Three-Phase Displacement Theory: Hyperbolic Models and Analytical Solutions

Ruben Juanes[†] and Knut–Andreas Lie[‡]

† Petroleum Engineering, School of Earth Sciences, Stanford University

‡ SINTEF ICT, Dept. Applied Mathematics

Contents

- Discussion of general conditions for relative permeabilities to ensure hyperbolicity
- Presentation of the analytical solution to the Riemann problem
- Implementation in a front-tracking method
- Streamline simulation results

Displacement Theory

Assumptions:

- One-dimensional flow
- Immiscible fluids
- Incompressible fluids
- Homogeneous rigid porous medium
- Multiphase flow extension of Darcy's law
- Gravity and capillarity are not considered
- Constant fluid viscosities

Displacement Theory cont'd

Mass conservation for each phase:

$$\partial_t(m_\alpha)+\partial_x(F_\alpha) = 0, \qquad \alpha = w, g, o$$

$$m_\alpha = \rho_\alpha S_\alpha \phi$$

$$F_\alpha = -\rho_\alpha k \lambda_\alpha \, \partial_x p$$

Saturations add up to one:

$$\sum_{\alpha=w,g,o} S_\alpha \equiv 1$$

Displacement Theory cont'd

If the fractional flow approach is used:

- "Pressure equation"

$$\partial_x(v_T) = 0$$

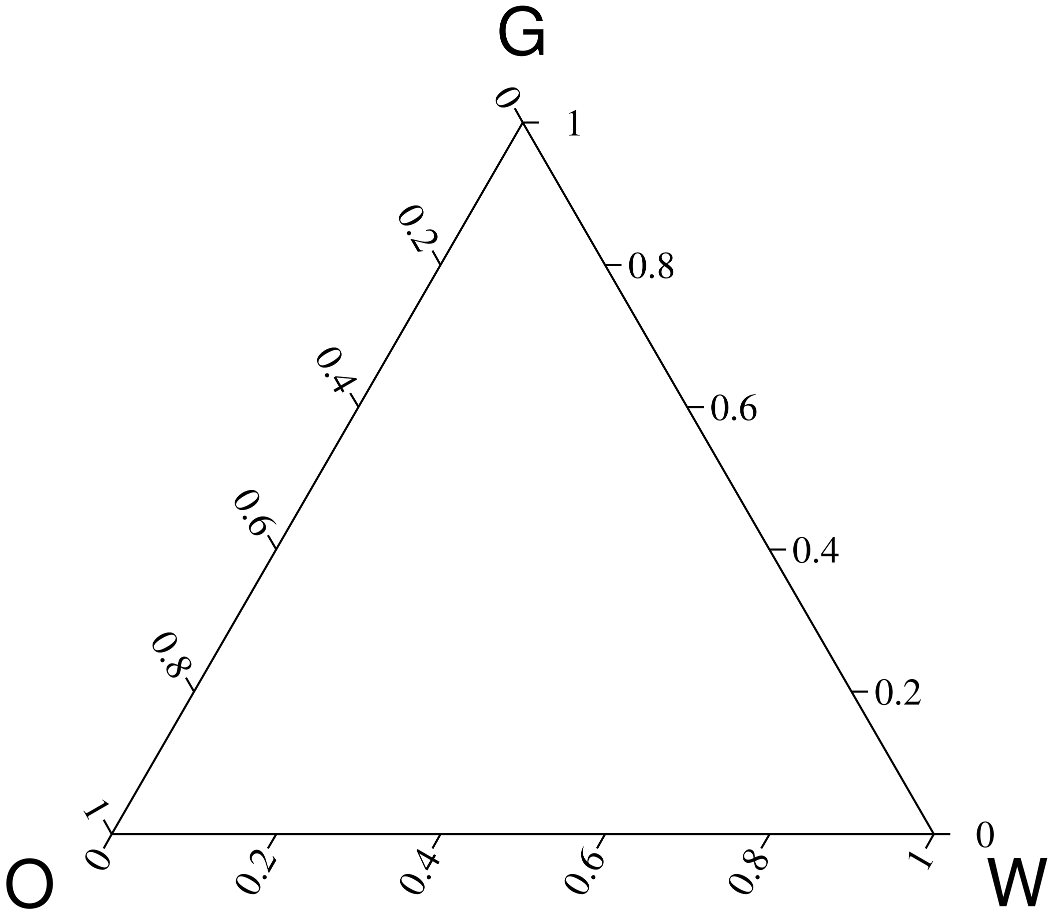
$$v_T = -\frac{1}{\phi} k \lambda_T \partial_x p$$

- System of "saturation equations"

$$\partial_t \begin{pmatrix} S_w \\ S_g \end{pmatrix} + v_T \partial_x \begin{pmatrix} f_w \\ f_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Displacement Theory cont'd

Saturation triangle:



Displacement Theory cont'd

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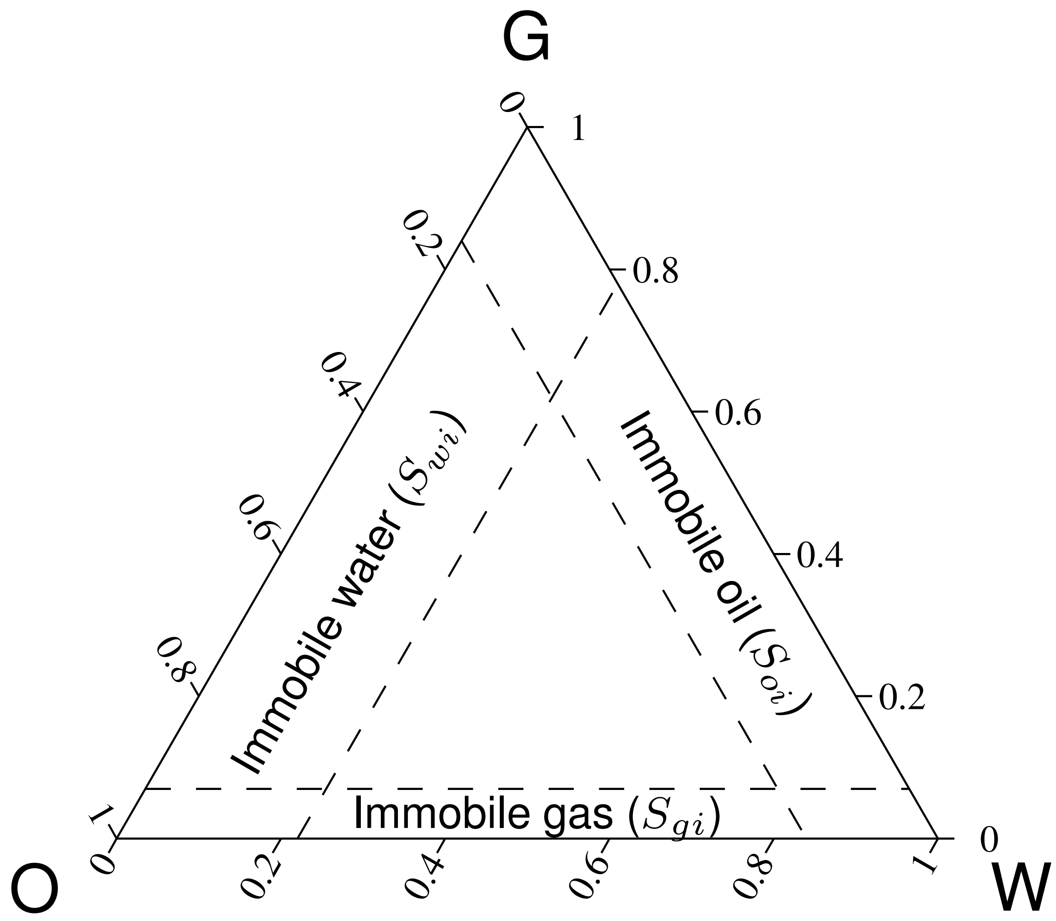

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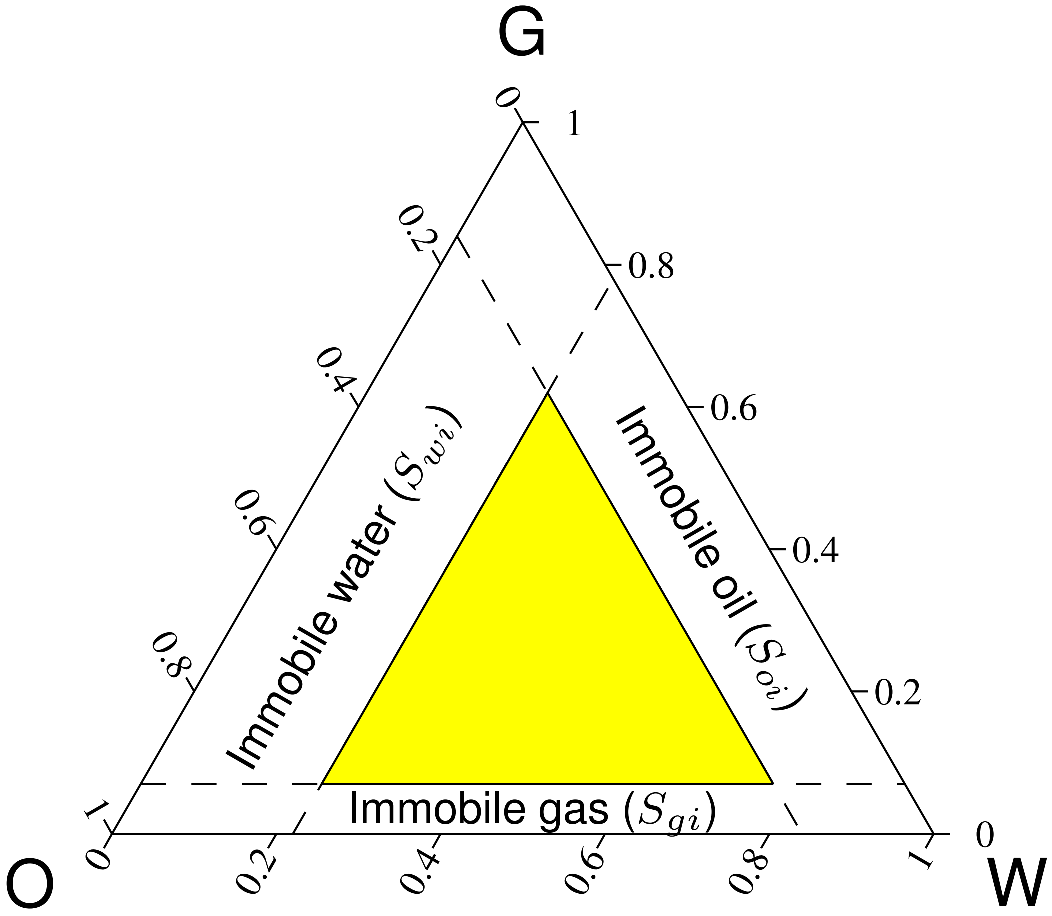

Reduced saturations:

$$\tilde{S}_\alpha := \frac{S_\alpha - S_{\alpha i}}{1 - \sum_{\beta=1}^{3} S_{\beta i}}$$

$$\downarrow$$

Renormalized triangle

Character of the System

The character of the system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + v_T \partial_x \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \partial_t \boldsymbol{u} + v_T \partial_x \boldsymbol{f} = \boldsymbol{0}$$

is determined by the eigenvalues $(\nu_1,\ \nu_2)$ and eigenvectors $(\boldsymbol{r}_1,\ \boldsymbol{r}_2)$ of the *Jacobian matrix:*

$$\boldsymbol{A}(\boldsymbol{u}) := \mathsf{D}_{\boldsymbol{u}} \boldsymbol{f} = \begin{pmatrix} f_{,u} & f_{,v} \\ g_{,u} & g_{,v} \end{pmatrix}$$

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Hyperbolic: The eigenvalues are *real* and the Jacobi matrix is *diagonalizable*. Strictly hyperbolic: distinct eigenvalues $\nu_1 < \nu_2$.

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Hyperbolic: The eigenvalues are *real* and the Jacobi matrix is *diagonalizable*. Strictly hyperbolic: distinct eigenvalues $\nu_1 < \nu_2$.

Elliptic: The eigenvalues are *complex*.

Character of the System cont'd

It was long believed that the system was strictly hyperbolic for *any* set of relative permeability functions

Character of the System cont'd

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However: existence of elliptic regions proved 1985–95

- Bell et al., Fayers, Guzmán and Fayers, Hicks and Grader, ..
- Shearer and Trangenstein, Holden et al, ...

Character of the System cont'd

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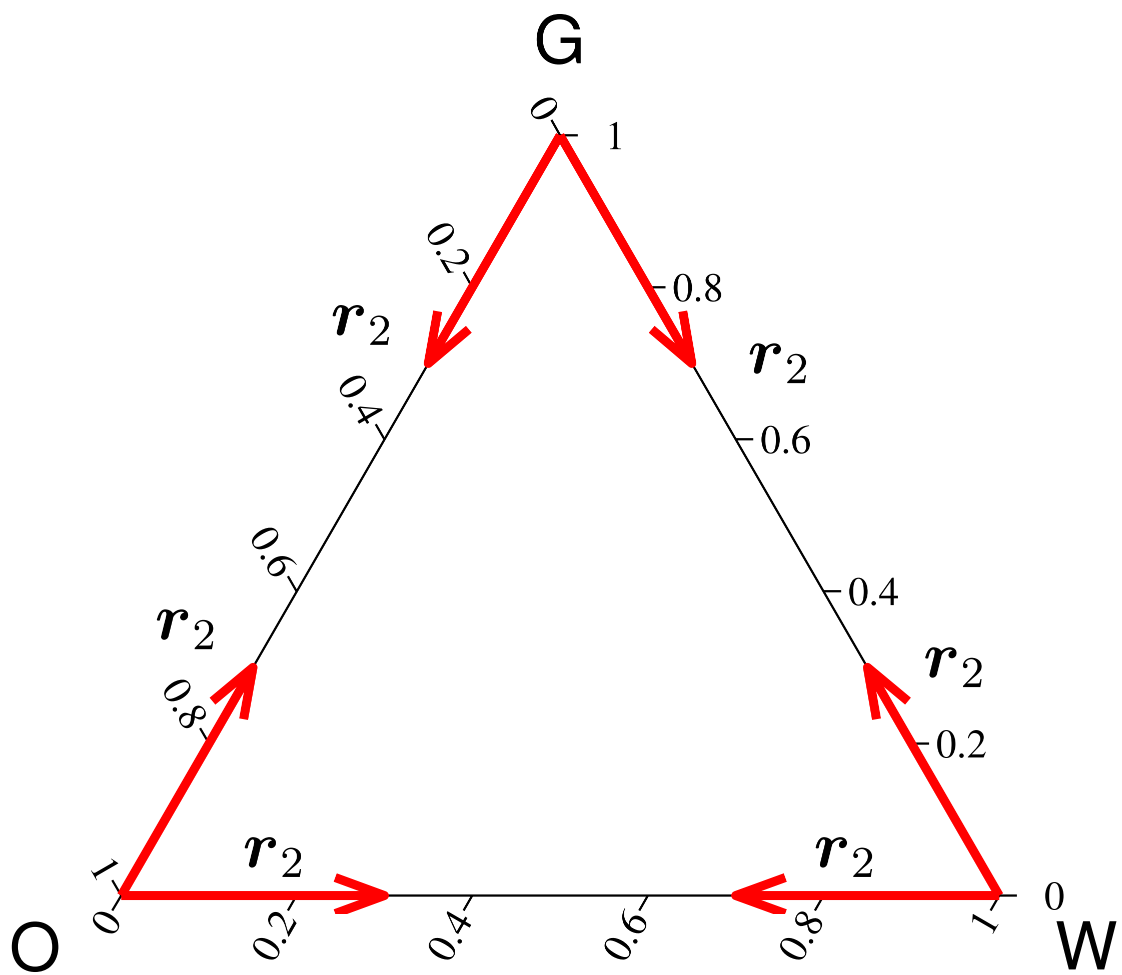
Approach in the existing literature:

- **Assume** "reasonable" conditions for relative permabilities on the edges
  - "Zero-derivative" conditions
  - "Interaction" conditions
- **Infer** loss of strict hyperbolicity inside the saturation triangle

Character of the System cont'd

Traditional assumed behavior of fast eigenvectors ($\boldsymbol{r}_2$)



Character of the System cont'd

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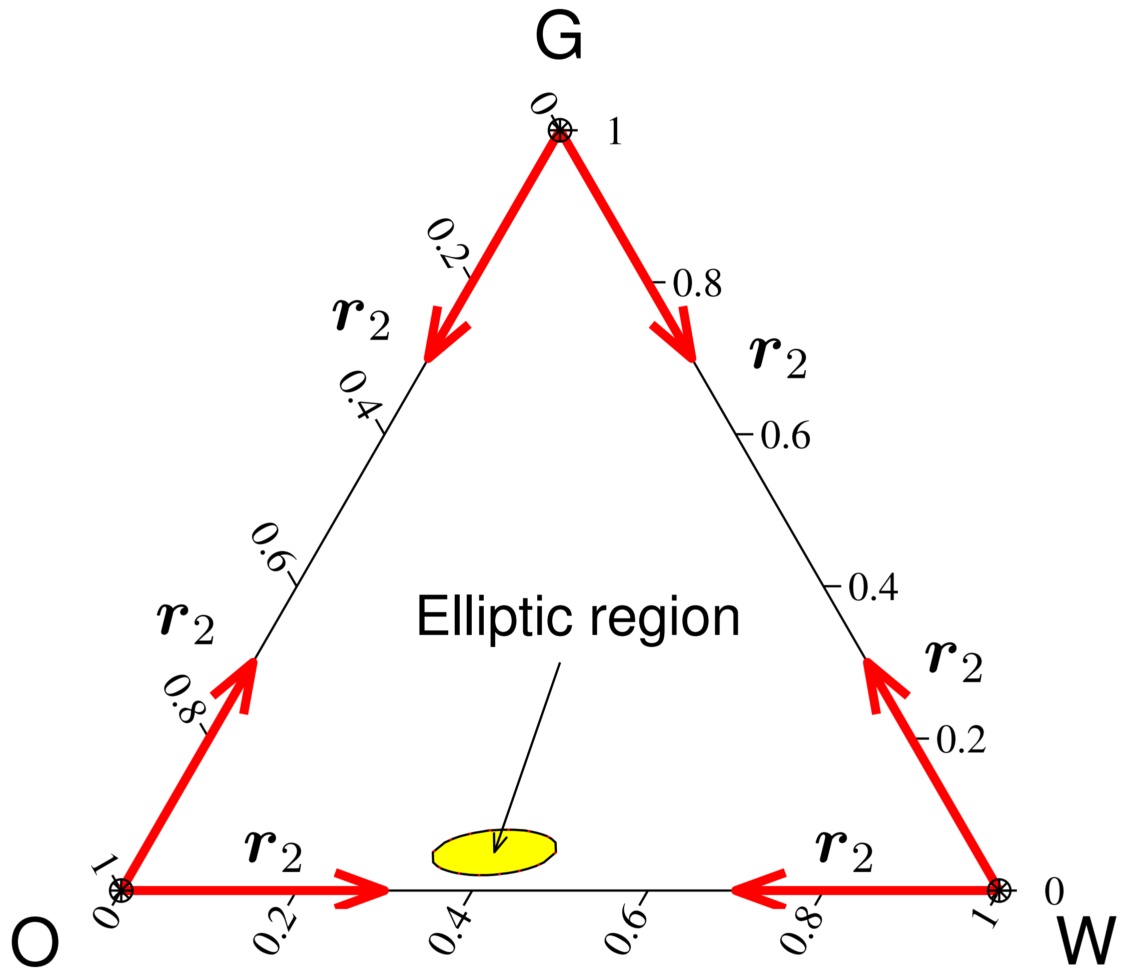

Character of the System cont'd

Consequences of ellipticity of the system:

- Flow depends on *future* boundary conditions
- The solution is *unstable*: arbitrarily close initial and injected saturations yield nonphysical oscillatory waves

Character of the System cont’d

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However:

- The elliptic region can be shrunk to an **umbilic point** only if interaction between phases is ignored:

$$k_{r\alpha} = k_{r\alpha}(S_\alpha), \quad \alpha = 1, \dots, 3$$

- This model is *not* supported by experiments and pore-scale physics
- Umbilic points still act as “repellers” for classical waves

Relative Permeabilities

Juanes and Patzek – New approach:

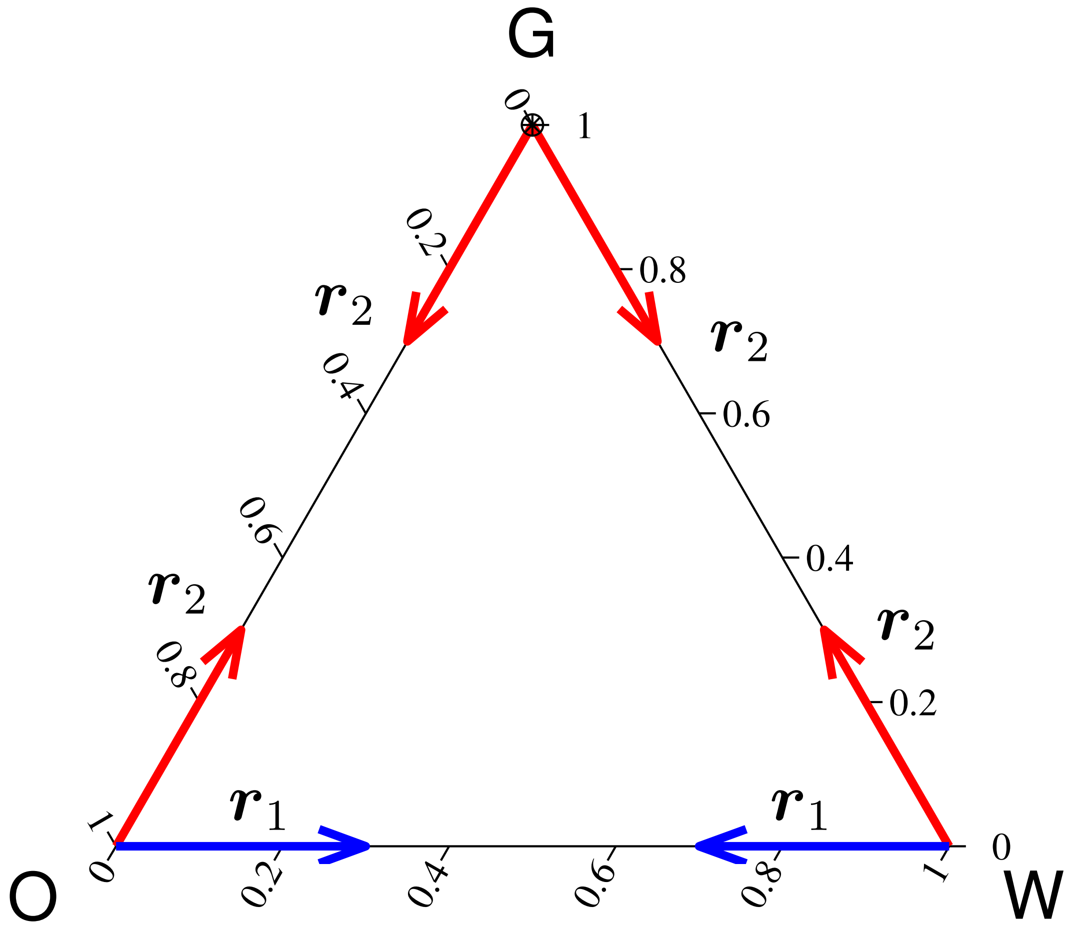
- **Assume** the system is strictly hyperbolic
- **Infer** conditions on relative permeabilities

Key observation:

- Whenever gas is present as a continuous phase, its mobility is much higher than that of the other two fluids
- Fast paths $\longleftrightarrow$ changes in gas saturation

Relative Permeabilities

Proposed behavior of eigenvectors ($\boldsymbol{r}_1$, $\boldsymbol{r}_2$)



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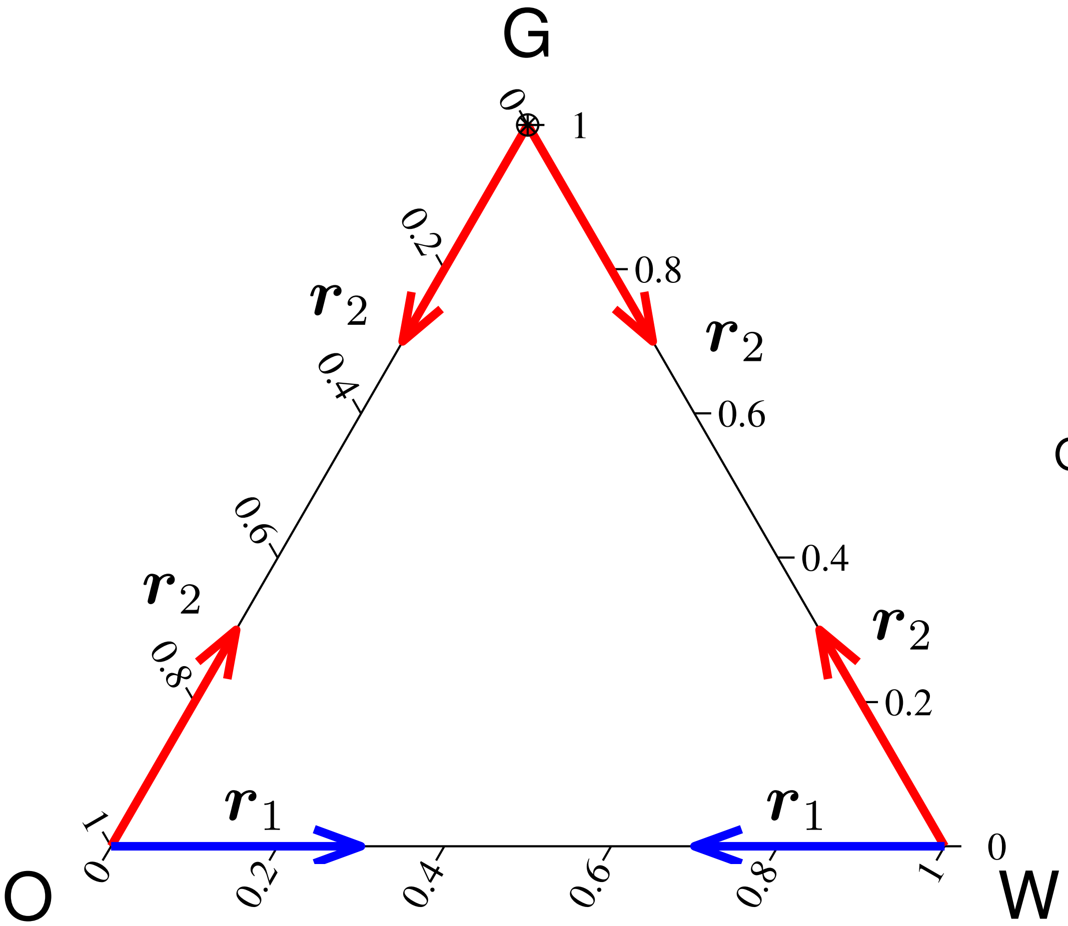
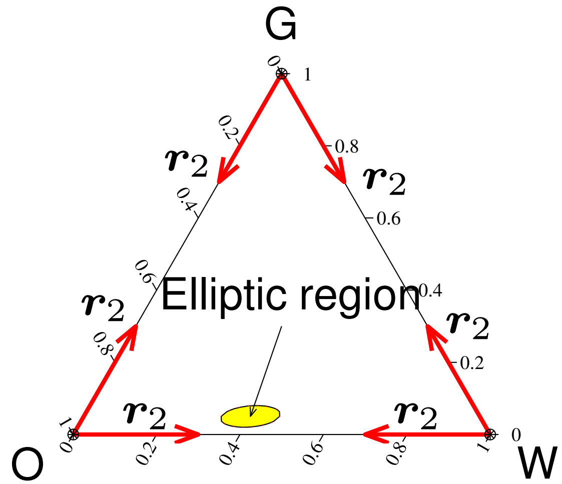




Relative Permeabilities cont'd

Two types of conditions:

- **Condition I.** Eigenvectors are parallel to each edge
- **Condition II.** Strict hyperbolicity along each edge

In particular, on the **OW edge:**

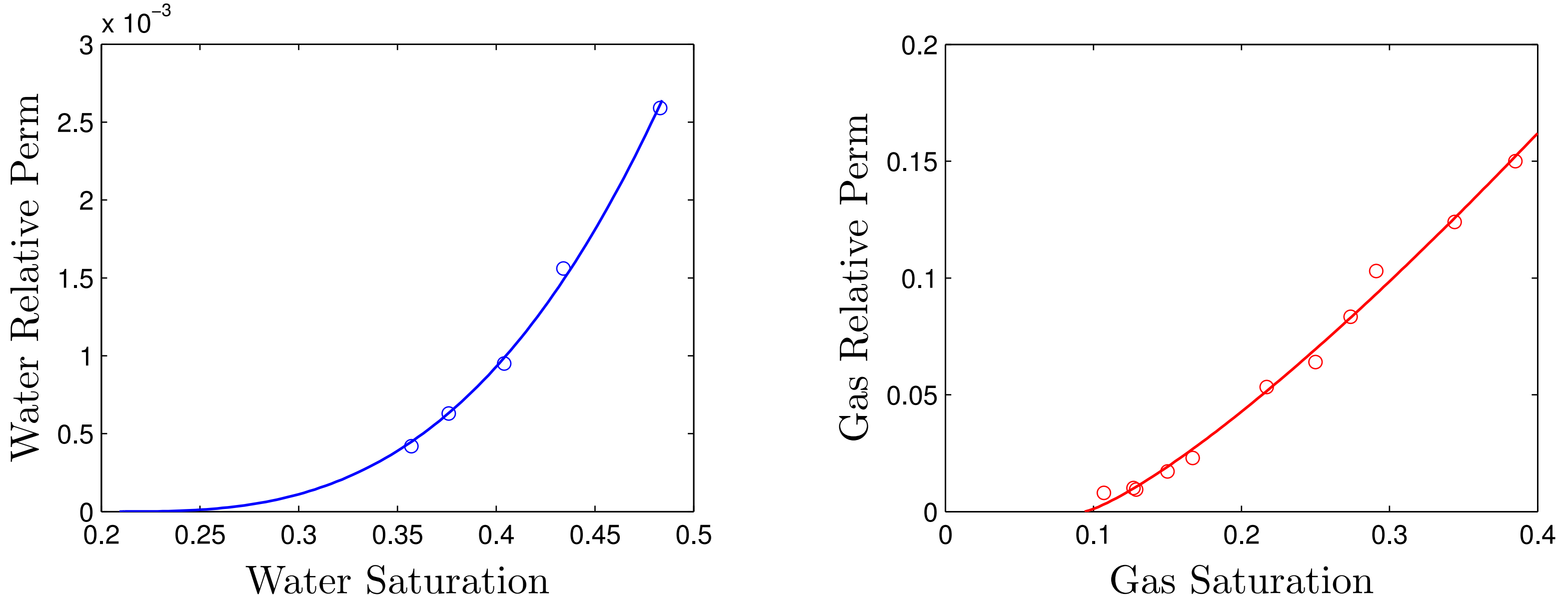
Condition	Frac. flows		Mobilities
I	$g_{,u} = 0$	$\Leftrightarrow$	$\lambda_{g,u} = 0$
II	$g_{,v} - f_{,u} > 0$	$\Leftrightarrow$	$\lambda_{g,v} > \lambda_{w,u} - \lambda_{T,u}\frac{\lambda_w}{\lambda_T}$

Condition II requires that the gas relative permeability has a *positive derivative* at its endpoint saturation.

Relative Permeabilities cont'd

Remarks:

- Necessary condition for strict hyperbolicity
- Can be justified from pore-scale physics (bulk flow vs. corner flow)
- Supported by experimental data (Oak's steady-state)



Relative Permeabilities cont'd

A simple model:

$$
\begin{aligned}
k_{rw}(u) &= u^2 \\
k_{rg}(v) &= \left(\beta_g v + (1-\beta_g)v^2\right), \quad \beta_g > 0 \\
k_{ro}(u,v) &= (1-u-v)(1-u)(1-v)
\end{aligned}
$$

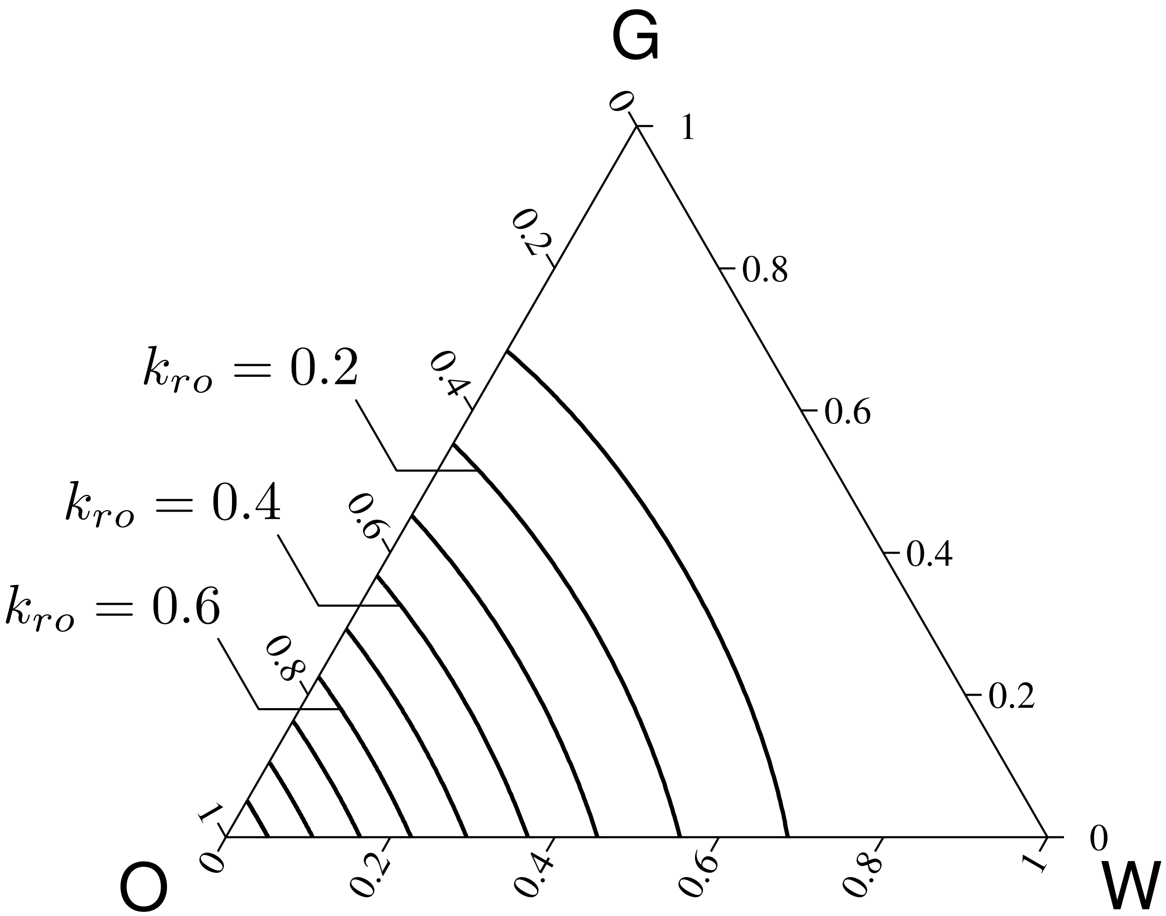
with reasonable values of viscosities:

$$\mu_w = 1, \quad \mu_g = 0.03, \quad \mu_o = 2 \text{ cp}$$

and a small value of the endpoint slope: $\beta_g = 0.1$

Relative Permeabilities cont'd

Oil isoperms:



Analytical Solution

Riemann problem: Find a weak solution to the 2×2 system

$$\partial_t \boldsymbol{u} + v_T \partial_x \boldsymbol{f} = \boldsymbol{0}, \quad -\infty < x < \infty,\ t > 0$$

$$\boldsymbol{u}(x,0) = \begin{cases} \boldsymbol{u}_l & \text{if } x < 0 \\ \boldsymbol{u}_r & \text{if } x > 0 \end{cases}$$

Previous work:

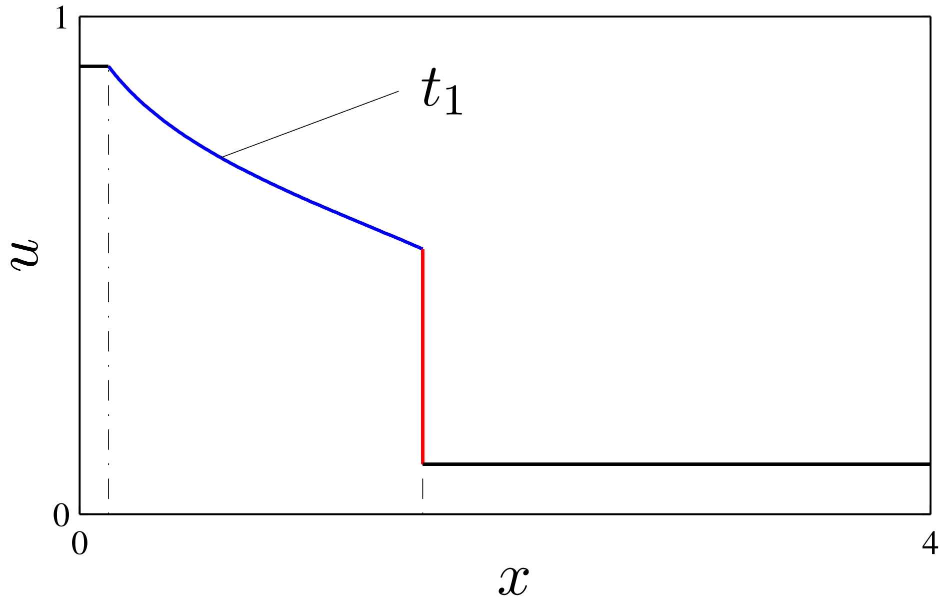
- Sequence of two successive two-phase displacements (Kyte et al., Pope, ..)
- Triangular systems (Gimse et al., ..)

New results by Juanes and Patzek:

- A *complete classification* all wave types
- Solution of Riemann problem (structure of waves)

Analytical Solution cont'd

Self-similarity ("stretching", "coherence"):



Analytical Solution cont'd

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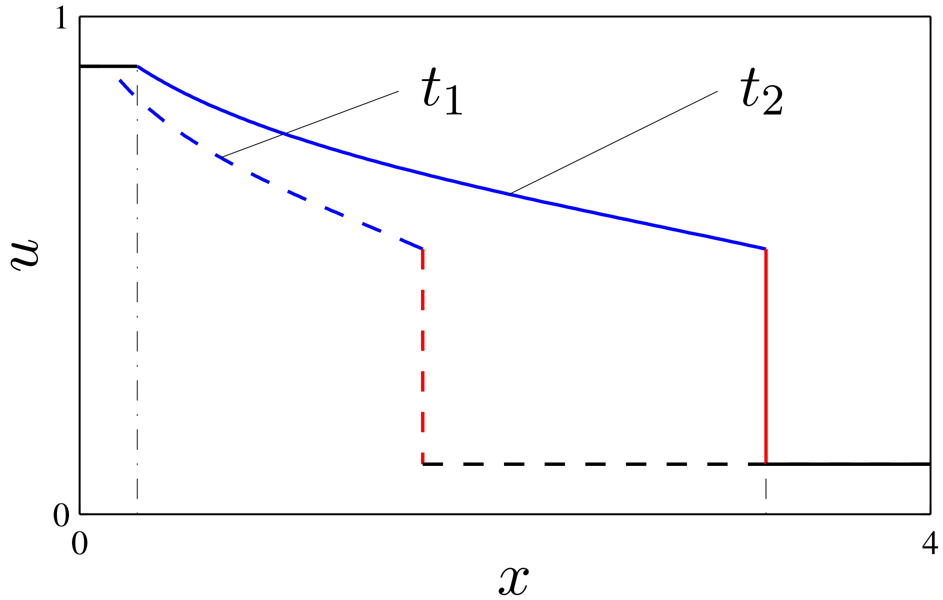

Analytical Solution cont'd

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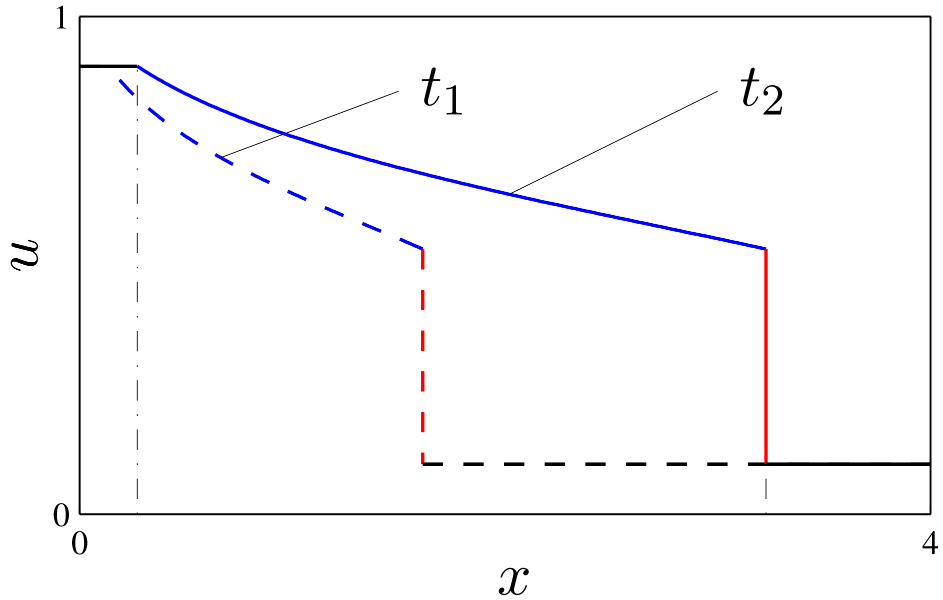


$$\boldsymbol{u}(x,t) = \boldsymbol{U}(\zeta), \quad \zeta := \frac{x}{\int_0^t v_T(\tau)\,d\tau}$$

Analytical Solution cont'd

Using self-similarity, the Riemann problem is a boundary value problem:

$$(\boldsymbol{A}(\boldsymbol{U}) - \zeta \boldsymbol{I})\boldsymbol{U}' = \boldsymbol{0}, \quad -\infty < \zeta < \infty$$

with boundary conditions

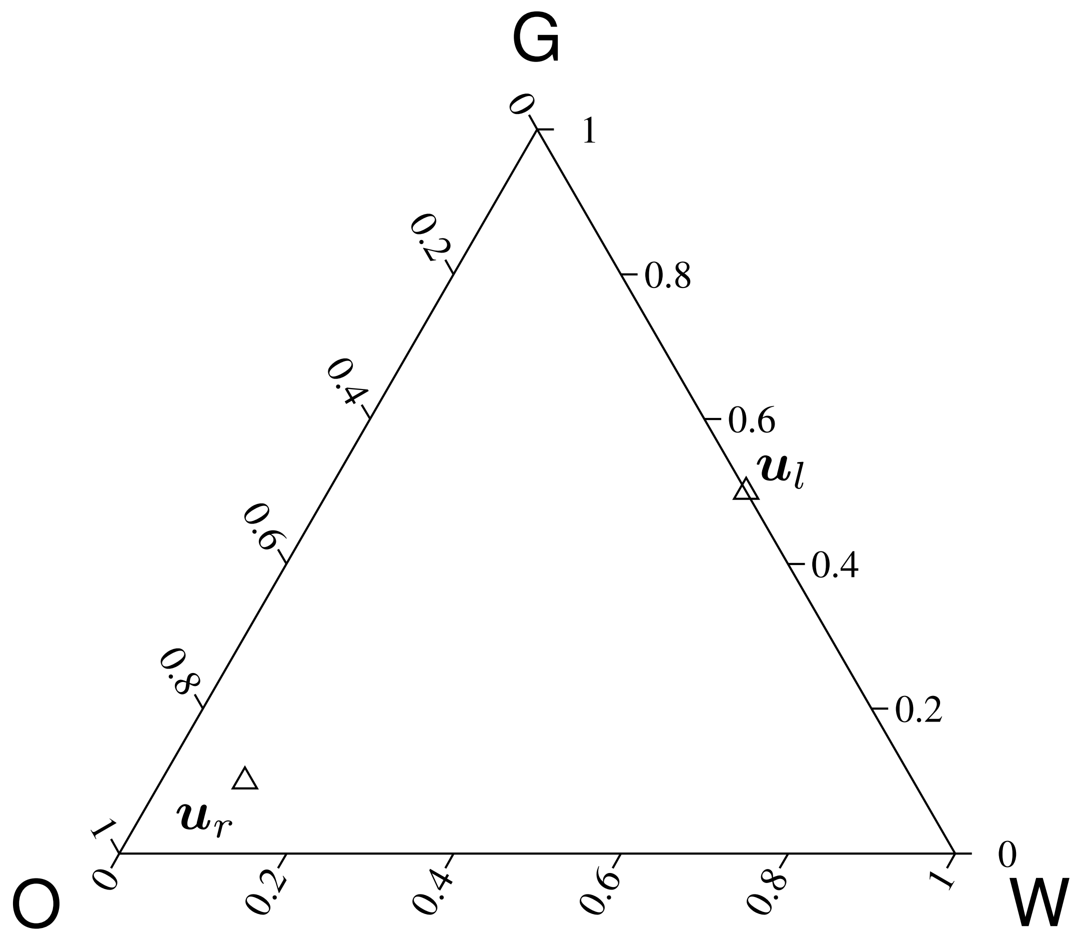
$$\boldsymbol{U}(-\infty) = \boldsymbol{u}_l, \quad \boldsymbol{U}(\infty) = \boldsymbol{u}_r$$

Strict hyperbolicity $\longrightarrow$ wave separation:

$$\boldsymbol{u}_l \xrightarrow{\mathcal{W}_1} \boldsymbol{u}_m \xrightarrow{\mathcal{W}_2} \boldsymbol{u}_r$$

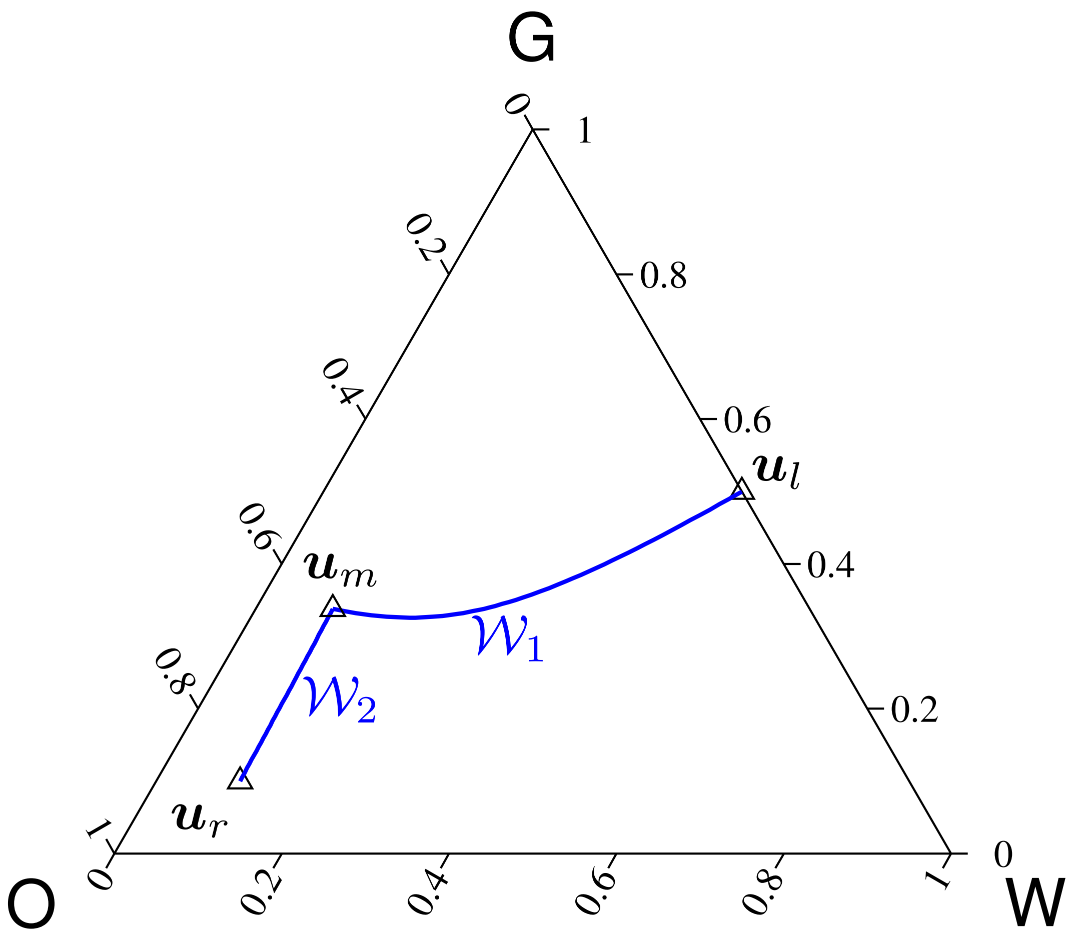
Analytical Solution cont'd

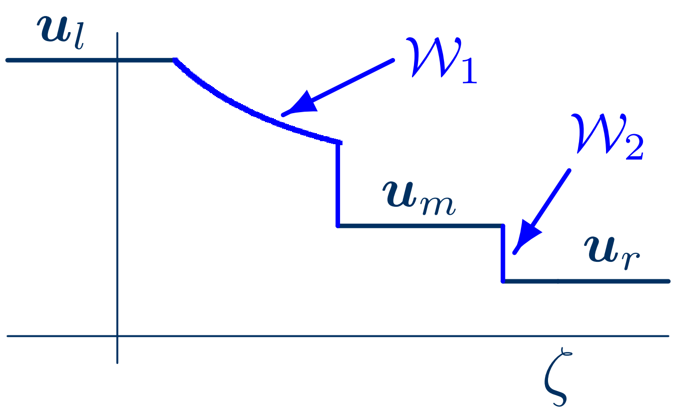
Schematic of Riemann solution



Analytical Solution cont'd

Schematic of Riemann solution





Wave Structure: Rarefactions

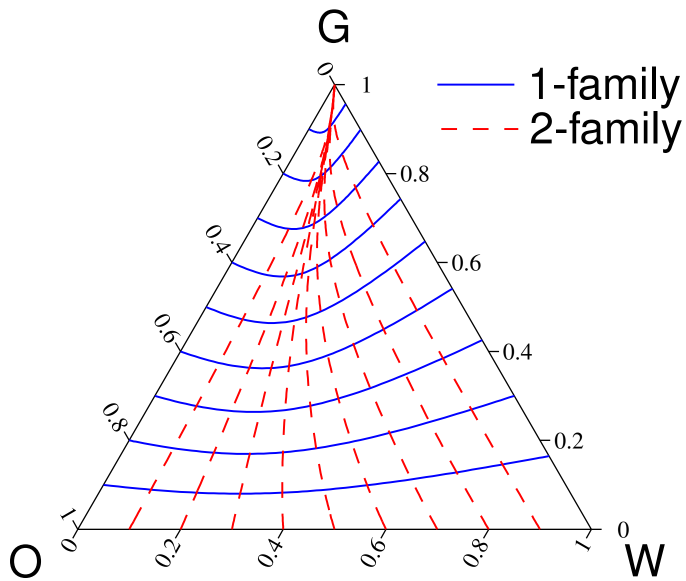
If the solution is smooth, it satisfies:

$$(\boldsymbol{A}(\boldsymbol{U}) - \zeta \boldsymbol{I})\boldsymbol{U}' = \boldsymbol{0}$$

$\nearrow$ $\nwarrow$

eigenvalue (ν_p) eigenvector ($\boldsymbol{r}_p$)

A smooth solution (**rarefaction**) must lie on a curve whose tangent is in the direction of the eigenvector (**integral curve**)



Wave Structure: Rarefactions cont'd

Admissibility of a rarefaction wave

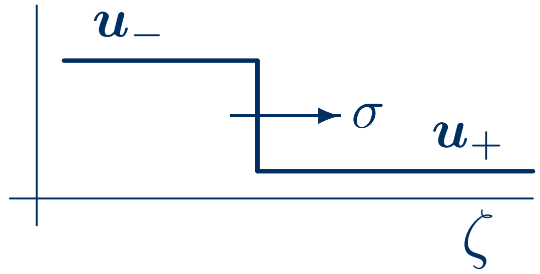
- To avoid a multiple-valued solution, ν_p must **increase monotonically** along the curve

$$\boldsymbol{u}_l \xrightarrow{\mathcal{R}_p} \boldsymbol{u}_r$$

- Thus, rarefaction curves $\mathcal{R}_p$ are **subsets** of integral curves $\mathcal{I}_p$

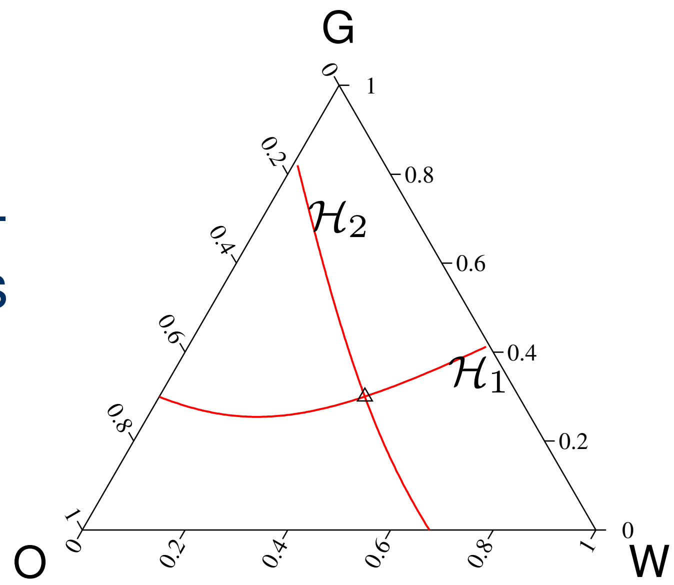
Wave Structure: Shocks

If the solution is discontinuous, it must satisfy the **Rankine-Hugoniot jump condition**:



$$\boldsymbol{f}(\boldsymbol{u}_+) - \boldsymbol{f}(\boldsymbol{u}_-) = \sigma \cdot \left[\boldsymbol{u}_+ - \boldsymbol{u}_-\right]$$

The set of states which can be connected satisfying the jump condition is called the **Hugoniot locus**



Wave Structure: Shocks cont'd

Admissibility of a shock wave

- Not every discontinuity satisfying the jump condition is a valid shock
- Characteristics of the p-family must go **into** the shock (**Lax entropy condition**):

$$\nu_p(\boldsymbol{u}_-) > \sigma_p > \nu_p(\boldsymbol{u}_+),$$

- Thus, shock curves $\mathcal{S}_p$ are **subsets** of Hugoniot loci $\mathcal{H}_p$

Wave Structure: Rarefaction-Shocks

Genuine nonlinearity: eigenvalues vary **monotonically** along integral curves

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This is **not** the case in multiphase flow, where each wave may involve rarefactions **and** shocks

Wave Structure: Rarefaction-Shocks

Genuine nonlinearity: eigenvalues vary **monotonically** along integral curves

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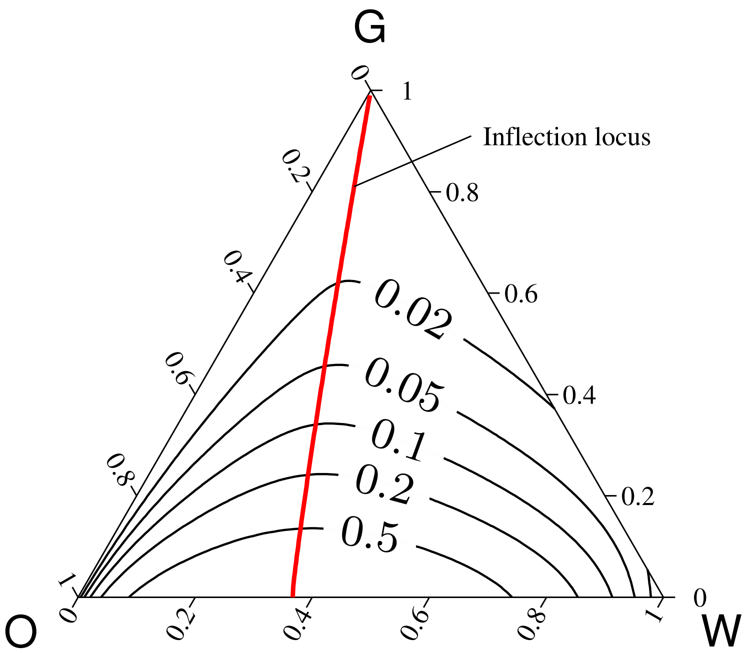
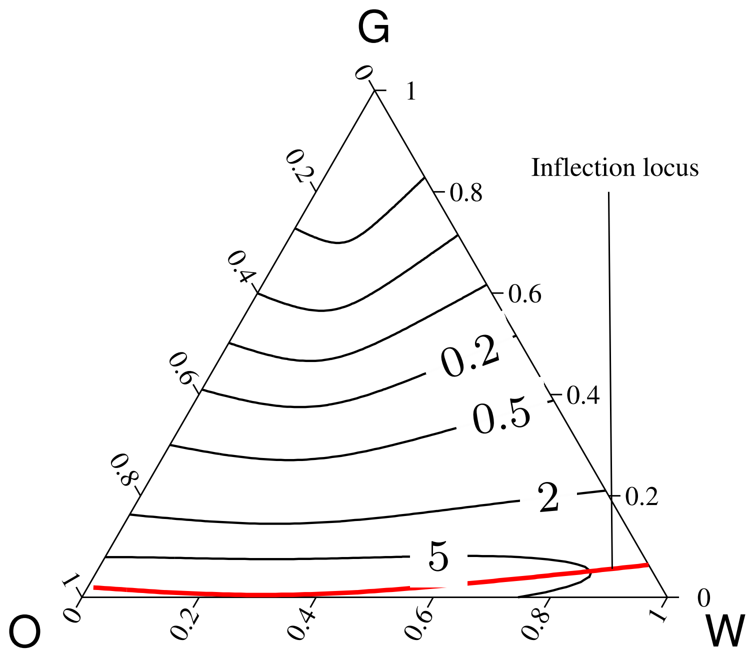
Inflection locus: set of points at which eigenvalues attain a local maximum when moving along integral curves

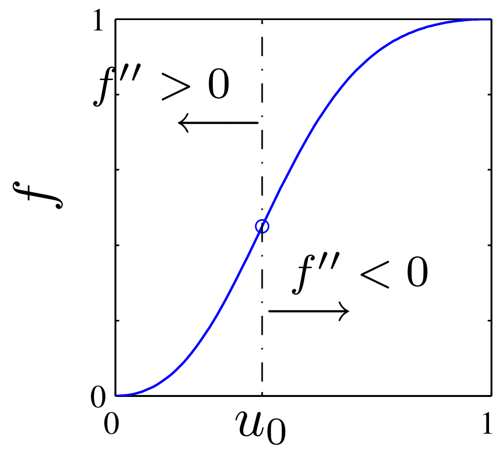




Wave Structure: Rarefaction-Shocks cont'd

Properties of the inflection loci:

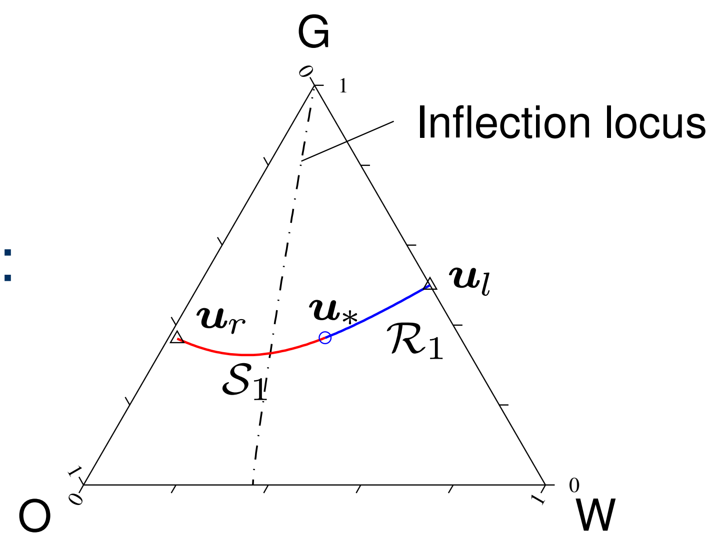
- **Single** curves, transversal to integral curves
- Correspond to **maxima** of eigenvalues



Consequences:

- **At most** one rarefaction and one shock
- Rarefaction is **always slower** than shock:

$$\boldsymbol{u}_l \xrightarrow{\mathcal{R}_p} \boldsymbol{u}_* \xrightarrow{\mathcal{S}_p} \boldsymbol{u}_r$$



Wave Structure: Rarefaction-Shocks cont'd

Admissibility of a rarefaction-shock wave

- Eigenvalue ν_p must increase monotonically along the curve

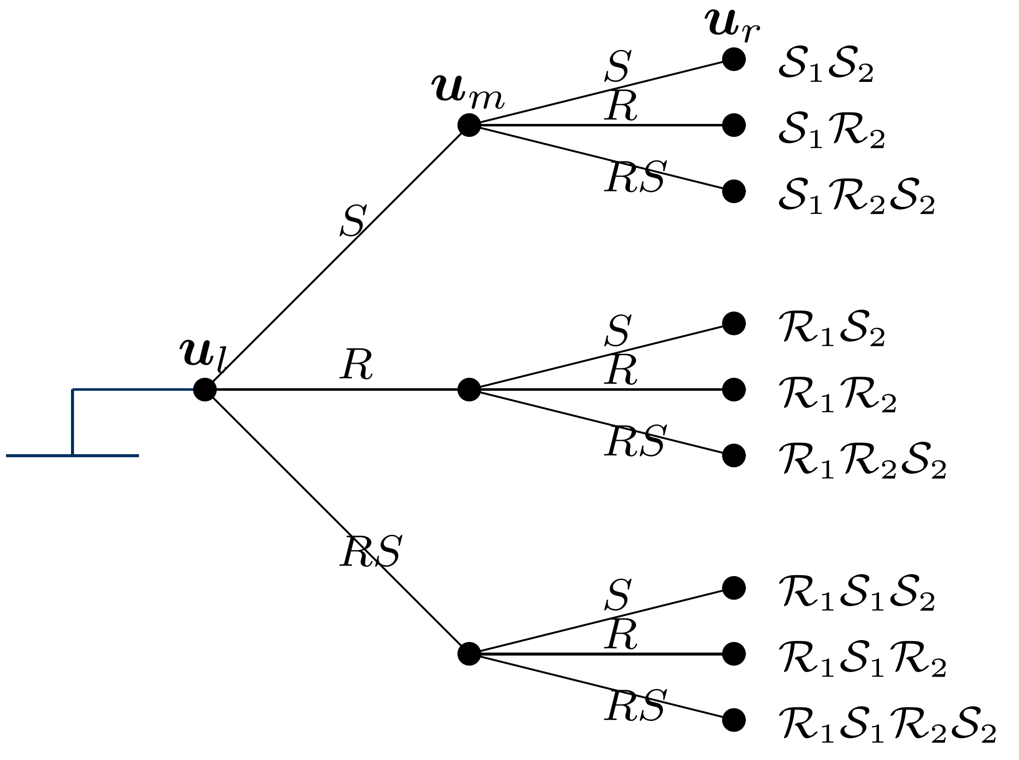
$$\boldsymbol{u}_l \xrightarrow{\mathcal{R}_p} \boldsymbol{u}_*$$

- The shock must satisfy the **extended-Lax entropy condition**:

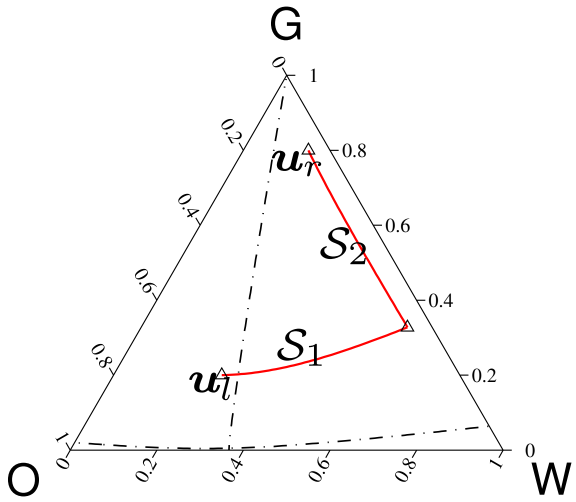
$$\nu_p(\boldsymbol{u}_*) = \sigma_p > \nu_p(\boldsymbol{u}_r)$$

Wave Structure cont'd

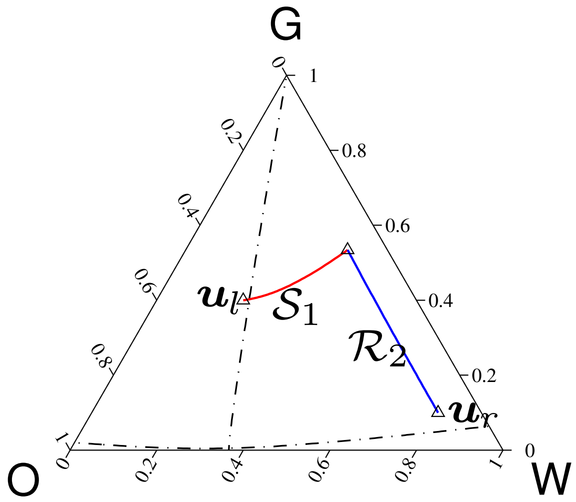
Complete set of solutions: 9 different wave configurations



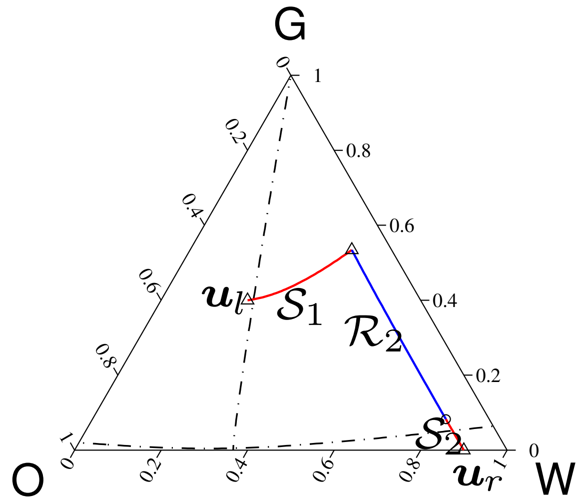
Complete Set of Solutions



(a) $\mathcal{S}_1\mathcal{S}_2$

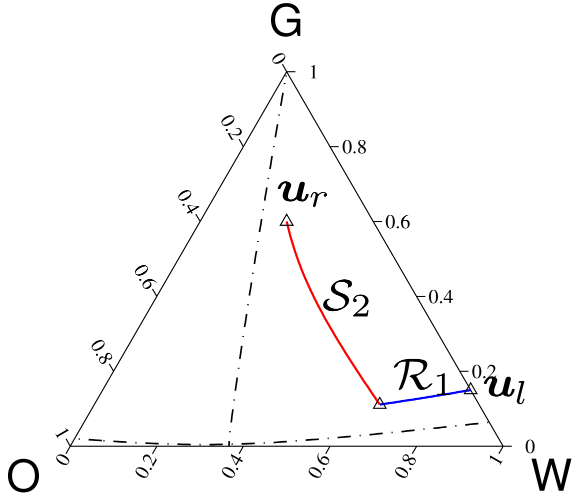


(b) $\mathcal{S}_1\mathcal{R}_2$

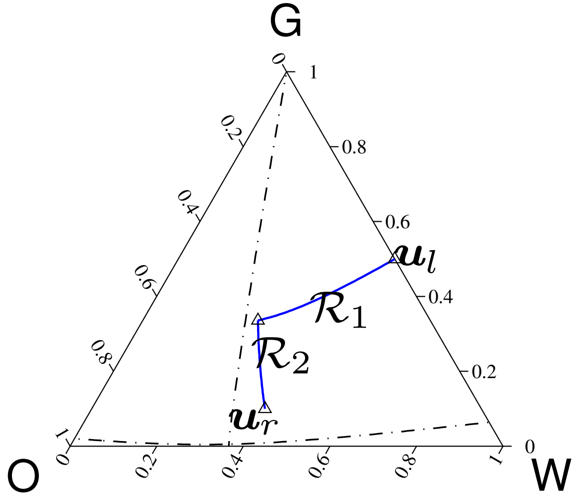


(c) $\mathcal{S}_1\mathcal{R}_2\mathcal{S}_2$

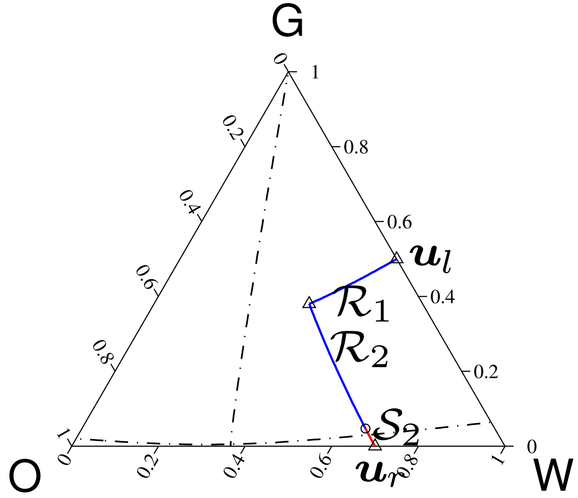
Complete Set of Solutions cont'd



(d) $\mathcal{R}_1\mathcal{S}_2$

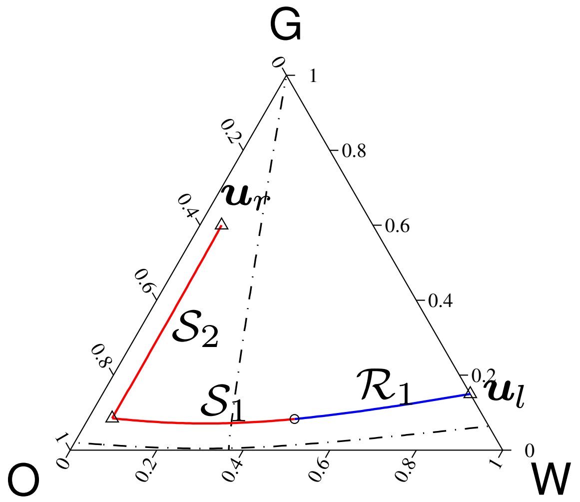


(e) $\mathcal{R}_1\mathcal{R}_2$

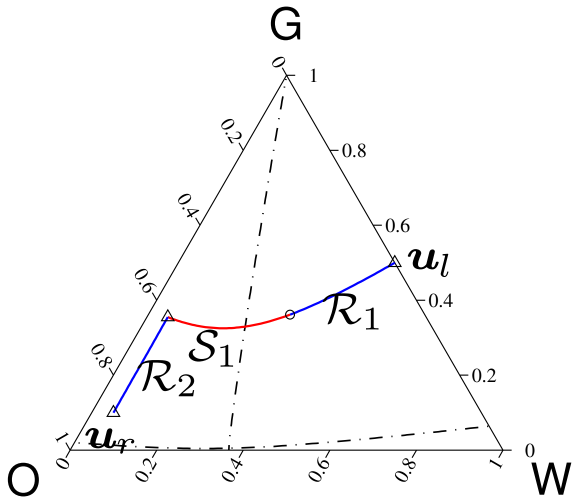


(f) $\mathcal{R}_1\mathcal{R}_2\mathcal{S}_2$

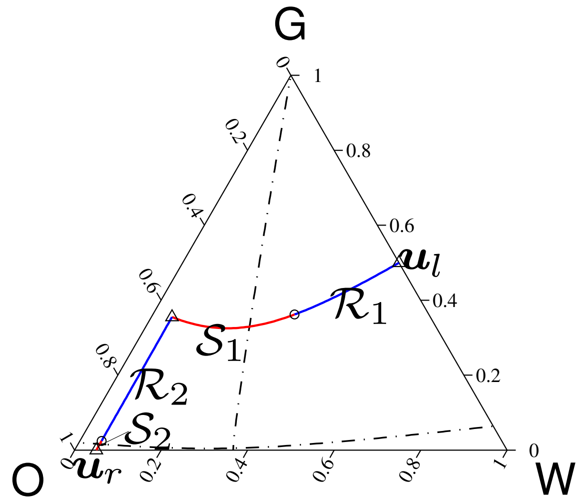
Complete Set of Solutions cont'd



(g) $\mathcal{R}_1\mathcal{S}_1\mathcal{S}_2$



(h) $\mathcal{R}_1\mathcal{S}_1\mathcal{R}_2$



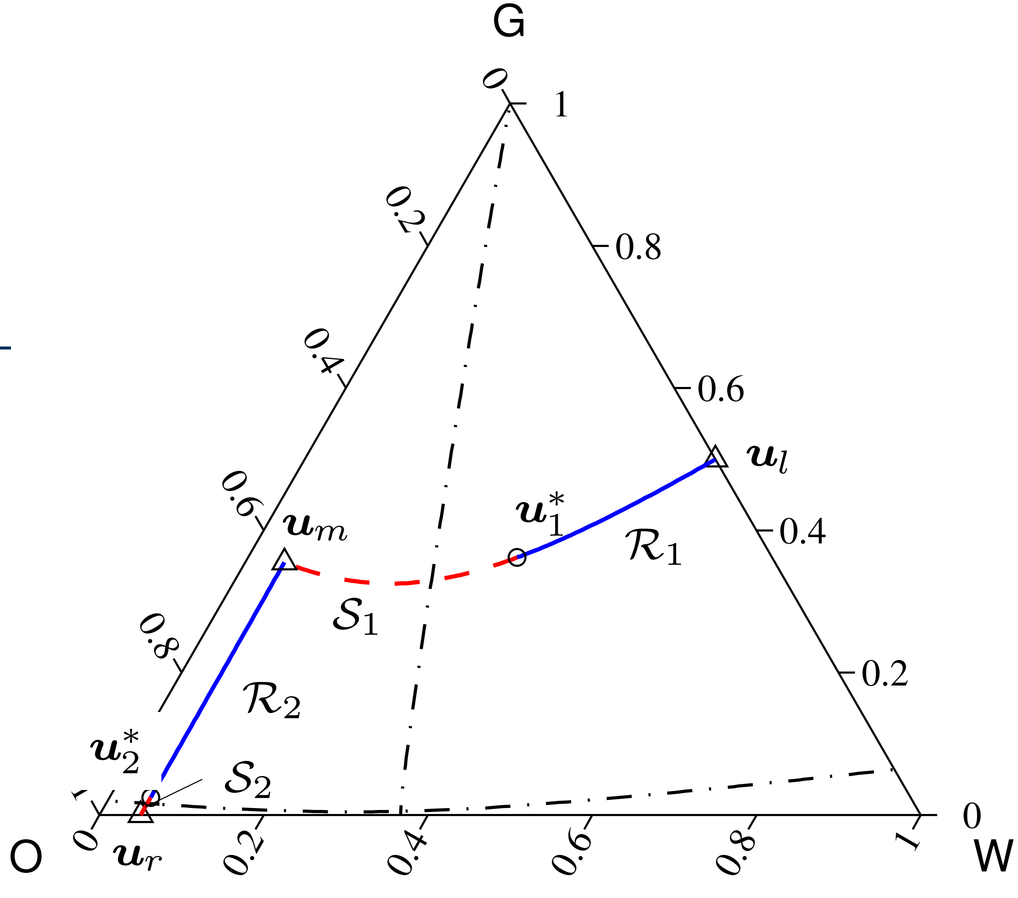
(i) $\mathcal{R}_1\mathcal{S}_1\mathcal{R}_2\mathcal{S}_2$

Example 1

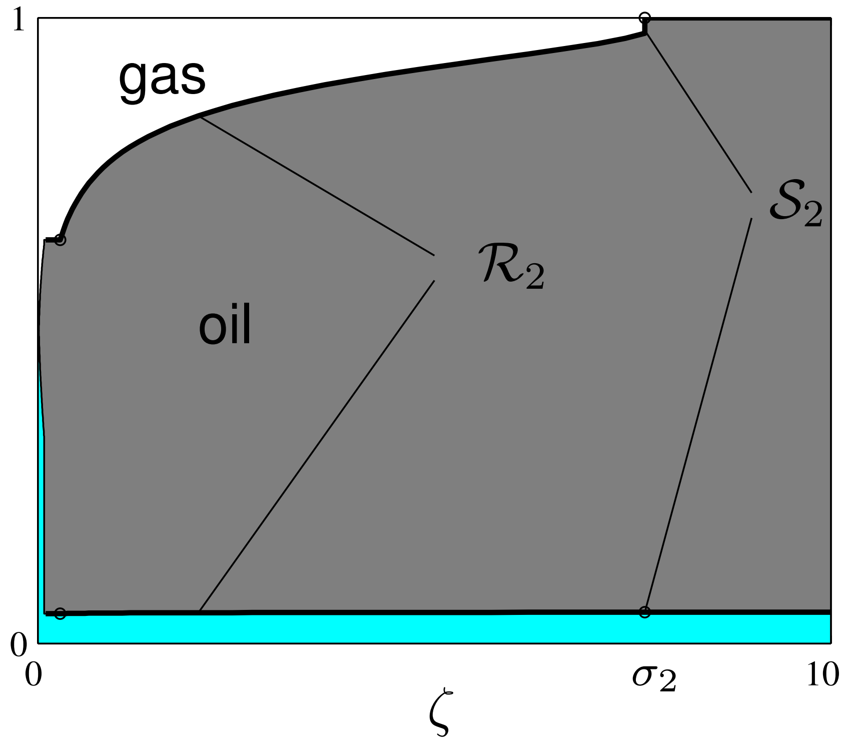
Injection of water and gas into an oil-filled core (with some mobile water)

Problem of great practical interest

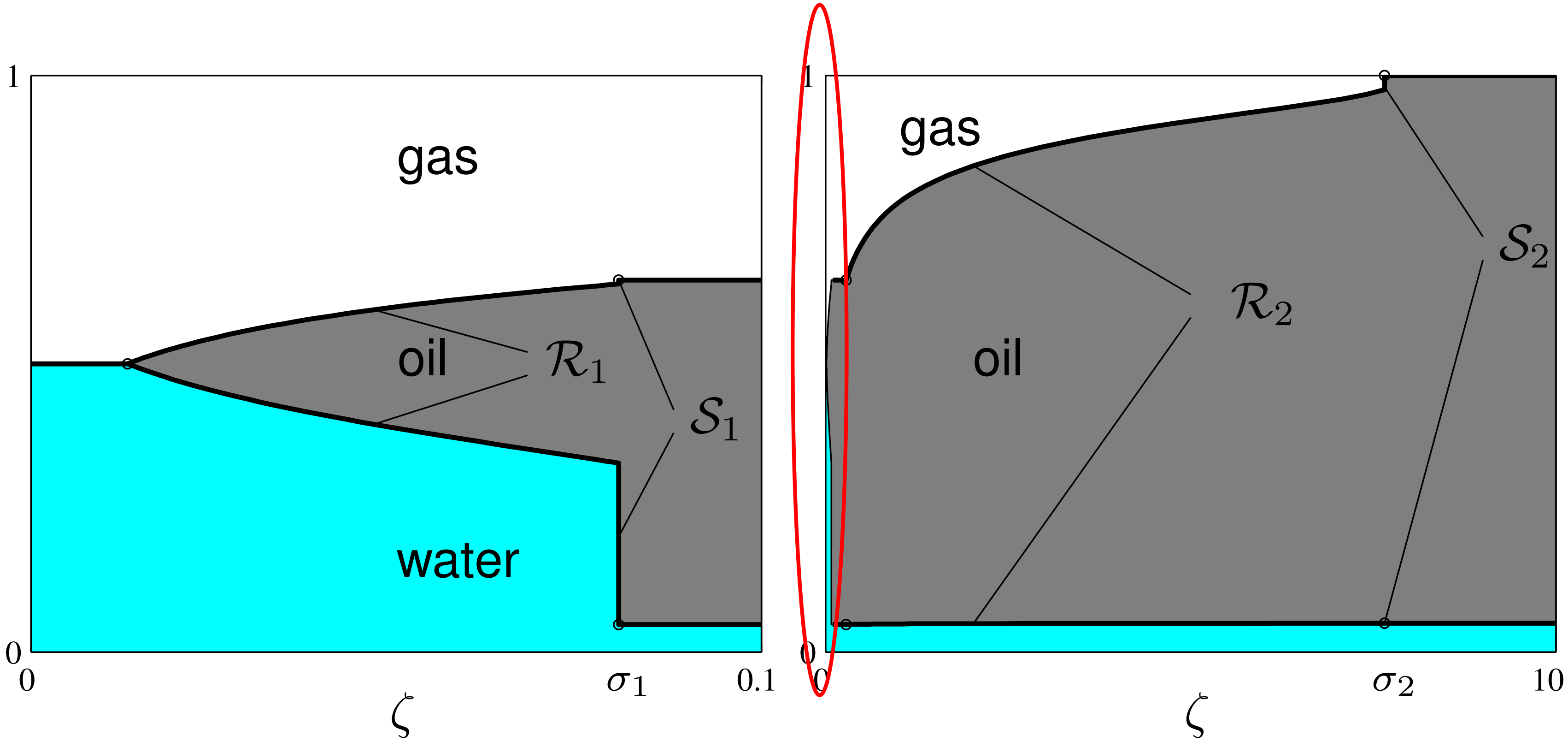
Injected saturation	Initial saturation
$S_g = 0.5$	$S_g = 0$
$S_o = 0$	$S_o = 0.95$
$S_w = 0.5$	$S_w = 0.05$



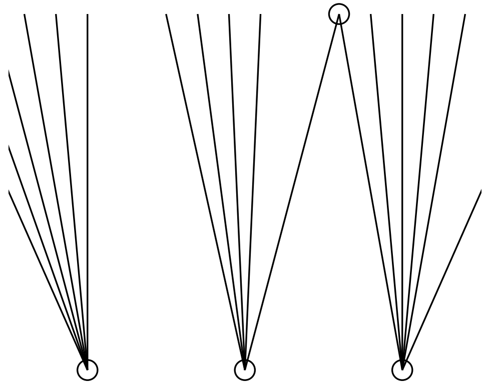
Example 1 cont'd

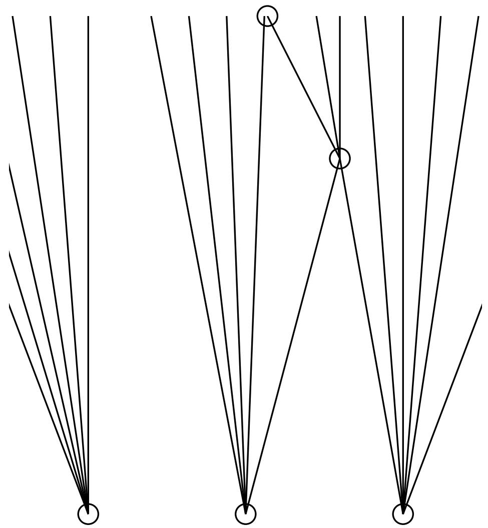


Example 1 cont'd



The Cauchy Problem: Front Tracking





Start ($t = 0$): piecewise constant inital data

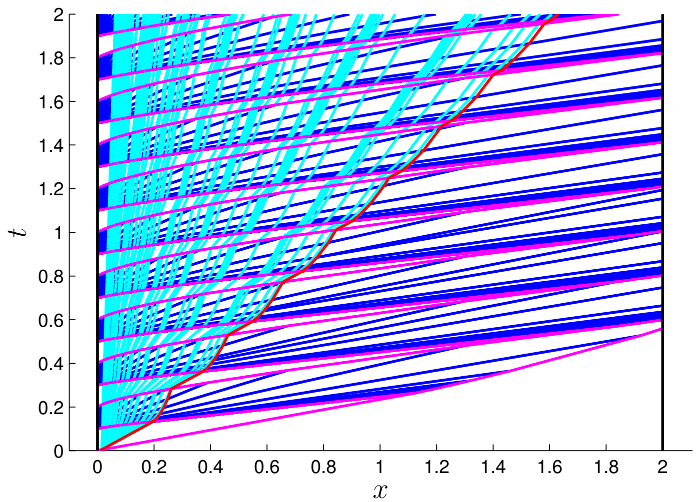
$\longrightarrow$ sequence of local Riemann problems

$\longrightarrow$ p.w discontinuities between (x,t)-rays

While $t < t_{end}$:

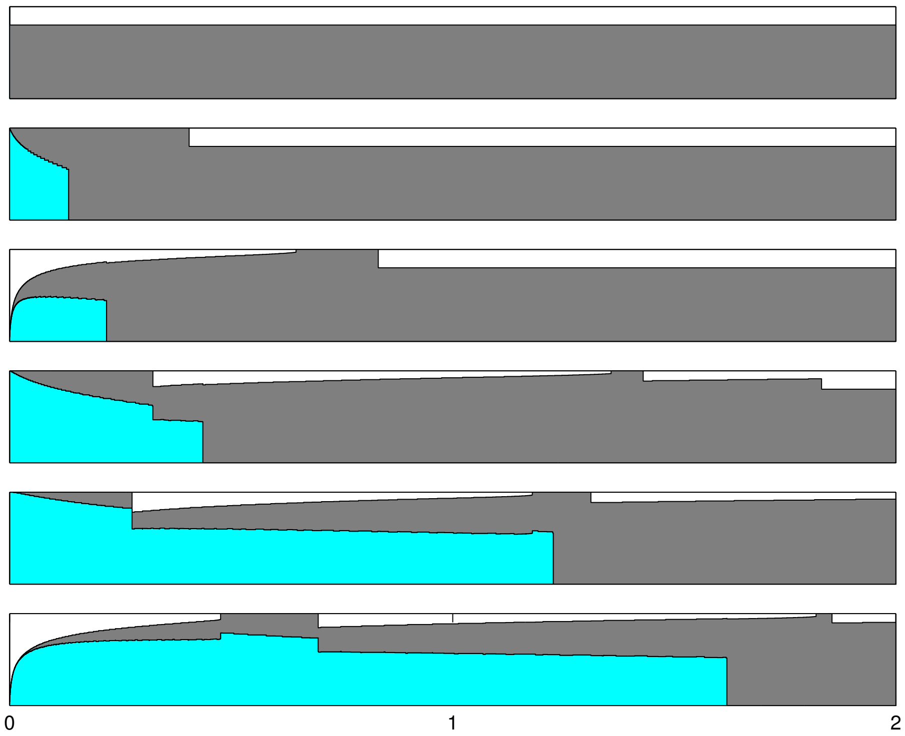
track discontinuities

solve Riemann problems



Example 2

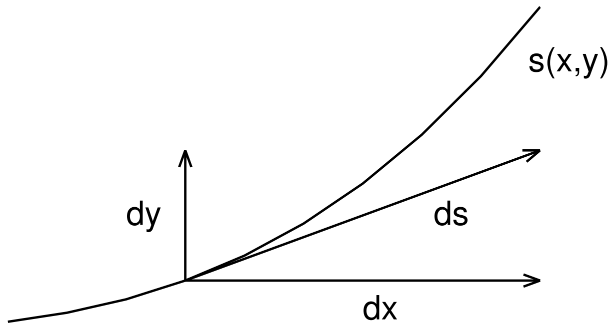
- Initially, reservoir filled with 80% oil and 20% gas
- Alternate cycles of water and gas injection
- Front-tracking solution (with $\Delta_u = 0.005$
- Half a million Riemann solves $\sim$ 5 sec on a desktop PC



Streamline Methods

Interpret the saturation equation $\phi\partial_t S + v \cdot \nabla f(S) = 0$ as an equation along streamlines using

$$\frac{v}{|v|} = \left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right]^T \quad \text{or} \quad v\cdot\nabla = |v|\frac{\partial}{\partial s}$$



Transformation using *time-of-flight* τ

$$|v|\frac{\partial}{\partial s} = \phi\frac{\partial}{\partial \tau}$$

gives a family of 1-D transport equations along streamlines

$$\partial_t S + \partial_\tau f(S) = 0.$$

Streamline Simulation

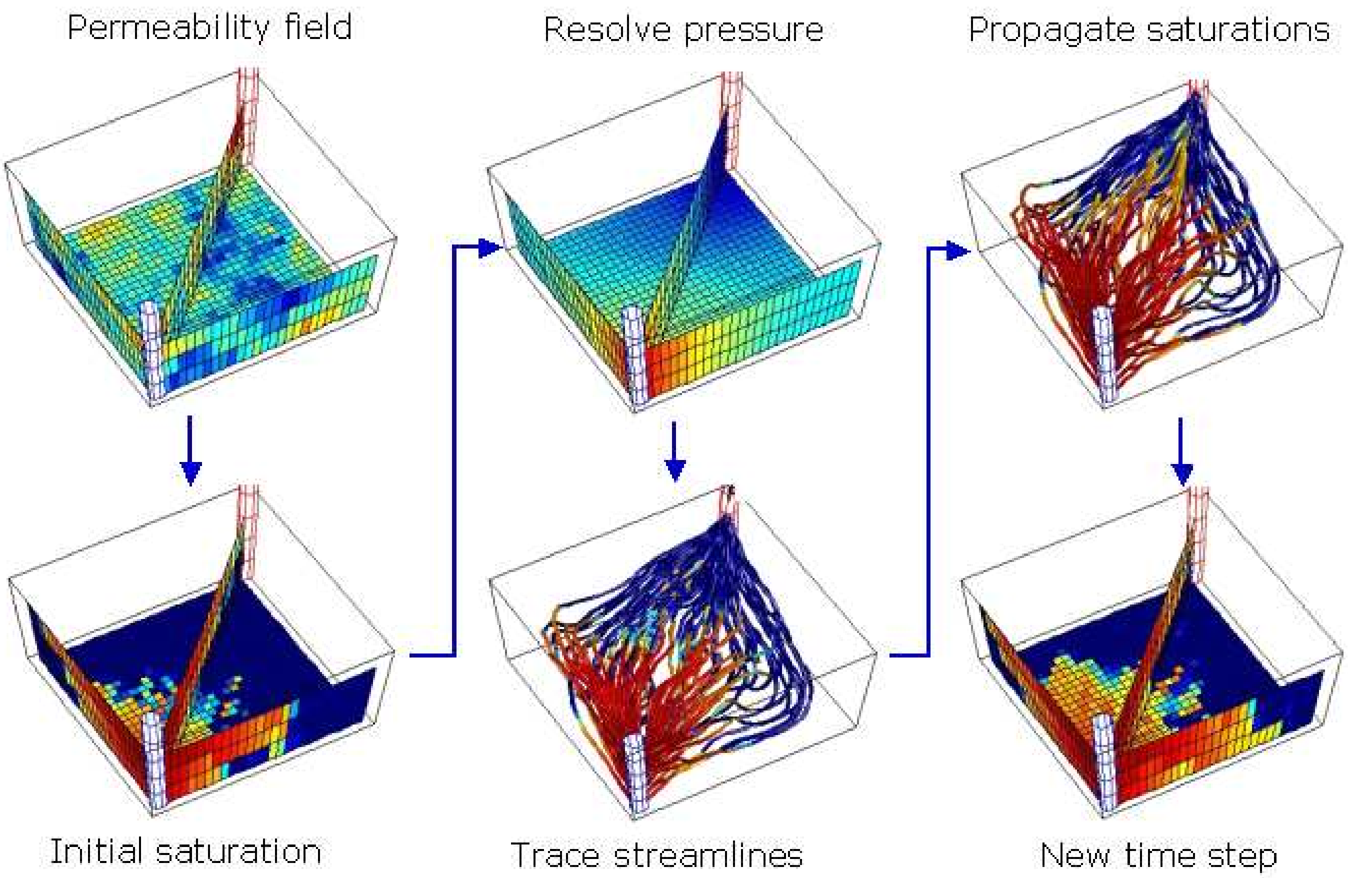
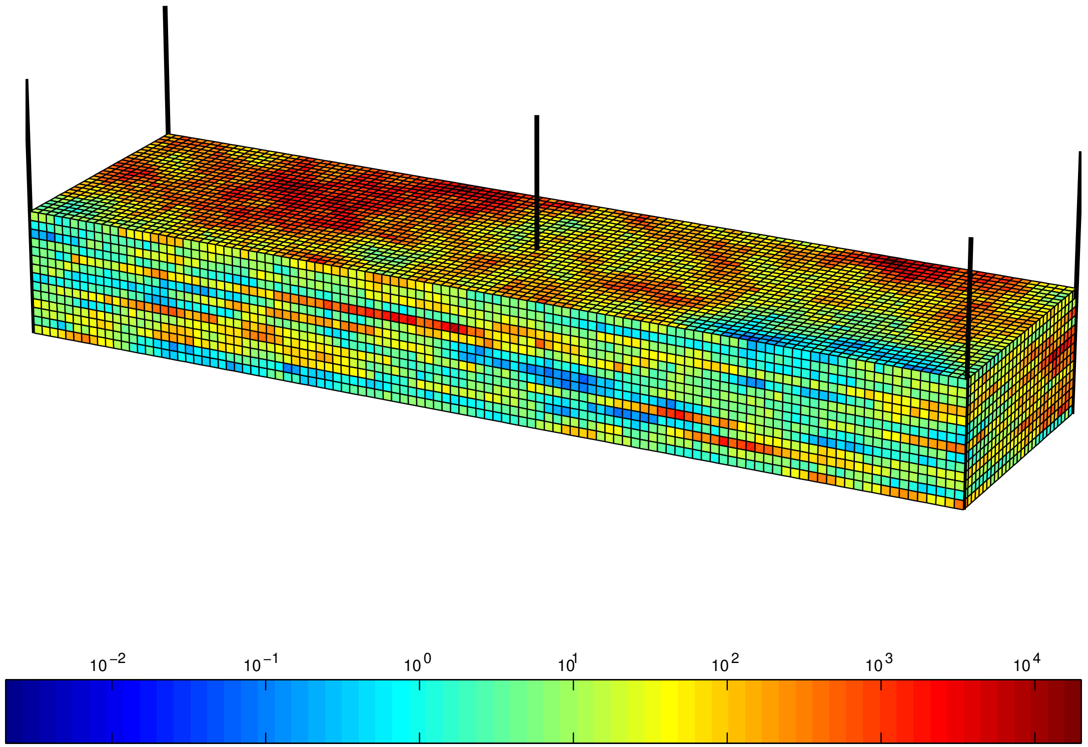


Figure from Yann Gautier

Example 4: SPE 10 Tarbert Formation

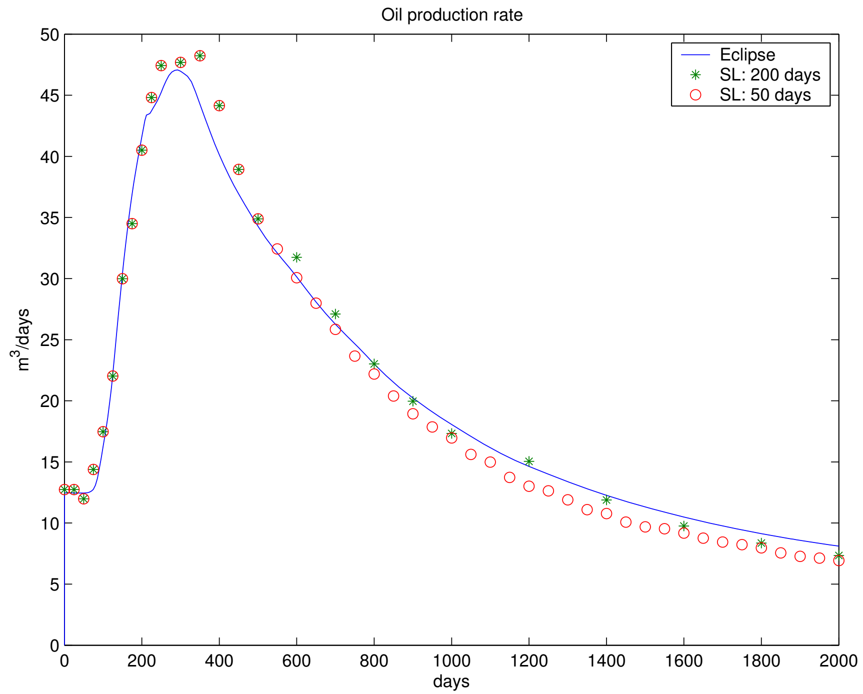


- $30 \times 110 \times 15$ upscaled sample from Tarbert formation
- Initial composition: $(S_w, S_g) = (0.0, 0.2)$
- 2000 days of production
- Either: continuous water injection
- Or: water-alternating-gas every 200 day

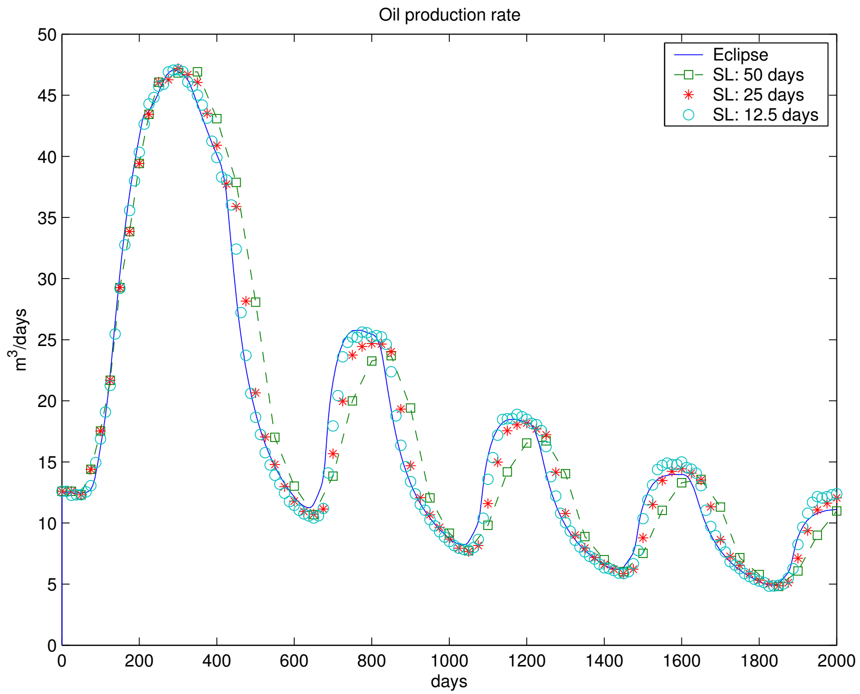
Data reduction is used to speed up front-tracking solution: weak interactions are treated as being of type $\mathcal{S}_1\mathcal{S}_2$.

Example 4 cont'd

Oil production:



water injection



water alternating gas injection

Runtimes: 8 hr 20 min for Eclipse, 2 hr 13 min for streamlines