Analytical Riemann Solvers for Two-Phase, Three-Component Flow in Porous Media

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What do we propose, and why?

Objective:

Accurate and efficient numerical solutions to realistic first-contact miscible displacements in porous media

Key technologies:

- Analytical solution to the Riemann problem
- A front-tracking algorithm to solve general 1D Cauchy problems
- A streamline simulator that decouples the 3D transport equations into a set of 1D problems along streamlines

Outline

1. Analytical Solution
   - Mathematical model of FCM displacements
   - Mathematical character of the system
   - Admissible wave structure
   - Complete catalogue of Riemann solutions

2. Numerical Simulation
   - The front-tracking algorithm
   - One-dimensional examples
   - Three-dimensional streamline simulations

Mathematical model of FCM displacements

Assumptions:

- Three components: water (w), oil (o) and solvent (h)
- Two phases: aqueous (w) and hydrocarbon (h)
- Perfectly miscible hydrocarbons, immiscible water
- Incompressible fluids and no volume change in mixing
- Rigid medium, neglible gravity and capillary effects

Conservation laws:

$$\begin{aligned} \partial_t S_w + \partial_x v_w &= 0 \\ \partial_t \Big((1 - S_w)(1 - \chi_g) \Big) + \partial_x \Big((1 - \chi_g) v_h \Big) &= 0 \\ \partial_t \Big((1 - S_w)\chi_g \Big) + \partial_x \Big(\chi_g v_h \Big) &= 0 \end{aligned}$$

S_α: saturation of phase α, χ_g: mass fraction

Mathematical model, cont'd

2×2 system of conservation laws

$$\partial_t S + \partial_x f = 0$$

$$\partial_t C + \partial_x \left(\frac{1-f}{1-S} C \right) = 0$$

where

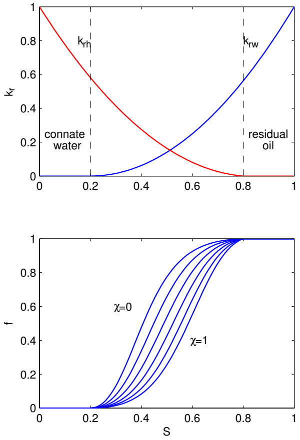
$$S \equiv S_w \quad : \text{ water saturation}$$

$$C \equiv (1 - S_w)\chi_g \quad : \text{ solvent concentration}$$

$$f = \frac{\frac{k_{rw}(S)}{\mu_w}}{\frac{k_{rw}(S)}{\mu_w} + \frac{k_{rh}(S)}{\mu_h(\chi)}} = f(S, C)$$

$$\mu_h = \left[\frac{1-\chi}{\mu_o^{1/4}} + \frac{\chi}{\mu_g^{1/4}} \right]^{-4}$$

Relative permeability and fractional flow curves



Mathematical character

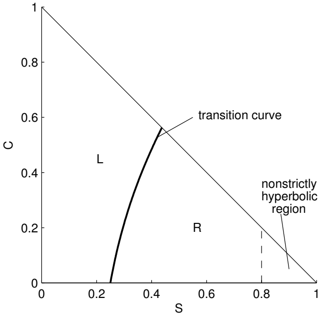
Mathematical character determined by Jacobian matrix: $f'(S, C)$
Two families: S–family (ν_s, r_s) and C–family (ν_c, r_c)

The system is hyperbolic but not strictly hyperbolic

- hyperbolic: real eigenvalues, Jacobi diagonalizable
- strictly hyperbolic: distinct eigenvalues

A transition curve divides the solution space in two regions

- Left: L: $\nu_s < \nu_c$
- Right: R: $\nu_s > \nu_c$



The Riemann problem

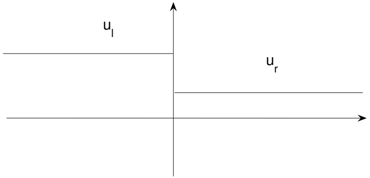
Find the weak solution of

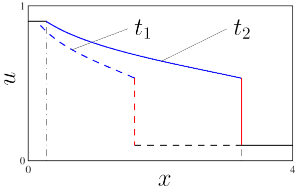
$$\partial_t u + \partial_x f(u) = 0,$$

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x \geq 0 \end{cases}$$

Self-similar solution

$$u(x,t) = U(\zeta), \quad \zeta = x/t$$





Similar problem: polymer flooding, previously analysed by

- Isaacson (1980)
- Johansen and Winther (1988, 1989)

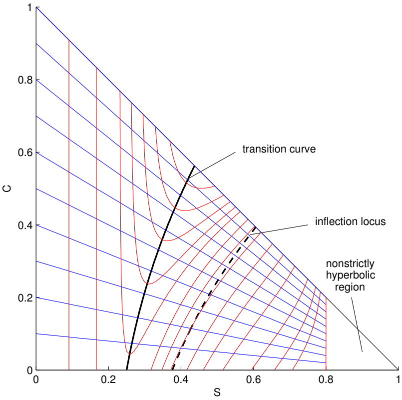
Integral curves

- A smooth solution must satisfy

$$F'(U)U' = \zeta U'$$

Eigenvalue ↗ ↖ Eigenvector

- Two families of solutions:
  - *S*–family: $\frac{C}{1-S} = \text{const}$ (tie-line)
  - *C*–family: $\nu_c = \text{const}$ (nontie-line)



Shock curves

- Discontinuous solutions must satisfy the Rankine–Hugoniot conditions:
$$f(u_l) - f(u_r) = \sigma \cdot (u_l - u_r)$$

- Two families of solutions (Hugoniot loci):
  - In general, shock and integral curves do not coincide, but they have second-order tangency
  - For the solvent system, shock and integral curves coincide
  - S–family: classical Buckley–Leverett wave for fixed solvent mass fraction χ (tie-line)
  - C–family: contact discontinuity of constant ν_c (nontie-line)

Admissible wave sequences

- For a strictly hyperbolic system, the solution comprises at most two waves

$$u_l \xrightarrow{\mathcal{W}_1} u_m \xrightarrow{\mathcal{W}_2} u_r$$

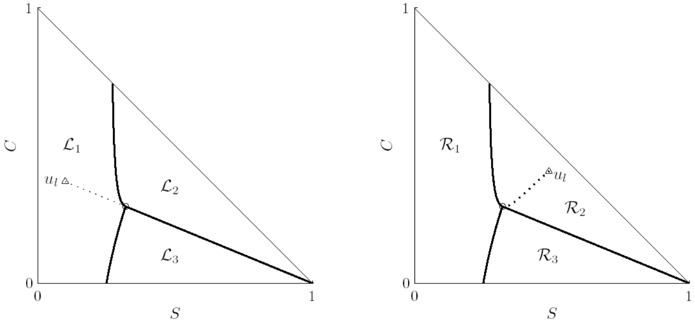
- For a nonstrictly hyperbolic system, the solution may involve more than two waves

$$u_l \xrightarrow{\mathcal{W}_1} u_m^{(1)} \xrightarrow{\mathcal{W}_t} u_m^{(2)} \xrightarrow{\mathcal{W}_2} u_r$$

↖ transitional wave

Riemann solution

- The global solution is obtained by joining admissible waves
- There are two cases (and three regions bounded by transitional curve, tie-line and nontie-line associated with each u_l):

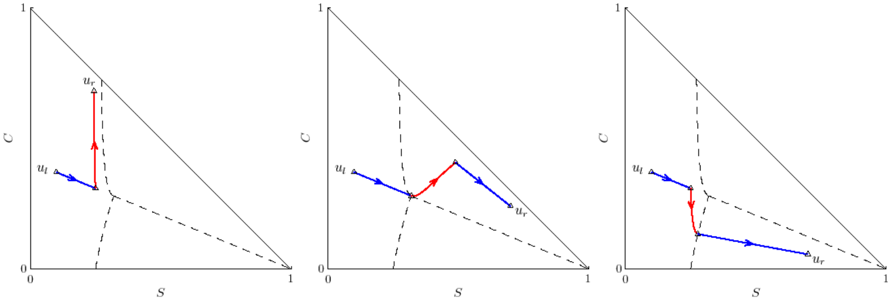


(a) $u_l \in \mathcal{L}$ (b) $u_l \in \mathcal{R}$

Recall: $\mathcal{L}$: $\nu_s < \nu_c$ $\mathcal{R}$: $\nu_s > \nu_c$

The case $u_l \in \mathcal{L}$

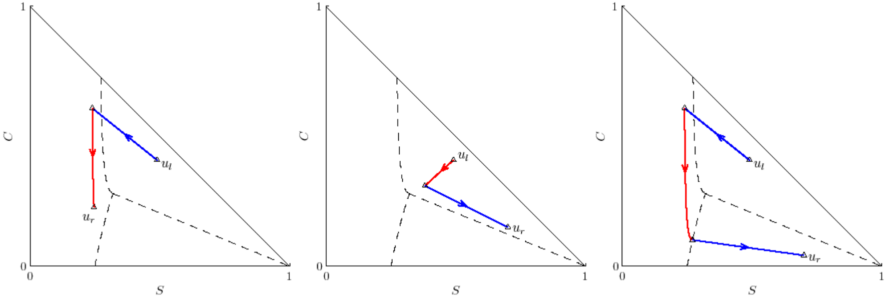
- $u_r \in \mathcal{L}_1$: $u_l \xrightarrow{\mathcal{S}} u_m \xrightarrow{\mathcal{C}} u_r$
- $u_r \in \mathcal{L}_2$: $u_l \xrightarrow{\mathcal{S}} u_m^{(1)} \xrightarrow{\mathcal{C}} u_m^{(2)} \xrightarrow{\mathcal{S}} u_r$
- $u_r \in \mathcal{L}_3$: $u_l \xrightarrow{\mathcal{S}} u_m^{(1)} \xrightarrow{\mathcal{C}} u_m^{(2)} \xrightarrow{\mathcal{S}} u_r$



(a) $u_l \in \mathcal{L}$, $u_r \in \mathcal{L}_1$ (b) $u_l \in \mathcal{L}$, $u_r \in \mathcal{L}_2$ (c) $u_l \in \mathcal{L}$, $u_r \in \mathcal{L}_3$

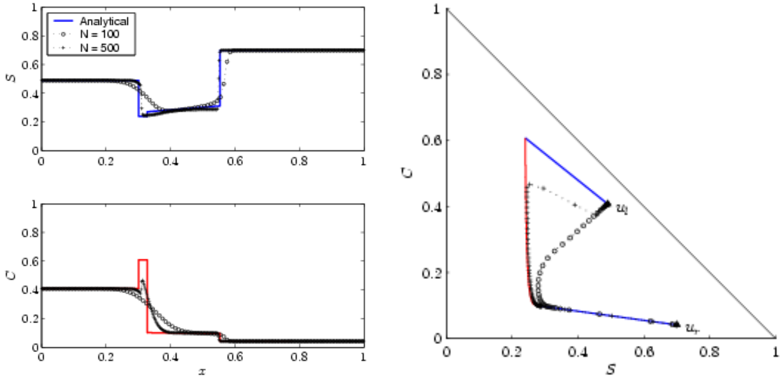
The case $u_l \in \mathcal{R}$

- $u_r \in \mathcal{R}_1$: $u_l \xrightarrow{\mathcal{S}} u_m \xrightarrow{\mathcal{C}} u_r$
- $u_r \in \mathcal{R}_2$: $u_l \xrightarrow{\mathcal{C}} u_m \xrightarrow{\mathcal{S}} u_r$
- $u_r \in \mathcal{R}_3$: $u_l \xrightarrow{\mathcal{S}} u_m^{(1)} \xrightarrow{\mathcal{C}} u_m^{(2)} \xrightarrow{\mathcal{S}} u_r$



(a) $u_l \in \mathcal{R}$, $u_r \in \mathcal{R}_1$ (b) $u_l \in \mathcal{R}$, $u_r \in \mathcal{R}_2$ (c) $u_l \in \mathcal{R}$, $u_r \in \mathcal{R}_3$

Example (Solution of type $\mathcal{R}_3$)

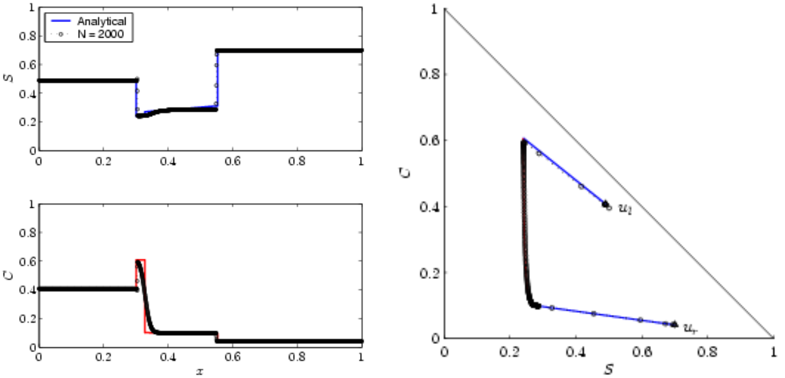


Slow convergence of finite-difference solutions for the transitional wave.
Scheme: single-point upwind, Crank-Nicolson with $\sigma \approx 2$

1D numerical examples

A Riemann problem with a transitional wave

Example (Solution of type $\mathcal{R}_3$)

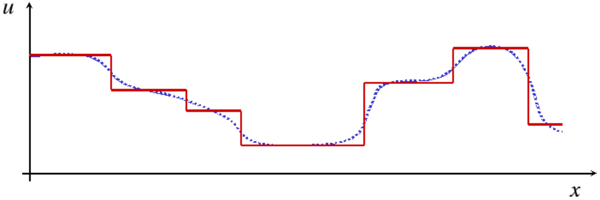


Slow convergence of finite-difference solutions for the transitional wave.
Scheme: single-point upwind, Crank-Nicolson with $\sigma \approx 2$

General 1D Cauchy problems

Solution to the Riemann problem is insufficient if

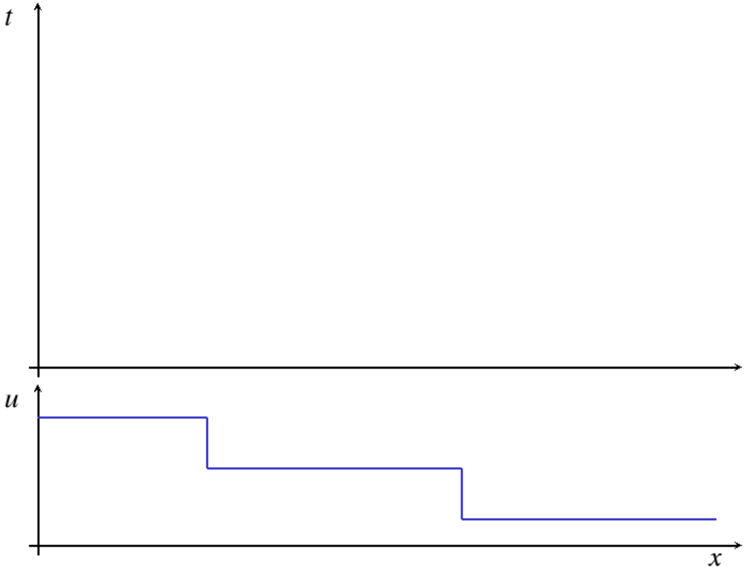
- Initial conditions are different from constant
- Variable injection saturations (e.g., WAG)



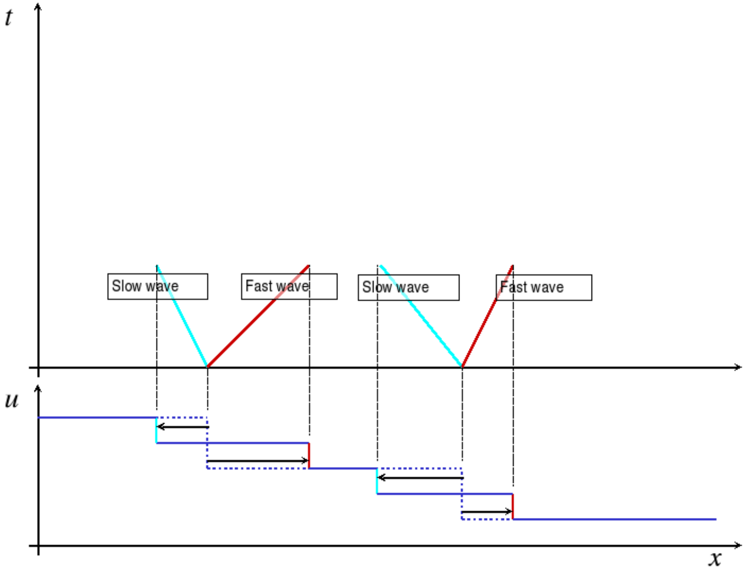
Front-tracking method:

- Piecewise constant approximation of solution
- Sequence of Riemann problems
- Riemann solutions discretised as step functions

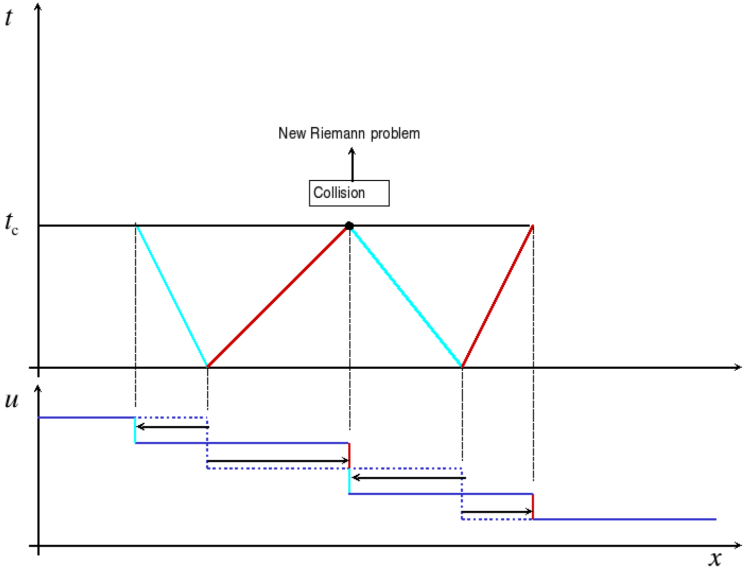
Front-tracking algorithm



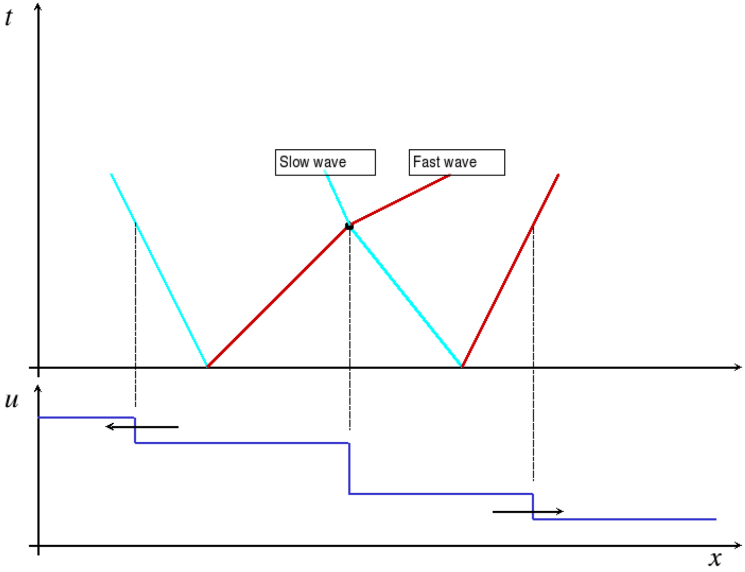
Front-tracking algorithm



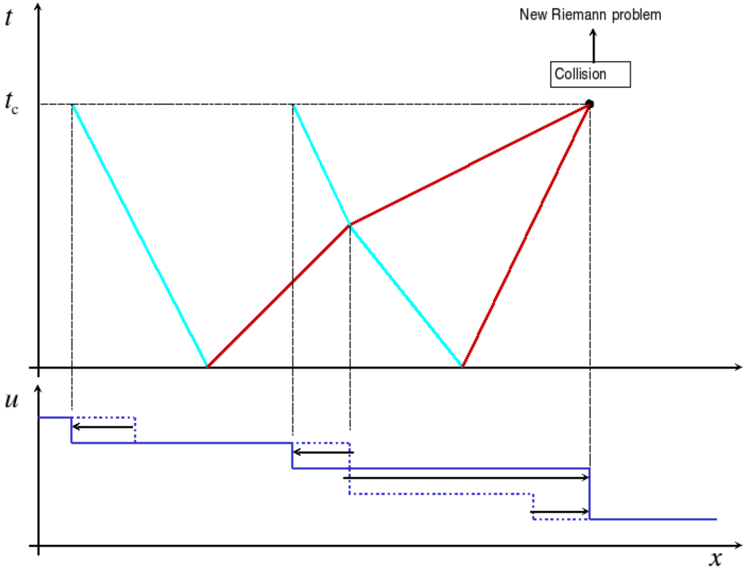
Front-tracking algorithm



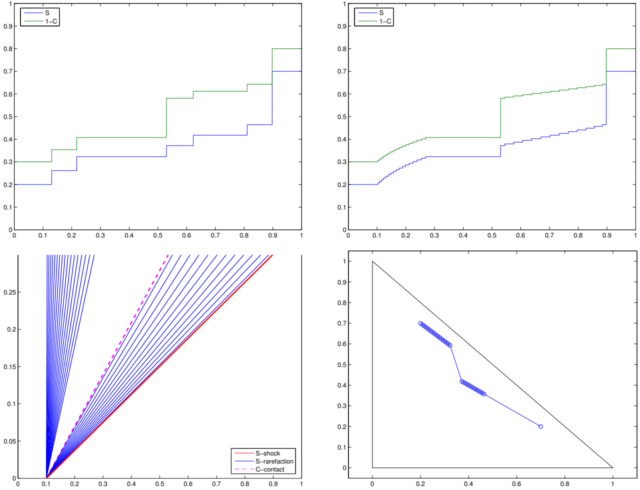
Front-tracking algorithm



Front-tracking algorithm



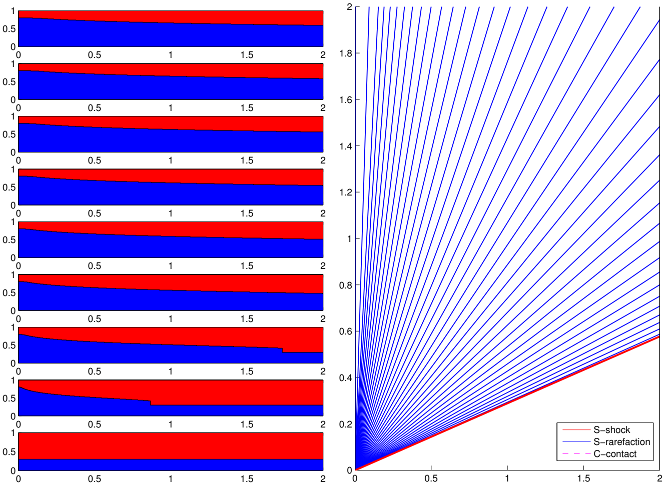
Example $(u_l \xrightarrow{\mathcal{S}} u_m^{(1)} \xrightarrow{\mathcal{C}} u_m^{(2)} \xrightarrow{\mathcal{S}} u_r)$



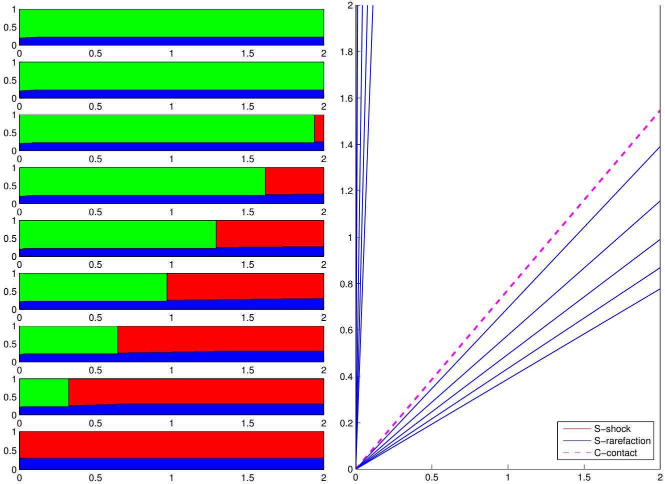
Example

- Initially, reservoir filled with 70% oil and 30% water
- Four different injection strategies:
  - continuous water injection
  - continuous gas injection
  - alternating solvent and water injection
  - alternating water and solvent injection
- Front-tracking solution with step size $\delta_u = 0.01$ for rarefactions

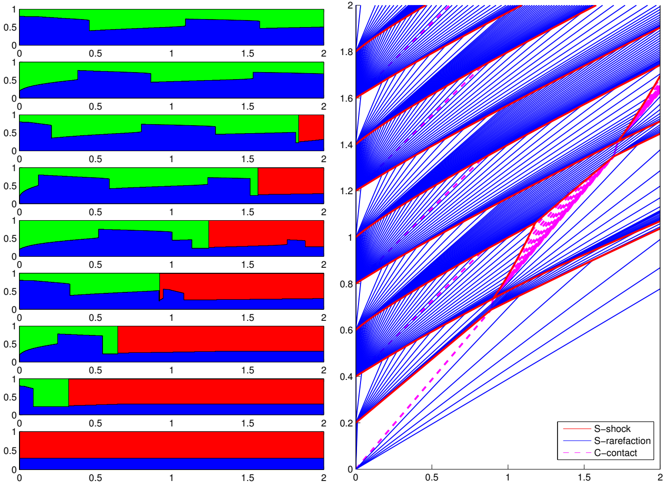
Example (water injection)



Example (Solvent injection)

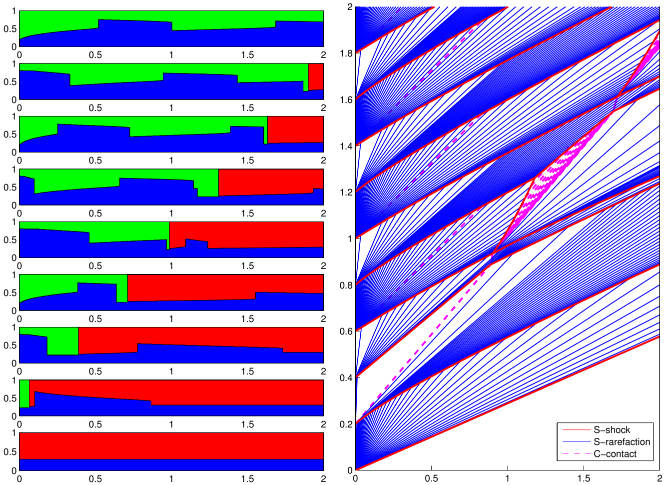


Example (alternating solvent and water injection)

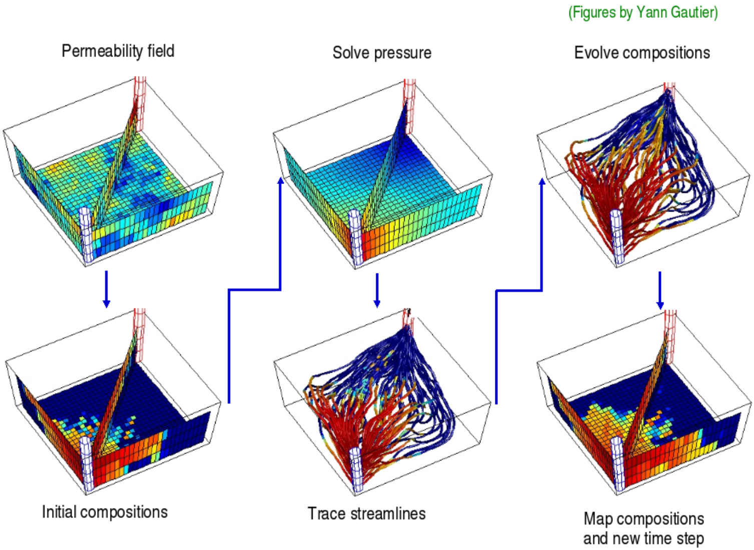


1D numerical examples
Linear reservoir

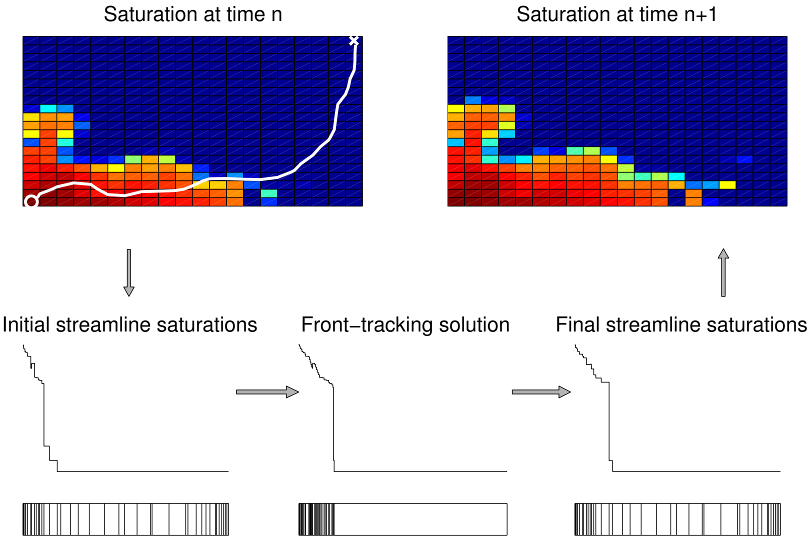
Example (alternating water and solvent injection)



In 3D: streamline simulation



Streamline step

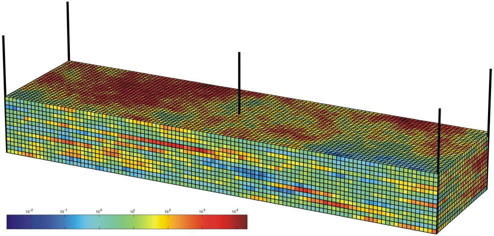


3D numerical simulations

Tarbert formation from 10th SPE Comparative Solution Project

Highly heterogeneous, shallow-marine formation

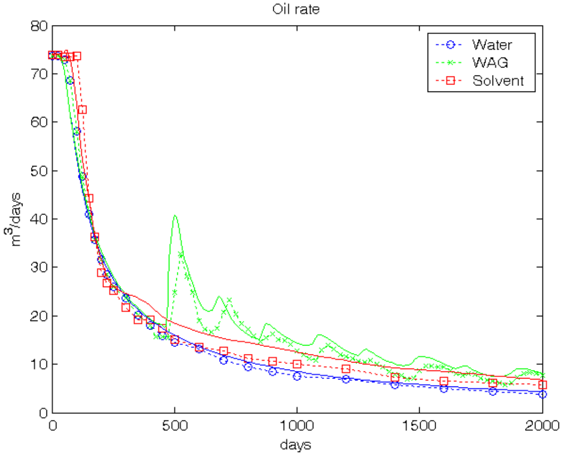
- Six orders of magnitude permeability variations
- Five vertical wells (1 injector, 4 producers)



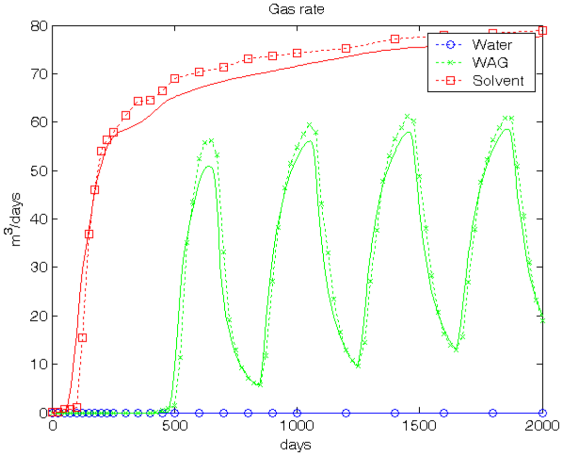
Three different injection schemes

1. continuous water injection
2. continuous gas injection
3. water-alternating gas injection

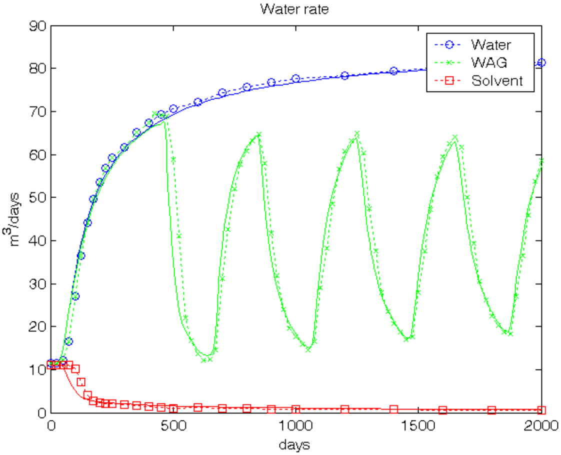
Example (Oil production)



Example (Gas production)



Example (Water production)



Conclusions

Efficient computational framework for first-contact miscible gas injection processes

- Analytical Riemann solver (6 solution types)
- Front-tracking algorithm
  - exact representation of discontinuities
  - unconditionally stable
  - grid-independent
- Streamline simulation
  - efficient
  - accurate? (work in progress)

Extensions: viscous fingering (Blunt & Juanes, 2005)