\text{new}} = \alpha_\ell K_{\text{old}}$. If more than one streamline pass through a grid cell, Wang and Kovscek [141] suggest to use a geometric average of the correction factors. This may be a crude method if the lengths of different streamlines passing through the same cell are not equal. Instead, one can use a sensitivity-weighted approach [26, 140]. The sensitivity of the effective permeability of streamline ℓ with respect to the permeability change in grid cell i is defined by direct differentiation of an expression for the effective permeability

$$g_{\ell,i} = \frac{\partial k_\ell}{\partial K_i}. \tag{63}$$

For instance, if (57) is used for the streamline-effective permeability, the following sensitivity is obtained

$$g_{\ell,i} = \frac{\tau_{\ell i} k_\ell^2}{\tau_\ell K_i^2}, \quad \tau_\ell = \sum_i \Delta\tau_{\ell,i}.$$

This sensitivity is weighted by the incremental time-of-flight for the streamline through the cell to obtain a weighted sensitivity

$$G_{\ell,i} = g_{\ell,i}\, \frac{\Delta\tau_{\ell,i}}{\sum_\ell \Delta\tau_{\ell,i}}.$$

The current approach requires no computation of sensitivity coefficients in the traditional sense.

An advantage of obtaining relative modifications, like in **Eqs. 61** and **62**, is that it results in cancellation of potential proportionality errors. Therefore, the real time or the time-of-flight is commonly used instead of the dimensionless PVI time defined by (59). Moreover, the proposed approach, using modifications (61) and (62), is independent of how the effective streamline permeabilities are defined; except if the sensitivities in (63) are used.

The original method of Wang and Kovscek [141] was later modified by Stenerud and Lie [121] to avoid some of the approximations inherent in the original method and bypass the need for solving a linear system for non-unit mobility ratios. In the modified method, one

obtains the following expression for the relative modifications with respect to mismatch in breakthrough time and effective pressure drop

$$r_\ell^{t,p} = \frac{\Delta k_\ell^{t,p}}{k_\ell^{\text{cal}}} = \frac{t_\ell^{\text{cal}}\Delta p^{\text{cal}} - t_\ell^{\text{obs}}\Delta p^{\text{obs}}}{t_\ell^{\text{obs}}\Delta p^{\text{obs}}}. \tag{64}$$

In addition, the method was extended to match porosity or permeability–porosity ratio, and to account for gravity along streamlines. Finally, a multiscale strategy was proposed, inspired by a work of Yoon et al. [150], see Section 5.2.6.

In Appendix D we show that under the same assumptions as used in [121, 141], but by relaxing the assumption of piston-like displacement to Buckley–Leverett profile, the same expressions for the relative modifications can be obtained: that is, (61) and (64). However, relaxing the assumption of piston-like displacement may make it harder to relate the breakthrough of individual streamlines to increments in the fractional-flow curve. The relaxation of the assumption of piston-like displacement is facilitated by the proportionality relation between the time-of-flight and the breakthrough time for the analytic Buckley–Leverett solution, see Section 3.2. This observation has also been made by other authors, see e.g., [3].

5.3.3. *The Agarwal–Blunt Method.* Agarwal and Blunt [3] extend the Wang–Kovseck method to compressible black-oil systems with gravity by using 'full physics' in the forward simulation that determines the match. As in [141], the key idea is to use a piston-like approach to sort streamlines with respect to breakthrough times and match permeability values along individual streamlines. To avoid inverting a matrix system, Agarwal and Blunt [3] use an alternative method to adjust effective permeability values. Assume that the permeability along a streamline is modified by a fixed amount α so that $k_{\text{new}} = \alpha k_{\text{old}}$. By (20) the time-of-flight along a streamline is proportional to an effective permeability–porosity ratio; i.e., $\tau \propto \frac{\phi}{k}$. Therefore, the new time-of-flight is given as $\tau_{\text{new}} = \tau_{\text{old}}/\alpha$. For piston-like displacement (or tracer-like flow), the time-of-flight and the arrival time of a saturation contour will coincide. For non-piston flow, the time-of-flight may be a good enough approximation to the arrival time, because a proportionality factor will anyway cancel out in the correction factor specified above. For instance, for the analytic Buckley–Leverett solution presented in Section 3.2, the arrival time of a saturation contour is proportional to the time-of-flight

$$t \propto \tau \propto \frac{\phi}{k}, \tag{65}$$

where the proportionality factor depends on the saturation value. To match a calculated breakthrough time t_i^{cal} with the observed time t_i^{obs} one may therefore modify the permeability by a factor $\alpha = t_i^{\text{cal}}/t_i^{\text{obs}}$.

In the first case, there is a fixed pressure drops between wells. Now, if the average permeability of the region is incorrect, the well rate at late times may be erroneous. The calculated water rate q_w is therefore first rescaled. We let $t_{\max}$ denote the latest time for which an observed rate is available and introduce an overall modification α_0 of the permeability field (i.e., a modification for all streamlines connected to the producer). To match the end-point we require that

$$\alpha_0 = q_w^{\text{obs}}(t_{\max})/q_w^{\text{cal}}(\alpha_0 t_{\max}) \approx q_w^{\text{obs}}(t_{\max})/q_w^{\text{cal}}(t_{\max}).$$

This gives a rescaling of the calculated water-rate curve. Then the water-rate axis is divided into increments $\alpha_0 q_\ell$. To match water rate $q_w = \alpha_0 \sum_{j=1}^{\ell} q_j$ at time $t_\ell^{\text{cal}}/\alpha_0$, we determine the corresponding time t_ℓ^{obs} from the data $q_w^{\text{obs}}(t_\ell^{\text{obs}})$. To align the two times, we must modify the permeability by a factor $\tilde{\alpha}_\ell = (t_\ell^{\text{cal}}/\alpha_0)/t_\ell^{\text{obs}}$. Thus the overall modification along streamline ℓ becomes

$$\alpha_\ell = \alpha_0\, \tilde{\alpha}_\ell = t_\ell^{\text{cal}}/t_\ell^{\text{obs}}.$$

If the total rate changes, a rescaling of the water rate is done by multiplying the observed and calculated water rates by $q^*/q_{\text{t}}(t)$, where q^* is the water rate when the streamline pattern was taken, and $q_{\text{t}}(t)$ is the time-varying total rate. The method described for fixed rates is then applied to the rescaled water rate. This is similar to matching fractional flow.

A similar modification is proposed for the case where the wells are constrained by total rate. Based on Darcy's law for effective permeability for streamline bundles between two wells, we can define the modification

$$\alpha_0 = \frac{\Delta p^{\mathrm{cal}}}{\Delta p^{\mathrm{obs}}},$$

where Δp^{cal} and Δp^{obs} are the calculated and observed pressure drops between an injector–producer pair when the streamline pattern was computed (e.g., at water breakthrough). For compressible flow, streamlines may originate and terminate away from the wells. Such streamlines are neglected by defining $\alpha_0 = 1$. The total modification along streamline ℓ is then

$$\alpha_\ell = \alpha_0 \cdot \frac{t_\ell^{\mathrm{cal}}}{t_\ell^{\mathrm{obs}}}.$$

To map the modified streamline properties back to the underlying grid, Agarwal and Blunt [3] use the following volume-weighted average in each grid cell

$$K_i^{\mathrm{new}} = \underbrace{\frac{\sum_{\ell=1}^{N_{\mathrm{sl},i}} q_\ell \, \Delta\tau_{\ell,i} \, \alpha_\ell}{\sum_{\ell=1}^{N_{\mathrm{sl},i}} q_\ell \, \Delta\tau_{\ell,i}}}_{\bar{\alpha}} K_i^{\mathrm{old}}. \tag{66}$$

Here $N_{\mathrm{sl},i}$ is the number of streamlines crossing grid cell i, $\Delta\tau_{\ell i}$ is the time-of-flight increment of streamline ℓ across grid cell i, and q_ℓ is the total flux of streamline ℓ.

Jang and Choe [78] apply the method of Agarwal and Blunt [3] as the second step of a two-step approach. The first step is a gradient-based minimization incorporating well pressures and permeability samples, where the necessary sensitivities are calculated by the adjoint method. The motivation of the first step is to get the order in which the streamlines break through more correct prior to the second step, which is intended to make the location of the permeability modifications more accurate. The two steps are iterated if necessary.

In [4] the prior method of Agarwal and Blunt [3] is extended to include modifications to the permeability histogram and the porosity of each well-pair. The first part of the method is as in [3]. Then, water rates at a fractional flow of 10% are matched by adjusting the porosity. Because breakthrough times scale with the effective porosity along the streamlines, the porosities in each producer-region are modified by

$$\phi^{\mathrm{new}} = \phi^{\mathrm{old}} \frac{t^{\mathrm{obs}}}{t^{\mathrm{cal}}}.$$

Finally, the histogram of the permeability is also modified to match the spread in the water rate. To preserve the rank order of the permeability values in a well-region, the following relation is applied

$$K^{\mathrm{new}} = K^{\mathrm{old}} \cdot \left[\frac{K^{\mathrm{old}}}{\bar{K}} \right]^{\xi}, \tag{67}$$

where $\bar{K}$ is a geometric average of the permeability for the well-region. To determine the exponents ξ for capturing heterogeneity, Agarwal and Blunt suggest the following approach: First the time spread Δt between 10% and 90% water rate or fractional flow is determined for both the observed and calculated curves, i.e., Δt^{obs} and Δt^{cal}, respectively. The 'fast' and the 'slow' streamlines are identified. The fast streamlines are the streamlines with breakthrough at 10% or lower fractional flow, and the slow streamlines have breakthrough at 90% or higher fractional flow. Streamline-effective permeabilities $\bar{k}_{\mathrm{f}}$ and $\bar{k}_{\mathrm{s}}$ are calculated by harmonic averaging for the 'fast' and the 'slow' streamlines bundles, respectively. It is then required that

$$\frac{\bar{k}_{\mathrm{f}}^{\mathrm{new}}}{\bar{k}_{\mathrm{s}}^{\mathrm{new}}} = \frac{\bar{k}_{\mathrm{f}}^{\mathrm{old}}}{\bar{k}_{\mathrm{s}}^{\mathrm{old}}} \cdot \frac{\Delta t^{\mathrm{obs}}}{\Delta t^{\mathrm{cal}}}.$$

By assuming that $\bar{K}$ is representative for the 'fast' and the 'slow' streamline bundles and applying (67) one obtains that

$$\left[\frac{\bar{k}_{\mathrm{f}}^{\mathrm{old}}}{\bar{K}}\right]^{\xi}\cdot\left[\frac{\bar{K}}{\bar{k}_{\mathrm{s}}^{\mathrm{old}}}\right]^{\xi}=\frac{\Delta t^{\mathrm{obs}}}{\Delta t^{\mathrm{cal}}}\quad\Leftrightarrow\quad\xi=\frac{\ln\left(\frac{\Delta t^{\mathrm{obs}}}{\Delta t^{\mathrm{cal}}}\right)}{\ln\left(\frac{\bar{k}_{\mathrm{f}}^{\mathrm{old}}}{\bar{k}_{\mathrm{s}}^{\mathrm{old}}}\right)}.$$

When ξ is determined, a new permeability distribution is calculated for each well region. The approach to modify the histogram is only applied at the first iteration of the history-matching procedure. Further, the other modifications for porosity and permeability described above are performed for the two first iterations. For the subsequent iterations Agarwal and Blunt instead use the Newton iteration

$$\mathbf{m}^{n+2}=\mathbf{m}^{n+1}-J^{n+1}\frac{\mathbf{m}^{n+1}-\mathbf{m}^{n}}{J^{n+1}-J^{n}},$$

where J is an objective function and $\mathbf{m}$ is the reservoir parameters to be modified.

In [3] the method presented above is applied to a portion of the Ekofisk field in the North Sea. Further, in [4] the extended methodology presented above is applied to an Arabian Gulf field.

Kretz et al. [89] matched the time-of-flight from an injector to the fluid front. The location of the fluid front (saturation front) is intended to be localized by 4D seismics, but only synthetic examples are presented in this paper. Motivated by the relation between time-of-flight and effective permeability, the correction factor for the first permeability modification is obtained by

$$\alpha_{\ell}^{1}=\frac{\tau_{\mathrm{4D}}^{1}}{\tau_{\mathrm{cal}}^{1}}.$$

By this correction factor the permeability is modified between the injector and the fluid front. For the consecutive iterations the following modification factor is applied between the previous front and the current front

$$\alpha_{\ell}^{n}=\frac{\tau_{\mathrm{4D}}^{n}-\tau_{\mathrm{4D}}^{n-1}}{\tau_{\mathrm{cal}}^{n}-\tau_{\mathrm{cal}}^{n-1}}.$$

To propagate the modifications to the underlying grid (66) is applied, i.e., the same approach as suggested by Agarwal and Blunt [3].

5.3.4. *Adding Geostatistics.* Caers et al. [26] present an extended version of the method in [141] in two spatial dimensions consisting of a two-step mapping of the effective streamline permeabilities back to grid-cell permeabilities. The first step consists of obtaining modifications of effective streamline permeabilities, and to propagate the modifications to the underlying grid by the same deterministic approach as described above [26, 141]. In the second step, the updated grid permeability is used as an initial seed for a Gauss–Markov iteration (McMC). This two-step mapping is iterated until the history match is converged. The overall method thus consists of an outer iteration and an inner geostatistical iteration. Flow simulations are only needed in the iterations of the outer loop. Through the use of this geostatistical framework, the authors are able to match streamline-effective permeabilities to the production data and at the same time honor prior geological information.

In [25] a similar method is presented, where gradual deformation [74] (see Section 5.4.4) is used instead of the Gauss–Markov random function method. Because the gradual-deformation method can be used with any sequential simulation algorithm [25], the assumption of a Gaussian random permeability field can be relaxed and the approach can be extended to include multi-point geostatistics to match more complex geological structures like fractures and channels. A straightforward application of gradual deformation would seek to minimize the misfit in the production data. A key point in Caers' method is to instead apply gradual deformation to minimize an objective function measuring the misfit with respect to the effective streamline

permeabilities derived by the original Wang–Kovscek method described above:

$$E(K^{\text{new}}) = \sum_{\ell=1}^{N_{\text{sl}}} \big(k_\ell^{\text{old}} - k_\ell(K^{\text{new}})\big)^2. \tag{68}$$

According to Caers [25], the required number of forward simulations is dramatically reduced (convergence typically requires only 5-10 flow simulations). However, a number of sequential simulations from a multivariate probability distribution have to be conducted in the inner loop that minimizes (68), but for large reservoir models the forward simulation will dominate the computational cost [25]. To apply this gradual-deformation approach for multi-well patterns, streamline-defined regions and a method for global optimization are required [25]. If a gradient-based method is used for the optimization, sensitivities for the objective function in (68) with respect to the gradual-deformation parameters are required. Caers [25] presents analytic sensitivities for this approach, in the case of Gaussian permeability field, by assuming that the streamline-effective permeability is given by the harmonic average in (57).

The overall methods in [25, 26] allow for drastic changes in the geological model as opposed to for instance the AHM method of Emanuel and Milliken [49] and the travel-time methods of He et al. [67], Vasco et al. [135], which seek minor changes to the model. Moreover, it should be mentioned that only synthetic two-dimensional cases are presented in [25, 26].

Ravalec-Dupin and Fenwick [116] present an alternative two-step method based upon the same ideas as Caers [25]. In the first step, the method of Agarwal and Blunt [3] is used to estimate corrections along bundles of streamlines (as opposed to individual streamlines in [3]). The effective permeability along a streamline bundle ℓ is here defined by (58); see the discussion in Section 5.3.1. The streamline bundles are identified by first sorting the streamlines according to breakthrough time and then segmenting the fractional flow curves so that all streamlines responsible for each given fractional flow increment are identified in a corresponding bundle. In the second step, the desired streamline-effective permeabilities are propagated back onto the underlying grid. To this end, the gradual-deformation method is used to minimize the misfit between the desired effective streamline permeabilities and those calculated for the given permeability field as above.

In [27, 53] Caers, Fenwick and coworkers present another extended version of the Wang–Kovscek method. Again, by the approximation in (65) a correction factor is defined by

$$\alpha = \frac{(\frac{\phi}{k})_{\text{sl}}^{\text{new}}}{(\frac{\phi}{k})_{\text{sl}}^{\text{old}}} = \frac{\tau^{\text{new}}}{\tau^{\text{old}}},$$

where $(\phi/k)_{\text{sl}}$ represents an effective permeability–porosity ratio for each streamline-bundle. The correction factor above can be used to modify the porosity, the permeability, or the permeability–porosity ratio in the effective region (along streamline, streamline-bundle, or producer zone)

Inspired by [141], the piston-like breakthrough in each streamline is related to the fractional-flow curve. The relative modification for a time-average producer zone is of amplitude type and defined by [27]

$$\alpha = 1 - \frac{1}{N_{\text{d}}} \sum_{j=1}^{N_{\text{d}}} \left(\frac{q_w^{\text{cal}}(t_j)}{q_t^{\text{cal}}(t_j)} - \frac{q_w^{\text{obs}}(t_j)}{q_t^{\text{obs}}(t_j)} \right) = 1 + \frac{1}{N_{\text{d}}} \sum_{j=1}^{N_{\text{d}}} \big(f_w^{\text{obs}}(t_j) - f_w^{\text{cal}}(t_j) \big). \tag{69}$$

Here, $q_w(t_j)$ and $q_t(t_j)$ are the water and total production rate at time t_j. This modification is equivalent to considering the whole flow zone as a single effective streamline bundle. To propagate the permeability modifications back to the underlying grid, direct sequential simulation (DSSIM) [81] with locally varying mean is applied. In contrast to common methods for sequential simulation, like sequential Gaussian simulation, that require transformation into a standard Gaussian space, DSSIM can be performed directly in data space. Journel [82] showed that the sequential simulation algorithm, without any prior transformation, succeeded in reproducing a covariance model, provided that the simulated values are drawn from local conditional distributions identifying the simple kriging mean and variance derived from that covariance model. This fundamental result is known as DSSIM; for more details on DSSIM

see e.g., [24]. However, DSSIM does not honor the prior histogram, which is not necessarily a disadvantage if the histogram is not known with sufficient accuracy. Further, the fact that the histogram is allowed to change, while at the same time honoring the covariance structure, gives flexibility to for instance perturb the local mean [27]. Application of the approach to a real field case is presented in [65].

In [53] the method of locally varying mean (LVM) presented in [27, 65] is extended, and combined with the probability perturbation method (PPM) of Hoffman and Caers [71]. While LVM perturbs large scale structures, PPM perturbs the small-scale variations, and in this manner the combination of the methodologies is considered a multiscale approach. The combination of LVM and PPM was first proposed in [27], but without any implementation. The two methodologies are applied sequentially. However, for the examples presented, PPM did not give any substantial contribution to the history matching. We will therefore not give any further description of PPM.

Under the assumption of piston-like displacement along each streamline, the correction factor for a bundle of streamlines with water breakthrough in the time interval from t_{j-1} to t_j is given by [53]

$$\alpha = 1 - \frac{\int_{t_{j-1}}^{t_j} q_w^{\mathrm{cal}} dt}{\int_{t_{j-1}}^{t_j} q_t^{\mathrm{cal}} dt} + \frac{\int_{t_{j-1}}^{t_j} q_w^{\mathrm{obs}} dt}{\int_{t_{j-1}}^{t_j} q_t^{\mathrm{obs}} dt} = 1 + \int_{t_{j-1}}^{t_j} f_w^{\mathrm{obs}} - f_w^{\mathrm{cal}} dt. \tag{70}$$

This equation is derived by assuming fixed streamlines for the time interval from t_{j-1} to t_j, and can be considered as an extension of (69). To account for non-piston-like displacement in the streamlines and that streamlines going through a single grid cell can be connected to multiple producers, the following weighting is applied

$$\bar{\alpha}_{\Delta t} = \frac{\sum_{k=1}^{N_{w,p}^{\Delta t}} \left(\alpha_k q_w^k + q_t^k - q_w^k\right)}{\sum_{k=1}^{N_{w,p}^{\Delta t}} q_t^k}. \tag{71}$$

Here $N_{w,p}^{\Delta t}$ is the number of production wells connected to the streamlines in a grid cell. The correction factor is calculated for each grid cell exclusively using rate information from the streamlines intersecting the cell. For piston-like displacement, $q_w^k = q_t^k$, so that the correction factor simply reduces to simple weighting by the total flow rate. If in addition a streamline formulation with equal total rate for each streamline is used the correction factor reduces to an arithmetic average. To average the correction factor over $N_{\Delta t}$ sets of streamlines, a time-weighted average is used

$$\bar{\alpha} = \frac{\sum_{j=1}^{N_{\Delta t}} \Delta t_j \bar{\alpha}_{\Delta t}}{t_{N_{\Delta t}} - t_1}, \quad \Delta t_j = t_j - t_{j-1}. \tag{72}$$

Gross [63, 64] presents another version of the LVM method. First, modifications in streamline-effective permeability are obtained and propagated to the pressure grid by a version of the methods presented above. Second, the permeability modifications are propagated to other reservoir properties accounting for prior cross-property correlations. Like in [27, 65] the modifications to streamline-effective permeability give locally varying means that are propagated to the underlying grid by direct sequential simulation (DSSIM). The permeability modifications are then propagated to the other properties by a kind of Monte Carlo sampling. The prior probability distribution for the reservoir properties is arranged into discrete classes. The properties can for instance be permeability (horizontal,vertical), porosity, facies, net-to-gross, etc. For continuous properties, the ranges of the properties have to be carefully subdivided into 'bins' (classes). The prior model can be computed in two manners: by direct user specification after pore-network studies, or by scanning a set of training models to establish frequencies for the different classes in local control volumes [64]. The latter approach is applied here. The probability distribution needed in the cross-property propagation is simply

$$P(c^{\mathrm{new}}) = P(c^{\mathrm{prior}} | K^{\mathrm{new}}),$$

where c is a class configuration. This posterior distribution is simple to obtain from a prior distribution by extracting the configurations with the correct permeability bin. A renormalization is then necessary by summing over the extracted classes. Further, it is also possible to keep other properties fixed, which will shrink the total number of classes to be extracted.

To obtain the correction factor α_i^t with respect to the fractional-flow data, an approach similar to the original Wang–Kovscek approach is used [63, 64]. Relative modifications $\Delta k_{i,R}^t$ are obtained by a time-streamline-average of relative modifications of type given in (61) for all streamlines contributing to a producer and intersecting a grid cell i. Correction factors for fractional flow are then obtained by

$$\alpha_i^t = 10^{\Delta k_{i,R}^t}.$$

To account for mismatch in the total production rate q_k at producer k and the pressure drop $\Delta p_{k,l}$ between an injector l and producer k, the following correction factor is used

$$\alpha_i^{p,q} = \frac{1}{|\Omega_i|} \sum_{(k,l)\in\Omega_j} \left[\frac{q_k}{\Delta p_{k,l}}\right]^{\text{obs}} \cdot \left[\frac{\Delta p_{k,l}}{q_k}\right]^{\text{cal}}.$$

Here Ω_i is the set of injector-producer pairs for the streamlines intersecting grid cell i.

Jang [77] adds geostatistics to the method of Agarwal and Blunt [3] described above, by a simple approach. A two-step approach is proposed, where the first step consists of the method of Agarwal and Blunt [3]. The second step consists of generating Gaussian realizations conditioned to randomly selected grid-cell permeabilities of the resulting permeability field of the first step. If a flow simulation reveals a sufficient decrease in the objective function, a realization is accepted. The two steps are iterated if necessary.

Finally, we outline an idea for a fully analytic Bayesian approach for propagating/downscaling the modifications in streamline effective properties to the simulation grid. This approach relies on two assumptions. The first assumption is that the streamline effective properties can be obtained by weighted arithmetic averages of the reservoir properties (possibly transformed, e.g., by the logarithm). Hence, the weighted geometric average for a set of parameters is equivalent to a weighted arithmetic average for the logarithm of the parameters. The second assumption is that the (possibly transformed) reservoir properties can be considered Gaussian. Then, there is a linear relation between the streamline-effective properties and the (possibly transformed) reservoir properties, which enables analytic determination of the posterior Gaussian distribution of the reservoir properties, conditioned on the history-matched streamline-effective properties. Details are given in Appendix E.

5.4. **Miscellaneous Methods.** In this section we describe how streamline-defined regions or streamline-derived sensitivities have been used to boost the performance of existing geostatistical methods that were originally introduced without any connection to streamlines. The methods we here pursue are within the realm of: Markov chain Monte Carlo (McMC), ensemble Kalman filter (EnKF), sequential self-calibration (SSC), and the gradual deformation method (GDM).

5.4.1. *Markov chain Monte Carlo.* In [98] a two-stage Markov chain Monte Carlo (McMC) approach is proposed. McMC is a sampling approach for sampling rigorously from the posterior distribution. The sampling consists of iterations over a two-step algorithm. In the first proposal step modifications to the reservoir parameters are drawn from a proposal distribution. In the second acceptance/rejection step one determines if the proposal should be kept based on the resulting simulated reservoir responses. The algorithm is shown to converge in probability distribution in the limit of infinitely many iterations. Thus, an extensive number of iterations may be required to get close to convergence to the posterior distribution. Moreover, rigorous sampling also requires a full flow simulation to be performed for each proposal step. Consequently, the accurate sampling has a high computational demand.

The two-stage approach proposed by Ma et al. [98] is intended to speed-up the McMC approach considerably without reducing the accuracy. The first stage consists of obtaining

approximate production responses analytically for the proposed modified reservoir configuration by applying streamline-derived sensitivities. The approximate production responses are obtained by a linear approximation in the vicinity of the current production response:

$$\mathbf{g}^*(\mathbf{m}) = \mathbf{g}(\mathbf{m}_p) + \mathbf{G}\delta\mathbf{m}.$$

Here $\mathbf{g}(\cdot)$ denote a forward model (i.e., the reservoir simulator), $\mathbf{G}$ is the sensitivity matrix, and $\delta\mathbf{m} = \mathbf{m} - \mathbf{m}_p$ is the proposed modifications to the reservoir parameters. Only if the approximate production responses achieve acceptance for the proposed reservoir parameter field, a full reservoir simulation is performed to really check for actual acceptance/rejection.

5.4.2. *Ensemble Kalman Filter.* Among the geostatistical methods discussed in this section, the ensemble Kalman filter (EnKF) [51] has gained the most interest lately. EnKF is very flexible with respect to the type of data incorporated, can do sequential data integration and state updates during the production period, assessment of uncertainty is available from the ensemble representing the posterior, and it has recently been applied to several field cases, see e.g., [52, 105].

The ensemble Kalman filter is a Monte Carlo approach for sequentially integrating data into a reservoir model represented by an ensemble of realizations. The method utilizes cross-covariances between measurements and model parameters estimated from the ensemble to update the ensemble members. It is advantageous to keep the ensemble size low for high computational efficiency. However, this will increase the error in the estimated cross-covariances and the updated model parameters. To speed up the estimation of covariances, Arroyo-Negrete et al. [9] used streamlines and Devegowda et al. [45] used streamline-based sensitivities. Both the streamline trajectories and the production-response sensitivities include information about spatial correlations between production responses and model parameters. Therefore, in [9, 45] it is suggested to rescale the estimated cross-covariances by streamline-based influence weights using two slightly different approaches. The first approach is a streamline-trajectory assisted approach where the authors investigate the simple choice of a binary weighting based on streamline-defined regions [9]. For measurements of fractional-flow type it is possible to further condition based on information of water-front movement, which is easy to keep track of for streamlines. The second approach is sensitivity assisted [45], where the weights consist of rescaled production-response sensitivities, for which the smallest sensitivity values are neglected. For both approaches the 'regions of influence' from all the ensemble members are stacked to obtain a 'common region of influence' for each well.

5.4.3. *Sequential Self-Calibration.* In a series of papers [130, 143, 144, 146, 147], Wen and coworkers extended the so-called sequential self-calibration (SSC) method [62, 145] by applying and deriving streamline-based sensitivities. Streamlines are used for two purposes: (i) fast forward flow solution and (ii) fast calculation of sensitivity coefficients.

The SSC method is used for inversion of dynamic data and is an iterative geostatistically-based method coupled with an optimization procedure. Key points in the SSC method are: (i) the concept of master points to reduce the degrees of freedom in the optimization problem; (ii) propagation of perturbations at master points to the permeability field by kriging to account for spatial correlations; and (iii) computation of sensitivities by a combined streamline-based and gradient-simulator approach as described in Sections 4.2 and 4.6. The SSC algorithm starts with a set of initial realizations generated by a geostatistical algorithm and then performs the following steps:

(1) Solve flow equations and calculate sensitivities
(2) Evaluate the objective function and exit if satisfactory match is obtained.
(3) Select master points
(4) Optimization: find optimal perturbation at master points
(5) Propagate the perturbations back to the entire field by kriging.
(6) Iterate Steps 1–5 until a satisfactory match is achieved.

Let us assume that the observations to be used in the history match are the pressures and fractional flow in production wells. Then the history match is obtained by minimizing the

following objective function

$$\mathcal{O} = \sum_{k=1}^{N_w^p} \sum_{j=1}^{N_d^p} W_{k,j}^p \left(p_{k,j}^{\text{obs}} - p_{k,j}^{\text{cal}} \right)^2 + \sum_{k=1}^{N_w^f} \sum_{j=1}^{N_d^f} W_{k,j}^f \left(f_{k,j}^{\text{obs}} - f_{k,j}^{\text{cal}} \right)^2. \tag{73}$$

Here $p_{k,j}$ and $f_{k,j}$ denote pressure and fractional flow, respectively, at well k at time t, and $W_{k,j}$ are weights. A gradient based-method is used to minimize the objective function $\mathcal{O}$. This requires the sensitivity coefficients of well pressure and fractional flow.

In [130], the SSC method is extended to include inversion of spatially distributed saturation data, e.g., from 4D-seismics. To include the saturation data, the following term is added to the objective function:

$$\sum_{\mathbf{x}} \sum_{j=1}^{N_d^S} W_j^S(\mathbf{x}) \left(S_j^{\text{obs}}(\mathbf{x}) - S_j^{\text{cal}}(\mathbf{x}) \right)^2.$$

Here S is the saturation field, W_j are weights, and $\mathbf{x}$ are the observation points in the reservoir domain. The calculation of analytic streamline-based saturation sensitivities is described in Section 4.5. A synthetic 3D example is presented where saturation data is incorporated, but where most weight is given to the fractional-flow data based on an assumption of less uncertainty.

In [130, 144] a two-stage multiscale approach is used. The first stage consists of using the method outlined above on an upscaled initial geostatistical realization. Then the coarse-scale history-matched model is geostatistically downscaled to obtain fine-scale realizations. Simulated annealing and sequential Gaussian simulation is used for the downscaling in [144] and [130], respectively. This approach is less CPU-intensive than simulated annealing, and should remedy features of the SA approach like the tendency of fuzziness (high nugget effect) [130]. Part of the motivation for the upscaling in Tran et al. [130] is that the saturation data, e.g., from 4D-seismics, often have lower resolution.

5.4.4. *Gradual Deformation Methods.* Gradual deformation [74, 117] is a parameterization method that reduces the number of unknown parameters considerably by seeking new realizations as linear combinations of independent or dependent [106] realizations drawn from a geostatistical probability distribution. The method is motivated by the fact that linear combinations of multi-Gaussian random functions remain multi-Gaussian random functions. The gradual deformation method (GDM) does not necessarily rely on a Gaussian probability distribution, but a sequential simulation algorithm is required [74]. Further, the gradual deformation can also be performed with respect to structural parameters like mean, variance, and correlation range [74].

Gautier et al. [58, 59] develop a GDM-based history-matching method for two-phase incompressible flows, where they use a Gauss–Newton method equipped with partly streamline-based sensitivity coefficients to minimize the objective function (or misfit functional). Further, the use of gradual deformation incorporates geostatistics and reduces the number of parameters. The sensitivity calculations are discussed in Section 4.7. Gautier et al. [59] present the method and preliminary results on synthetic examples. The objective function obtained by this approach is highly irregular with many local minima, which makes the optimization problem harder. In [58] the optimization problem is discussed in more detail and some methods for smoothing the objective function are presented.

Barthelemy et al. [16] investigate a methodology for local gradual deformation with regions defined by streamlines. Rather than solving a global minimization problem, the authors investigate the applicability of gradual deformation for the local independent streamline-defined regions in parallel. Further, the authors propose a definition of an objective function, inspired by the travel-time matching described in Section 5.2, based on temporal moments of the production data. The proposed objective function was consistent with the conventional amplitude based objective function, but did not remove the irregularity considerably.

Finally, we remark that two gradual-deformation approaches for history-matching effective permeabilities along streamlines and streamline-bundles [25, 116] were presented in Section 5.3.4.

6. Streamline-based ranking of geostatistical realizations

Ranking is the process of reducing the number of realizations of a reservoir model, while spanning the probability space to a maximum extent. To rank the realizations a criterion correlated to production/economical outcome is typically used. Ranking is closely related to history-matching because it can be performed both before, during, and/or after the history matching itself. It is therefore natural to briefly describe two streamline-based methods for ranking.

Wang and Kovscek [142] present a method for ranking geostatistical models with respect to production data. First, a single model is history-matched with respect to production data. Then, multiple realizations are ranked with regard to streamline properties like time-of-flight, flow rate etc., by comparing with the history-matched model. According to the authors, this gives a fast method for generating multiple models that incorporate production data.

Idrobo et al. [75] investigate and discuss the use of an approximation for the swept volume based on the streamline coordinates (τ, ψ, χ) introduced in Section 3. Calculations of the swept volume are facilitated by the fact that the Jacobian with respect to the spatial coordinates (x, y, z) takes a simple form [41, 85]

$$\left\| \frac{\partial(\tau,\psi,\chi)}{\partial(x,y,z)} \right\| = \nabla\tau \cdot (\nabla\psi \times \nabla\chi) = \nabla\tau \cdot \mathbf{u} = \phi.$$

Here the time-of-flight equation (10) and the property $\mathbf{u} = \nabla\psi \times \nabla\chi$ are applied. Thus, the time-of-flight coordinates preserve the pore volume by $\phi\, dx\, dy\, dz = d\tau\, d\psi\, d\chi$. Further, the volume swept at a time t can be approximated by [41, 85]

$$V_{\text{swept}}(t) = \int\int\int \theta\big(t - \tau(x,y,z)\big)\, \phi\, dx\, dy\, dz = \int\int\int \theta\big(t - \tau(\psi,\chi)\big)\, d\tau\, d\psi\, d\chi, \tag{74}$$

where θ is the Heaviside function. Notice in particular, that this derivation is exact for piston-like displacement. Now the integral in (74) can be approximated by [75]

$$V_{\text{swept}} = \sum_{\ell} \int_{\Psi_\ell} \theta(t-\tau) q_\ell d\tau,$$

where q_ℓ is the volumetric flux assigned to streamline Ψ_ℓ. Considering a 3D grid, an indicator variable can be defined at each cell based on the time-of-flight. A cell is marked 'unswept' if the time-of-flight at the cell is greater than the time of interest, and 'swept' if the time-of-flight is less than or equal to the time of interest. Summing the pore volumes of the 'swept' cells, an approximation for the total swept volume at the time of interest can be obtained, and thereby also the sweep efficiency by dividing by the total pore volume. To account for changing well conditions, the time-of-flight distribution is calculated for the different well-configurations.

An indicator for recovery can be used for ranking stochastic reservoir models, because realizations in the range from pessimistic to optimistic can be chosen for closer studies and determination of uncertainty. In [75] it is demonstrated that the proposed swept volume indicator is strongly correlated with waterflood recovery, and is thus proposed as a ranking criteria. However, it is not specified for which mobility ratios the numerical experiments are conducted. In [11] the above swept-volume indicator for ranking and uncertainty is used for analysis of a Middle Eastern carbonate reservoir.

7. Discussion

In the last section, we discuss and compare the different methods for streamline-based history-matching and point out similarities and differences, and discuss some potential problems, restrictions, advantages, etc. The history-matching methods presented in this review are the assisted history matching (AHM), (generalized) travel-time inversion ((G)TTI) methods, and methods for matching streamline effective properties (SLEP). In addition, we have

discussed streamline-based sensitivity calculations and reviewed various geostatistical history-matching methods where streamlines have been used to boost the performance. The discussion in this section is mainly focused on the three streamline-based approaches: AHM, (G)TTI, and SLEP.

Simplicity. The AHM approach [49, 104] is in many aspects the least sophisticated in the sense that it merely provides useful functionality to reservoir engineers that perform manual or semi-automated history matches. AHM appears to be fairly simple to implement on top of any existing streamline simulator, as this would not imply changing the flow solver itself, and would very likely offer useful functionality to reservoir engineers, see the histories of success in e.g., [29, 97]. However, the approach is manual, and as such not particularly suited to form the basis for an automated history-matching approach. An advantage of AHM, compared with traditional manual history-matching methods, is that the modifications made to the reservoir parameters seem to be more targeted, and thus results in a better preservation of prior geologic information.

Modifications to Reservoir Parameters. An obvious similarity for the three methods is that they more or less explicitly impose targeted modifications to the reservoir parameters along streamline paths. This is a general feature of all streamline methods, which both can be considered a strength and a weakness. The strength is that making targeted modifications along the streamlines is what is sought when the principal physical effects are aligned with the streamlines, which is the conditions for which streamline simulation is particularly well suited. In reality, however, traverse physical effects may be important in the forward simulation, but these effects are strongly convoluted in the production data, and may thus be hard to account for by any history-matching method.

The GTTI and SLEP methods are similar in the sense that the mismatches $\Delta\widetilde{t}$ and α_ℓ, respectively, are propagated to modifications in the reservoir parameters. However, the approaches for propagating the mismatch in data for the SLEP and the GTT methods are quite different. Even for methods within the SLEP class, there are large variations in how the modifications are propagated. Still, the two methods can theoretically be defined in terms of the propagation approach of the other method, see Appendix F.

The GTTI method tends, like the AHM method, to keep the modifications small, smooth, and targeted to preserve the prior geology description and the geologic realism. As discussed above, this is motivated by the general low spatial resolution of the production data and the need for stabilizing the under-determined inversion process. Some of the SLEP methods are constrained to geostatistical information. On the other hand, the approaches presented in [25, 26], for instance, are claimed to allow for quite drastic alterations to the reservoir model. The key to enable small targeted modifications for GTTI is the application of production-response sensitivities, which are not applied for AHM and SLEP.

Complexity of Flow Model. The streamline-based sensitivities for the travel-time inversion methods are derived assuming the analytical Buckley–Leverett profile described in Section 3.2 along each streamline. However, the minimization problem of the travel-time inversion methods is general without any assumptions on the flow profile. The methods for adjusting effective streamline properties (SLEP) are mainly derived by assuming piston-like displacement. However, in Appendix D we show that similar (or sometimes identical) expressions can be derived by replacing the piston-like displacement front by an analytic Buckley–Leverett profile. Generally, the application of the analytic Buckley–Leverett solution is only an approximation, because pressure updates, varying saturation along streamlines, and transverse flow effects are not explicitly accounted for. However, this approximation often turns out to be sufficiently accurate in practice to perturb the iteration in the correct direction. (If necessary, some of the inversion methods can be accompanied by more accurate forward simulations using 'full physics' to evaluate the quality of the match derived in each iteration).

To a certain extent, both GTTI and the SLEP methods account for changing pressure/velocity distribution during the forward simulation, and thereby changing streamline distribution. For GTTI, pressure updates are implicitly accounted for because an optimal time-shift for the

whole production curve is obtained, and because the sensitivities are calculated for the different pressure steps by the respective streamline distributions. For the early SLEP methods one or a few representative streamline distributions are used. However, for the association of the breakthrough of individual streamlines to increments in fractional-flow curves, it is implicitly assumed that the streamline trajectories are fixed in time. Exactly how information from more than one streamline distribution should be used seems to be a subjective implementation issue in the earlier approaches of SLEP. On the other hand, the more recent SLEP approach of Fenwick et al. [53] uses different streamline distributions to match different parts of the production curve (see (72)), which is similar to how sensitivities are obtained in GTTI. Moreover, multiple streamline distribution can be used in the AHM approach to guide the manual modifications.

Assessment of uncertainties in history-matching is particularly important for the purpose of predictions. This often requires a vast number of realizations and forward simulations. Properly implemented streamline simulators can be much faster than conventional finite-difference simulators, see e.g., [122], and is therefore well suited for this purpose. However, streamline simulators are under some conditions considered approximate simulators, even though the physics to be accounted for and the robustness of streamline simulators have improved considerably recently [40]. An alternative may then be to run a vast number of forward runs with a fast "approximate" simulator (i.e., streamline simulator), and a small number with the slower trusted simulator for calibration [108].

Convergence Properties. Both the original and the generalized travel-time inversion have shown quasilinear properties, while traditional amplitude matching exhibits orders of magnitude more non-linear behavior [34]. Most of the SLEP approaches are of travel-time type since the mismatch in arrival time of saturation fronts (breakthrough) are matched. However, multiple data points are usually incorporated for SLEP. On the other hand, the modifications in (69) and (70) are of amplitude type. Whether the different SLEP approaches share similar convergence properties or not, has not yet been addressed systematically. However, fast convergence is at least observed for the very simple test examples in the early travel-time-like SLEP approaches [121, 141].

Types of Data to be Matched. What kind of data can be incorporate and what kind of reservoir parameters can be modified by the different methods? For SLEP methods, water-cut, total well rate, pressure drops, and saturation-front (from seismics) have been incorporated. The parameters matched, are so far restricted to permeability, porosity, or permeability-porosity ratio, mainly because the approximation $t \propto \tau \propto \phi/k$ is heavily exploited in the SLEP methods. However, other quantities involved in the expressions relating streamline-effective properties with the data, like the end-point mobility ratio or residual saturations (see e.g., Eq. 13 in [121]), may be matched for each producer region.

For GTTI, on the other hand, tracer concentration, fractional-flow, and gas-oil ratio data have been incorporated. For these quantities, arrival-time sensitivities of some contributing quantity can be computed and related to the time-of-flight. Thus, it is the time-of-flight sensitivities that determine which reservoir parameters that can be modified: permeability, porosity, and relative permeability curves (mobility ratios), see Section 4.2. The matrix system in (47) is quite general, so as long as sensitivities relating mismatch in data and perturbations in reservoir parameters can be computed, any data type can potentially be incorporated. The Bayesian system in (56) is also general, but requires in addition a specification of a covariance structure for the reservoir parameters. A weakness of the current GTTI formulation is that well pressure has not yet been incorporated. It may be especially important to constrain jointly on pressure observations and other dynamic data when free gas is present. Streamline-based sensitivities can be obtain analytically for several reservoir responses, based on a single forward simulation, but streamline-based well/grid-pressure sensitivities have not been derived. However, sensitivities for the response of the propagation of a sharp 'pressure front' is derived, see Section 4.7. In Appendix G we outline two potential approaches for obtaining well-pressure sensitivities, which will allow for incorporating well pressures in GTTI.

Finally, the literature contains three papers that discuss how time-lapse seismics can be incorporated. The method in [130] is within the sequential self-calibration scope, the method described in [89] is inspired by the SLEP methods, while the method described in [136] is derived within the TTI scope.

Finally, we will just refer to a few papers [12, 54, 55] presenting streamline-based history-matching work flows applied to real field cases.

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Appendix A. Sensitivities for Relative Permeability

In Section 4.2 we presented time-of-flight sensitivities for various reservoir parameters. Kulkarni and Datta-Gupta [91] derived similar sensitivities for parameters involved in describing relative permeability curves. They present sensitivities for two different representations of the relative permeability curves.

In the first case, the oil and water relative permeability are represented by power functions (Corey curves)

$$k_{ro} = k_{ro}^o S_{no}^{\alpha_o}, \qquad k_{rw} = k_{rw}^o S_{nw}^{\alpha_w}.$$

Here $S_{n\alpha}$ is the normalized saturation and $k_{r\alpha}^o$ is the end-point relative permeability.

Alternatively, one may use a B-spline expansion for each curve. This gives more flexibility to the function representation because the assumption of a particular shape of the function is relaxed. For oil, the B-spline expansion is given by

$$k_{ro} = \sum_{j=1}^{N} c_{o,j} B_{o,j}^m(S_{no}),$$

where $c_{o,j}$ is the jth B-spline coefficient and $B_{w_j}^m(S_{no})$ is the jth B-spline of polynomial order m. A similar B-spline representation is used for the relative permeability functions of water.

Considering the relative permeability functions for oil we obtain the following sensitivities for slowness $s(\mathbf{x})$, given by (20), with respect to k_{ro}^o, α_o and $c_{o,j}$

$$\begin{aligned}
\frac{\partial s}{\partial k_{ro}^o} &= \frac{\partial s}{\partial \lambda_t}\frac{\partial \lambda_t}{\partial k_{ro}^o} = -\frac{s}{\lambda_t}\frac{S_{no}^{\alpha_o}}{\mu_o},\\
\frac{\partial s}{\partial \alpha_o} &= \frac{\partial s}{\partial \lambda_t}\frac{\partial \lambda_t}{\partial \alpha_o} = -\frac{s}{\lambda_t}\frac{k_{ro}^o S_{no}^{\alpha_o}\ln S_{no}}{\mu_o},\\
\frac{\partial s}{\partial c_{o,j}} &= \frac{\partial s}{\partial \lambda_t}\frac{\partial \lambda_t}{\partial c_{o,j}} = -\frac{s}{\lambda_t}\frac{B_{o,j}^m}{\mu_o},
\end{aligned}$$

where μ_o is the oil viscosity. Similarly, sensitivities are obtained with respect to the parameters for the water relative permeability. Time-of-flight sensitivities can now be obtained by (21).

Appendix B. Gaussian Linearity and Analytic Conditional Distribution

In this section we will show how a conditional Gaussian distribution can be determined analytically for a linear model. To this end, we start by considering a stochastic variable $\mathbf{m} \in \mathbb{R}^p$ from a multivariate Gaussian distribution

$$[\mathbf{m}] \sim \mathcal{N}_p(\mu_m, \mathbf{\Sigma}_{mm}). \tag{A-1}$$

Further, we assume a linear relation $\mathbf{d} = \mathbf{A}\mathbf{m} + \mathbf{u}$ between $\mathbf{m}$ and another stochastic variable $\mathbf{d} \in \mathbb{R}^k$, where $[\mathbf{u}] \sim \mathcal{N}_k(\mathbf{0}, \mathbf{\Sigma}_u)$ is an error term that is assumed to be independent of $\mathbf{m}$. Standard Gaussian theory then gives [80]

$$[\mathbf{d}] \sim \mathcal{N}_k(\mu_d, \mathbf{\Sigma}_{dd}), \tag{A-2}$$

where the expectation and covariance are

$$\mu_d = \mathbf{A}\mu_m, \qquad \mathbf{\Sigma}_{dd} = \mathbf{A}\mathbf{\Sigma}_{mm}\mathbf{A}^T + \mathbf{\Sigma}_u.$$

Further, a combination of (A-1) and (A-2) gives the joint probability distribution

$$\begin{bmatrix} \mathbf{m} \\ \mathbf{d} \end{bmatrix} \sim \mathcal{N}_{p+k}\left(\begin{bmatrix} \mu_m \\ \mu_d \end{bmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{mm} & \mathbf{\Sigma}_{md} \\ \mathbf{\Sigma}_{dm} & \mathbf{\Sigma}_{dd} \end{bmatrix} \right),$$

where $\mathbf{\Sigma}_{dm}$ describes the covariance between $\mathbf{d}$ and $\mathbf{m}$,

$$\mathbf{\Sigma}_{dm} = \mathrm{Cov}\{\mathbf{d}, \mathbf{m}\} = \mathbf{A}\mathbf{\Sigma}_{mm},$$

and $\mathbf{\Sigma}_{md}$ is the transpose of $\mathbf{\Sigma}_{dm}$. Finally, from the joint distribution, the posterior distribution for $\mathbf{m}$ given $\mathbf{d}$ can be derived:

$$[\mathbf{m}|\mathbf{d}] \sim \mathcal{N}_p(\mu_{m|d}, \mathbf{\Sigma}_{m|d}),$$

where the conditional expectation and covariance are

$$\mu_{m|d} = \mu_m + (\mathbf{A}\mathbf{\Sigma}_{mm})^T[\mathbf{A}\mathbf{\Sigma}_{mm}\mathbf{A}^T + \mathbf{\Sigma}_u]^{-1}(\mathbf{d} - \mathbf{A}\mu_m),$$
$$\mathbf{\Sigma}_{m|d} = \mathbf{\Sigma}_{mm} - (\mathbf{A}\mathbf{\Sigma}_{mm})^T[\mathbf{A}\mathbf{\Sigma}_{mm}\mathbf{A}^T + \mathbf{\Sigma}_u]^{-1}\mathbf{A}\mathbf{\Sigma}_{mm}.$$

Computing the inverse of the $(k \times k)$ matrix can be very costly for large k.

Appendix C. One-dimensional upscaling

Assume one-phase flow in a uni-directional system partitioned by $N+1$ nodes into N (sub) cells, with a total length of Δx and a pressure drop of Δp. Further, assume no gravity. For each node there is associated a pressure p_i. The distance between node i and node $i+1$ is denoted Δx_i. The pressure drop over the reservoir is equal to the sum of the pressure drops between two consecutive nodes, i.e., $\Delta p = \Delta p_1 + \Delta p_2 + \ldots + \Delta p_N$, and the total length is $\Delta x = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_N$. An average/effective Darcy velocity over the one-dimensional system is then given by Darcy's law:

$$\bar{u} = -\frac{K}{\mu}\frac{\Delta p}{\Delta x}.$$

Summing the contributions of the subintervals gives

$$-\frac{\bar{u}\Delta x}{K} = \frac{1}{\mu}\Delta p = \frac{1}{\mu}\Big(\Delta p_1 + \Delta p_2 + \ldots + \Delta p_N\Big) = -\left(\frac{u_1\Delta x_1}{K_1} + \frac{u_2\Delta x_2}{K_2} + \ldots + \frac{u_N\Delta x_N}{K_N}\right).$$

Hence the upscaled/effective permeability is given by the following weighted harmonic average

$$K = \frac{\bar{u}\Delta x}{\frac{u_1\Delta x_1}{K_1} + \frac{u_2\Delta x_2}{K_2} + \ldots + \frac{u_N\Delta x_N}{K_N}}.$$

If the sub-cell velocities are available and used in this expression, it can be considered as flow based upscaling. Further, if the sub-cell velocities are not available, it is possible to argue that

the effective velocity $\bar{u}$ can be considered equal in each interval, i.e., $\bar{u} = u_1 = u_2 = \ldots = u_N$, which gives

$$K = \frac{\Delta x}{\frac{\Delta x_1}{K_1} + \frac{\Delta x_2}{K_2} + \ldots + \frac{\Delta x_N}{K_N}}. \tag{A-3}$$

Assume instead a set of parallel one-dimensional uni-directional flow systems with a single permeability each (like homogeneous layers or streamlines). The parallel layers are numbered by $i \in \{1, 2, \ldots, N\}$ and have a common pressure drop Δp and length Δx. Further, the velocity u can be described by the volumetric flow rate q divided by the cross-sectional area A. Similarly as above, Darcy's law over all layers then reads

$$-\frac{KA}{\mu}\frac{\Delta p}{\Delta x} = \bar{q} = q_1 + q_2 + \ldots + q_N = -\frac{1}{\mu}\frac{\Delta p}{\Delta x}\big(K_1A_1 + K_2A_2 + \ldots + K_NA_N\big).$$

The upscaled/effective permeability is therefor given by the weighted arithmetic average

$$K = \frac{1}{A}\big(K_1A_1 + K_2A_2 + \ldots + K_NA_N\big) = \frac{\bar{u}}{\bar{q}}\left(K_1\frac{q_1}{u_1} + K_2\frac{q_2}{u_2} + \ldots + K_N\frac{q_N}{u_N}\right).$$

Hence, a flow-based upscaling is possible also for this scenario.

Appendix D. Modifications for 1D Buckley–Leverett Displacement

In this section we will derive the effective permeability modifications for a 1D Buckley–Leverett displacement. Assuming Riemann initial data, the arrival time t_ℓ of a saturation front S_{wf} is discussed in Section 3.2, that is,

$$t_\ell = \frac{\tau_\ell}{\widetilde{f}'(S_{wf})}.$$

Inserting (20) for the time-of-flight and assuming a streamline effective permeability k_ℓ gives

$$t_\ell = \frac{1}{\widetilde{f}'}\int_\Psi \frac{\phi(\mathbf{x})}{\lambda_t k_\ell |\nabla p|}dr \quad \Leftrightarrow \quad k_\ell = \frac{1}{t_\ell \widetilde{f}'}\int_\Psi \frac{\phi(\mathbf{x})}{\lambda_t |\nabla p|}dr. \tag{A-4}$$

By assuming that the streamline paths of the prior permeability field and the quantities involved in the integral are exact (also assumed in [141]), the relative modifications are then given by

$$r_\ell^t = \frac{\Delta k_\ell^t}{k_\ell^{\text{cal}}} = \frac{k_\ell^{\text{obs}} - k_\ell^{\text{cal}}}{k_\ell^{\text{cal}}} = \frac{t_\ell^{\text{cal}} - t_\ell^{\text{obs}}}{t_\ell^{\text{obs}}}.$$

Hence, (61) is recovered. If also the effective pressure drop Δp for an injector–producer pair is to be matched, we can use an effective pressure gradient of $\Delta p / L_\ell \approx |\nabla p|$, where L_ℓ is the length of streamline ℓ. The relative modification of (64) is then obtained by the same approach.

Appendix E. Analytic Gaussian Upscaling/Downscaling

Assume there are N grid parameters $\mathbf{M} = \{M_i\}$ (possibly transformed) contributing to s effective/upscaled parameters $\mathbf{m} = \{m_\ell\}$. Further, assume that the effective parameter m_ℓ can be calculated by the weighted arithmetic average

$$m_\ell = \frac{1}{\sum_i w_{\ell i}}\Big[w_{\ell 1}M_1 + w_{\ell 2}M_2 + \ldots + w_{\ell N}M_N\Big] + u_\ell, \tag{A-5}$$

where u_ℓ is a random error term that will be described below. Hence, the weights are zero if a grid parameter does not contribute to the effective parameter. All the s effective parameters can then be calculated by

$$\mathbf{m} = \mathbf{A}\mathbf{M} + \mathbf{u}, \tag{A-6}$$

where

$$\mathbf{A} = \left\{A_{\ell i} = \frac{w_{\ell i}}{\sum_i w_{\ell i}}\right\} \quad \text{and} \quad \mathbf{u} = \{u_\ell\}.$$

Assuming multivariate Gaussian reservoir parameters $[\mathbf{M}] \sim \mathcal{N}_N(\mu_M, \mathbf{\Sigma}_M)$ and Gaussian noise error $[\mathbf{u}] \sim \mathcal{N}_s(\mathbf{0}, \mathbf{\Sigma}_u)$, it follows from Gaussian linearity that also the effective parameters $\mathbf{m}$ are Gaussian, see Appendix B. Following the derivation given in Appendix B the conditional distribution for $\mathbf{M}|\mathbf{m}$ is Gaussian and given analytically by

$$[\mathbf{M}|\mathbf{m}] \sim \mathcal{N}_N(\mu_{M|m}, \mathbf{\Sigma}_{M|m}), \tag{A-7}$$

$$\mu_{M|m} = \mu_M + (\mathbf{A}\mathbf{\Sigma}_{MM})^T[\mathbf{A}\mathbf{\Sigma}_{MM}\mathbf{A}^T + \mathbf{\Sigma}_u]^{-1}(\mathbf{m} - \mathbf{A}\mu_M), \tag{A-8}$$

$$\mathbf{\Sigma}_{M|m} = \mathbf{\Sigma}_{MM} - (\mathbf{A}\mathbf{\Sigma}_{MM})^T[\mathbf{A}\mathbf{\Sigma}_{MM}\mathbf{A}^T + \mathbf{\Sigma}_u]^{-1}\mathbf{A}\mathbf{\Sigma}_{MM}. \tag{A-9}$$

The computation of the inverse $(s \times s)$ matrix may be computer intensive. However, some kind of sequential local downscaling (e.g., for streamline-based regions) may be applied to reduce the dimension s.

For permeability it is sometimes assumed a log-Gaussian model, which entails that $M_i = \log K_i$ is Gaussian. The approach outlined above can then be applied if the effective permeability for streamline/coarse cell ℓ can be represented by the weighted geometric average (see discussion in Section 5.3.1)

$$k_\ell = u_\ell \cdot \left(\Pi_i K_i^{w_{\ell i}}\right)^{1/\sum_i w_{\ell i}},$$

where u_ℓ is a log-Gaussian approximation error. Because taking the logarithm on each side of the geometric average gives

$$\begin{aligned}\log k_\ell &= \frac{1}{\sum_i w_{\ell i}} \log(\Pi_i K_i^{w_{\ell i}}) + \log u_\ell \\ &= \frac{1}{\sum_i w_{\ell i}} \Big[w_{\ell 1} \log K_1 + w_{\ell 2} \log K_2 + \ldots + w_{\ell N} \log K_N\Big] + \log u_\ell,\end{aligned}$$

which is on a linear form equivalent to (A-5). Further, applying the logarithm of the permeability ensures a positive permeability. For porosity it is sometimes assumed a Gaussian distribution directly, i.e., $M_i = \phi_i$, but a porosity within the interval $[0, 1]$ is then not guaranteed. However, the variability in porosity is usually much smaller than for permeability. In addition, it is possible to use some transformation that ensures a porosity within range, e.g., a logit transformation.

Appendix F. Equivalence Between SLEP and GTTI

First, we consider GTTI propagation of the SLEP modifications. For SLEP modifications $\delta\mathbf{k}$ are obtained (as described above). Further, sensitivities with respect to grid permeability $\mathbf{G} = \partial\mathbf{k}/\partial\mathbf{K}$ are given by for instance (63). Hence, the streamline-effective permeabilities can be propagated to obtain grid-permeability modifications $\delta\mathbf{K}$ by the minimization system in (47) or (56) that were originally used for the deterministic and the Bayesian version of the GTTI, respectively. We may also normalize the modifications and the sensitivities to obtain relative modifications instead. Further, it should be noted that the size of the inverse system generally is larger than for GTTI, unless only one streamline-bundle is used for each production well.

Second, we consider SLEP propagation of the GTTI modifications. The time-shift $\Delta\widetilde{t}$ propagated in GTTI can be propagated by the same methodology as used in SLEP by specifying the modifications along a streamline-bundle ℓ by $\alpha_\ell = f(\Delta\widetilde{t}, \ldots)$. Here f is some function of the time-shift and possibly other variables. Inspired by (61), an example of a modification factor would be $\alpha_\ell = f(\Delta\widetilde{t}, \bar{t}^{\text{obs}}) = 1 + \Delta\widetilde{t}/\bar{t}^{\text{obs}}$. To obtain a time-shift for each streamline-bundle it is possible to partition the fractional-flow curve into several segments vertically, like done in SLEP, to obtain one time-shift for each segment (streamline-bundle).

Appendix G. Potential Strategies for Incorporating Well Pressures in GTTI

For streamline-effective properties, the effective pressure drop can be explicitly related to the permeability/porosity as shown in Appendix D above. Therefore, sensitivities are not

really needed in this case, but can of course be obtained by differentiation of the explicit expressions; i.e. to obtain $\partial \Delta p / \partial k_\ell$. By the chain rule we obtain

$$\frac{\partial \Delta p}{\partial K_i} = \sum_{\ell=1}^{N_{\text{sl}}^i} \frac{\partial \Delta p}{\partial k_\ell} \cdot \frac{\partial k_\ell}{\partial K_i},$$

which makes it possible to define GTTI also to match effective pressure drops. Here $\partial k_\ell / \partial K_i$ is given by for instance (63). Ideally, however, compressibility should also have been incorporated in the expressions for the relation between the permeability and the pressure drop, but it may be better with an approximate pressure constraint than no constraint at all.

Another potential approach for incorporating well pressure (or well rate) in GTTI is to use

$$\frac{\partial p}{\partial m_i} = \frac{\partial p}{\partial q} \cdot \frac{\partial q}{\partial t} \cdot \frac{\partial t}{\partial m_i} = \frac{\partial p}{\partial q} \cdot \frac{\partial q}{\partial m_i}.$$

Here $\partial p / \partial q$ can potentially be obtained by differentiating a well model (e.g., the Peacemann well model like in [59]). The term $(\partial q / \partial t)^{-1}$ is discussed in [7] and Section 4.3 and the arrival-time sensitivity $\partial t / \partial m_i$ is discussed in Section 4.3. Again, it may be better with an approximate pressure constraint than no constraint at all.

Appendix H. Nomenclature

Symbols:

- reservoir property: m
- absolute grid permeability: K
- streamline effective permeability: k
- porosity: ϕ
- pressure: p
- total mobility: λ_t
- viscosity: μ
- density: ρ
- time-of-flight: τ
- bi-streamfunctions: ψ, χ
- streamline: Ψ
- fractional flow function: f
- sensitivity matrix: $\mathbf{G}$
- number of grid cells: N
- number of streamlines: N_{sl}
- number of data: N_d
- number of wells: N_w
- constant: const

Indices:

- streamline: ℓ
- grid cell: i
- time step: j
- well: k
- master point: d
- total: t
- phase: α

Sub/super scripts:

- water: w
- oil: o
- gas: g
- calculated: cal
- observed: obs