# Non-Uniform Coarse Grids and Multiscale Mixed FEMs.

J. Aarnes Stein Krogstad K.-A. Lie

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# Outline.

# Model Equations.

Elliptic pressure equation:

$$v = -\lambda(S) K \nabla p$$
$$\nabla \cdot v = q$$

Hyperbolic saturation equation:

$$\phi \frac{\partial S}{\partial t} + \nabla \cdot (v f(S)) = q_w$$

- Total velocity:

$$v = v_o + v_w$$

- Total mobility:

$$\lambda = \lambda_w(S) + \lambda_o(S) = k_{rw}(S)/\mu_w + k_{ro}(S)/\mu_o$$

- Saturation water: $S$
- Fractional flow water:

$$f(S) = \lambda_w(S)/\lambda(S)$$

# Motivation.

Small scale variations in the permeability can have a strong impact on large scale flow and should be resolved properly.

- the pressure may be well resolved on a coarse grid
- the fluid transport should be solved on the finest scale possible

Thus: a multiscale method for the pressure equation should provide velocity fields that can be used to simulate flow on a fine scale.

The MsMFEM basis functions *predicts* what the global flow looks like locally.

- problems can occur if large scale structures penetrate the local domains

Thus: the shape of the local domains (the coarse grid) should adapt to important large scale features.

## Mixed formulation:

Find $(v, p) \in H_0^{1,\text{div}} \times L^2$ such that

$$\int (\lambda K)^{-1} u \cdot v \, dx - \int p \nabla \cdot u \, dx = 0, \qquad \forall u \in H_0^{1,\text{div}},$$

$$\int l \nabla \cdot v \, dx = \int q l \, dx, \quad \forall l \in L^2.$$

## Multiscale discretization:

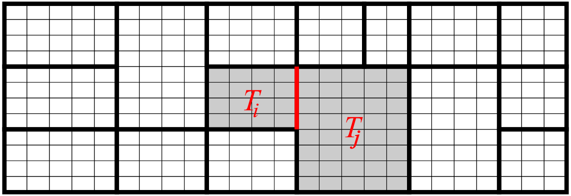
Seek solutions in low-dimensional subspaces

$$U^{ms} \subset H_0^{1,\text{div}} \text{ and } V \in L^2,$$

where local fine-scale properties are incorporated into the basis functions.

# Grids and Basis Functions.

We assume we are given a *fine* grid with permeability and porosity attached to each fine grid block.




We construct a *coarse* grid, and choose the discretization spaces $V$ and $U^{ms}$ such that:

- For each coarse block $T_i$,there is a basis function $\phi_i \in V$.
- For each coarse edge $\Gamma_{ij}$, there is a basis function $\psi_{ij} \in U^{ms}$.

# Basis Functions (Local Version).

## Velocity:

For each coarse edge $\Gamma_{ij}$ we define a basis function $\psi_{ij} = -\lambda K \nabla \phi_{ij}$ with

$$\nabla \cdot \psi_{ij} = \begin{cases} w_i(x), & \text{for } x \in T_i \\ -w_j(x), & \text{for } x \in T_j, \end{cases}$$

with BCs $\psi_{ij} \cdot n = 0$ on $\partial(T_i \cup \Gamma_{ij} \cup T_j)$, and the weight-functions $w_i, w_j$ has average 1.

## Pressure:

For each $T_i$ we define

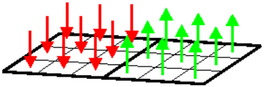
$$\phi_i = \begin{cases} 1, & \text{for } x \in T_i \\ 0, & \text{otherwise.} \end{cases}$$

If $\int_{T_i} q dx \neq 0$ ($T_i$ contains a source), then

$$w_i(x) = \frac{q(x)}{\int_{T_i} q(\xi)\, d\xi}.$$

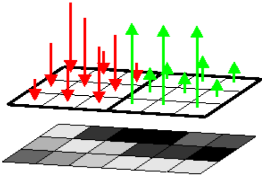
Otherwise we may choose

$$w_i(x) = \frac{1}{|T_i|},$$



or to avoid high flow through low-perm regions

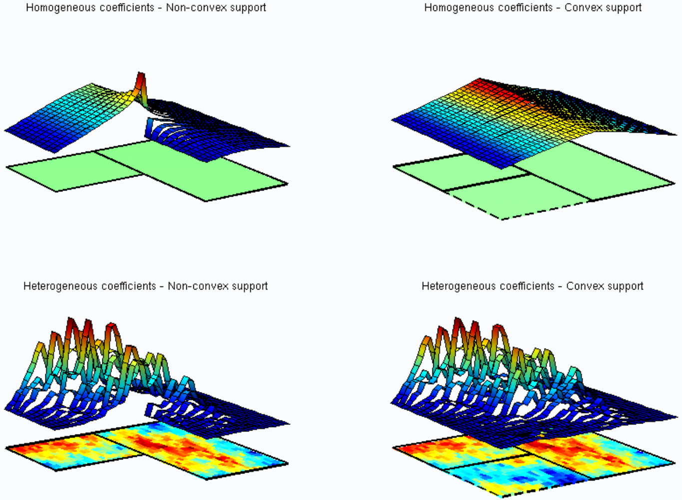
$$w_i(x) = \frac{\text{trace}(K(x))}{\int_{T_i} \text{trace}(K(\xi))\, d\xi}.$$



The latter is more accurate - even for strong anisotropy.

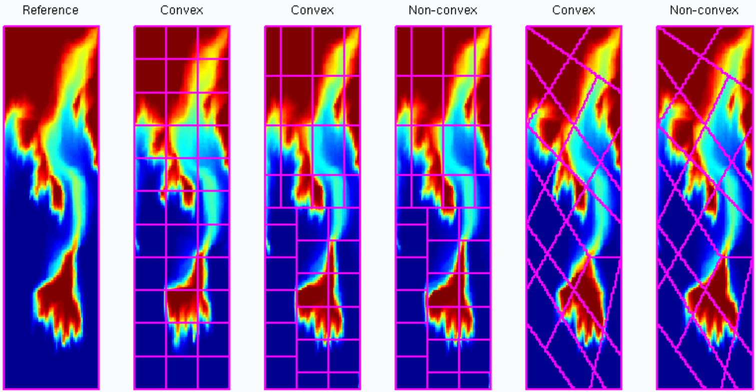
# Basis Functions for Velocity.

## X-component of 2D basis functions:



# Impact of Convex versus Non-Convex Support.

## Heterogenous media:

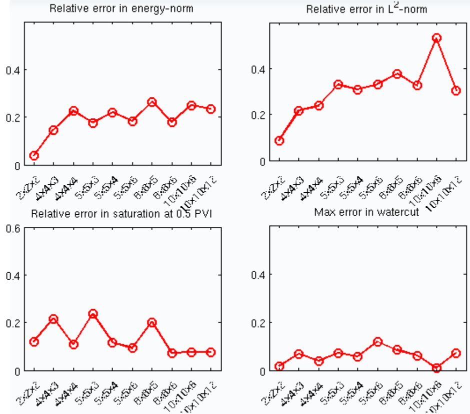


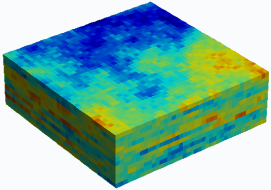

We use *tight* support.

Question:

Does refining the coarse grid increase accuracy?

Error-measures for various coarse mesh-sizes.



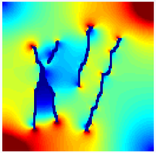



Chen and Hou 2002:
error is bounded by

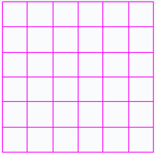
$$\mathcal{O}(H + \sqrt{\epsilon} + \sqrt{\frac{\epsilon}{H}})$$

Certain large scale features may cause problems:

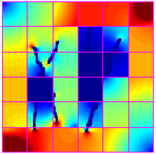
## Traversing barriers.



Reference.

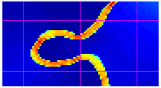


Coarse Grid.

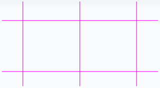


MsMFEM.

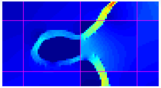
## Cross-flow.



Reference.



Coarse Grid.

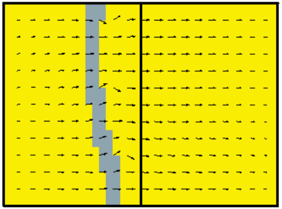


MsMFEM.

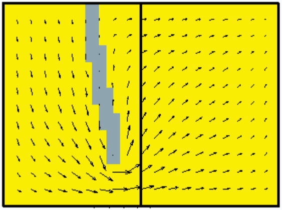
Can be ameliorated through coarse grid refinement/adaptation.

# Traversing Barriers.

Problems occur when a basis function forces flow through a barrier:



Potential problem



No problem

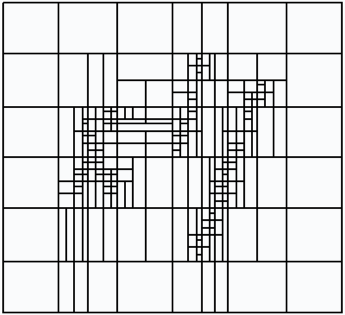
Problem-cases can be detected automatically through the indicator

$$v_{ij} = \psi_{ij} \cdot (\lambda K)^{-1} \psi_{ij}.$$

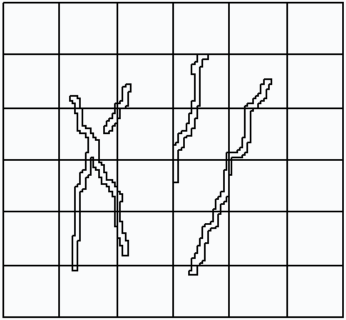
If $v_{ij}(x) > C$ for some $x \in T_i$, then split $T_i$, and generate basis functions for the new faces.

# Traversing Barriers.

### Automatic approach:



### Direct approach:

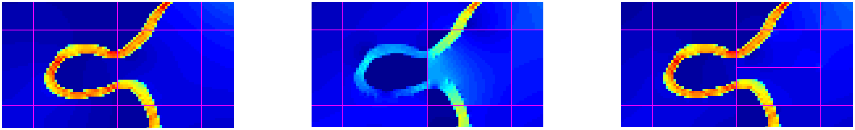


# Cross-Flow.

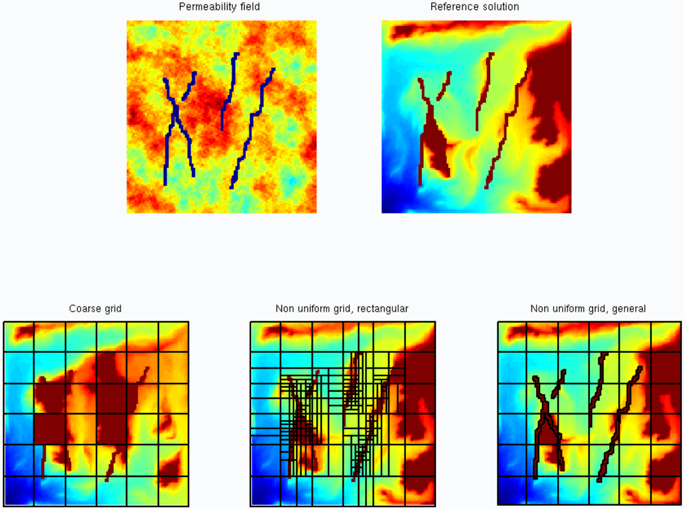
- The velocity field $v$ experiences cross-flow over $\Gamma_{ij}$ if the quantity

$$\frac{\int_{\Gamma_{ij}} |v \cdot n| \, ds}{\left| \int_{\Gamma_{ij}} v \cdot n \, ds \right|}$$

  is large. Edges of potential cross-flow can be detected by solving a local pressure equation in $\Omega_{ij}$.
- The problem is solved by splitting one of the neighboring blocks.

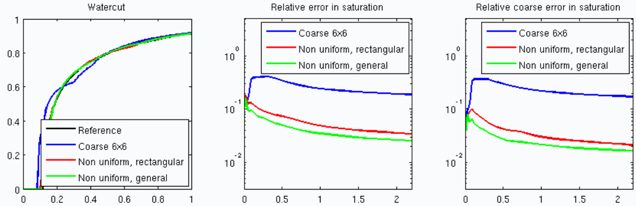


## Saturation plots ( 0.5 PVI):

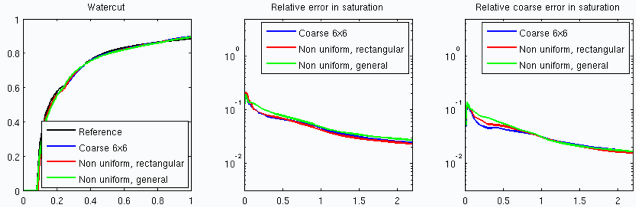


# Numerical Example: Barriers in 2D.

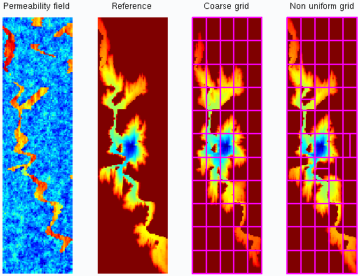
## Watercut and saturation errors:



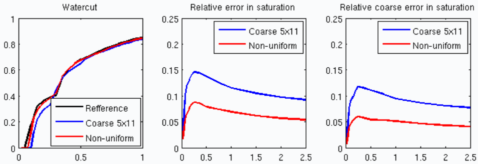

## ... and with barriers removed (same background perm):



## Saturation plots:

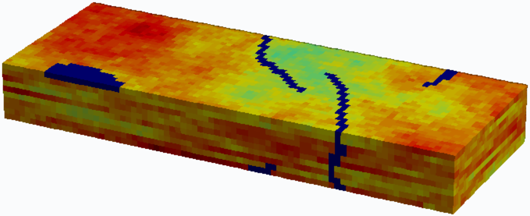



## Watercut and saturation errors

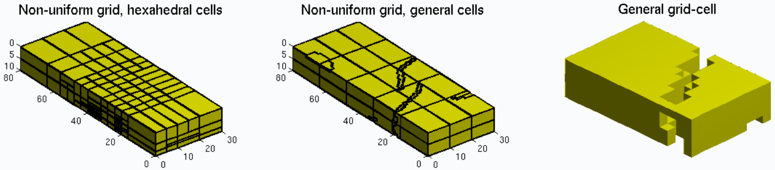


# Numerical Example: Barriers in 3D.

## Permeability field:

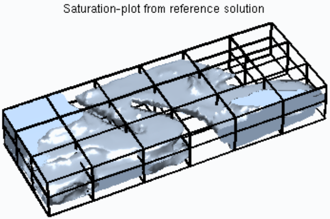


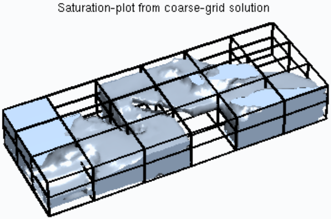
## Grids:



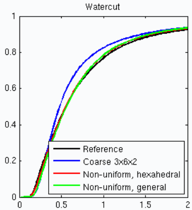
# Numerical Example: Barriers in 3D.

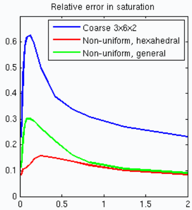
## Saturation plots:

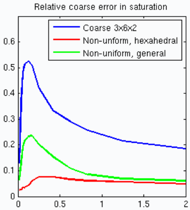



## Watercut and saturation errors:



# Concluding Remarks.

## In this talk:

- A local version of the MsMFEM.
- Potential weaknesses can be fixed through coarse-grid adaptation.
- Great flexibility and robustness w.r.t. shapes of the coarse grid-blocks.

## Extensions and further work:

- More general *fine* grids.
- Adapting the coarse grids to fractures and faults.
- Non-uniform coarse grids around wells.