How to Make Implicit Upwind Schemes as Efficient as Streamline Methods

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Topic of this talk

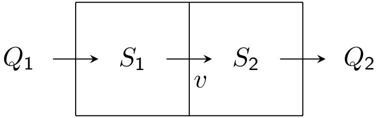
Aim:
To construct fast implicit solver for

$$\phi \frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla f(S) = Q(S), \qquad (1)$$

assuming no gravity and no capillary pressure.

Method:
Decompose (1) in sequence of local problems.

Implicit First-Order Upwind Scheme 1D



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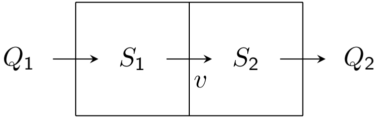
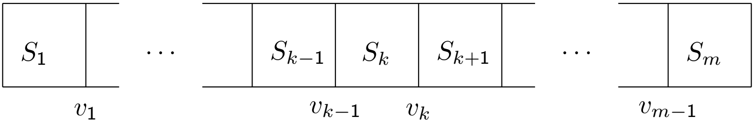


$$
\begin{aligned}
\frac{\phi}{\Delta t}(S_1^n - S_1^{n-1}) - \frac{v}{\Delta x} f(S_1^n) &= Q_1(S_1^n),\\
\frac{\phi}{\Delta t}(S_2^n - S_2^{n-1}) + \frac{v}{\Delta x} f(S_1^n) &= Q_2(S_2^n),
\end{aligned}
$$

Implicit First-Order Upwind Scheme 1D

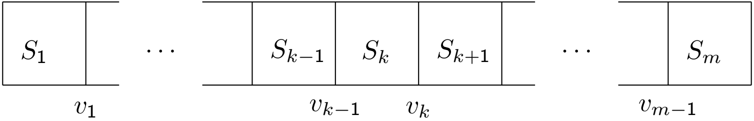


Consider the scheme

$$\frac{\phi}{\Delta t}(S_k^n - S_k^{n-1}) + \frac{1}{\Delta x}\Big(v_{k-1} f(S_{k-1}^n) - v_k f(S_k^n)\Big) = Q_k(S_k^n),$$

where we have assumed $v_k > 0$.

Implicit First-Order Upwind Scheme 1D



Consider the scheme

$$\frac{\phi}{\Delta t}(S^n - S^{n-1}) + \frac{1}{\Delta x}\begin{bmatrix} -v_1 & & & \\ v_1 & -v_2 & & \\ & \ddots & \ddots & \\ & & v_{m-1} & 0 \end{bmatrix}\begin{bmatrix} f(S_1^n) \\ \vdots \\ f(S_m^n) \end{bmatrix} = Q(S).$$

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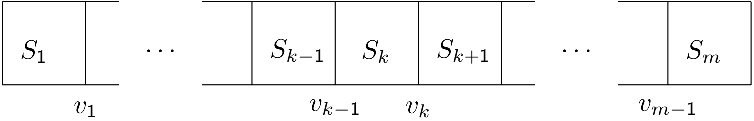

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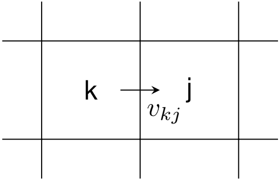
where we have assumed $v_k > 0$.

Lower triangular matrix $\Longrightarrow$ equations can be solved in sequence.

Implicit First-Order Upwind Scheme 2D

Then consider the same scheme in 2D with m grid cells and fluxes given by the (sparse) $m \times m$-matrix v.



$$\frac{S_k^n - S_k^{n-1}}{\Delta t} - \frac{1}{h}\Big(\sum_j \max(v_{kj},0) f(S_k^n) + \sum_j \min(v_{kj},0) f(S_j^n)\Big) = Q_k(S_k^n).$$

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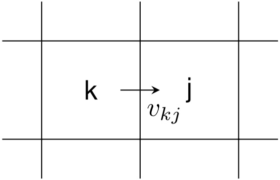

Again, this can be written in matrix notation

$$\frac{1}{\Delta t}(S^n - S^{n-1}) + V \begin{bmatrix} f(S_1^n) \\ \vdots \\ f(S_m^n) \end{bmatrix} = Q(S^n).$$

where

$$V_{kk} = -\frac{1}{h}\sum_j \max(v_{kj}, 0), \qquad V_{kj} = -\frac{1}{h}\min(v_{kj}, 0).$$

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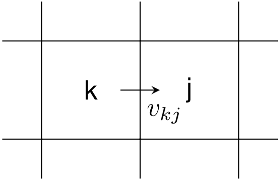

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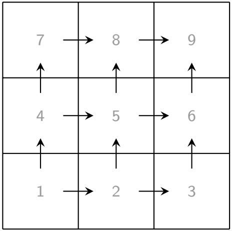
Is V triangular?

“Homogeneous” Quarter five-spot

What does V look like?



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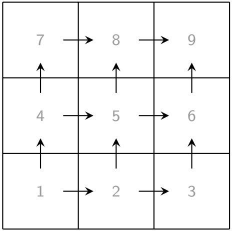
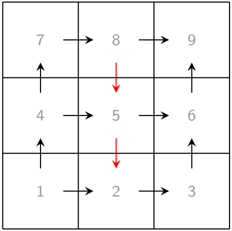


$$V = \begin{bmatrix} * & & & & & & & & \\ * & * & & & & & & & \\ & * & * & & & & & & \\ * & & & * & & & & & \\ & * & & * & * & & & & \\ & & * & & * & * & & & \\ & & & * & & & * & & \\ & & & & * & & * & * & \\ & & & & & * & & * & * \end{bmatrix}$$

Lower triangular

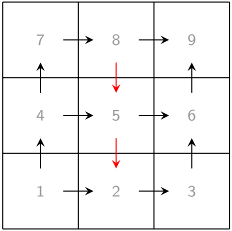
“Heterogeneous” Quarter five-spot

What does V look like now?



“Heterogeneous” Quarter five-spot

What does V look like now?



$$V = \begin{bmatrix} * & & & & & & & & \\ * & * & & & {\color{red}*} & & & & \\ & * & * & & & & & & \\ * & & & * & & & & & \\ & & & * & * & & & {\color{red}*} & \\ & & * & & * & * & & & \\ & & & * & & & * & & \\ & & & & & & * & * & \\ & & & & & * & & * & * \end{bmatrix}$$

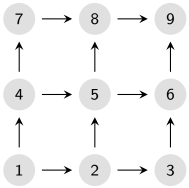
Not lower triangular

Graph Interpretation

What can be done?

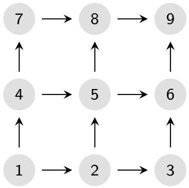
Graph Interpretation Continued

“Homogeneous” Quarter five-spot as directed graph



Graph Interpretation Continued

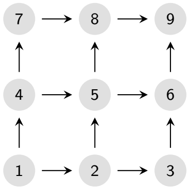
“Homogeneous” Quarter five-spot as directed graph



Can this directed graph be flattened such that all arrows point to the right?

Graph Interpretation Continued

"Homogeneous" Quarter five-spot as directed graph



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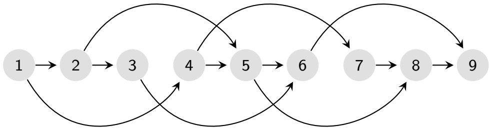
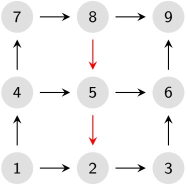


Yes!

Graph Interpretation Continued

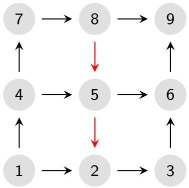
"Heterogeneous" Quarter five-spot as directed graph



What about this directed graph?

Graph Interpretation Continued

“Heterogeneous” Quarter five-spot as directed graph



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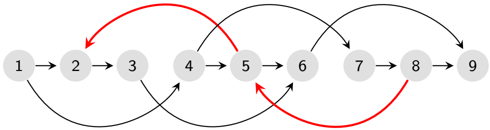


Topological sorting

Topological sorting:

Find sequence of vertex numbers $(p_1, \ldots, p_m)$ such that

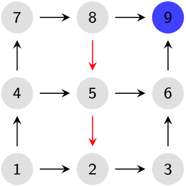
$$p_i < p_j$$

whenever there is a directed edge from vertex i to vertex j.

A *topological sort* of the vertices in a directed graph can be found in linear time as the post-order of the depth-first traversal of the reversed graph.

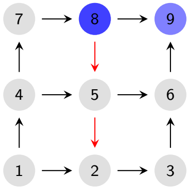
Graph Interpretation Continued

"Heterogeneous" Quarter five-spot as directed graph



Graph Interpretation Continued

"Heterogeneous" Quarter five-spot as directed graph



Graph Interpretation Continued

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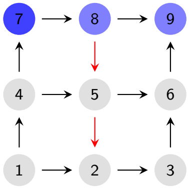

Graph Interpretation Continued

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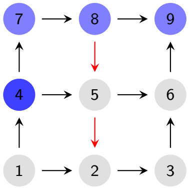

Graph Interpretation Continued

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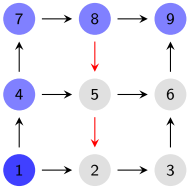

Graph Interpretation Continued

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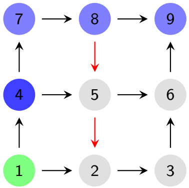

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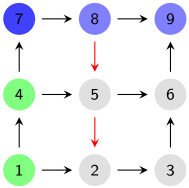

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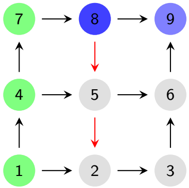

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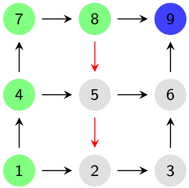

Graph Interpretation Continued

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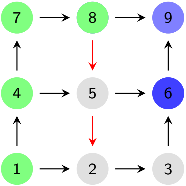

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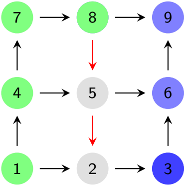

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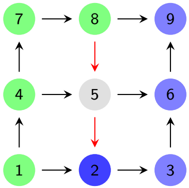

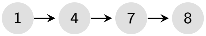

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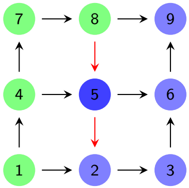

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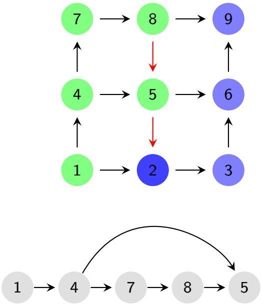

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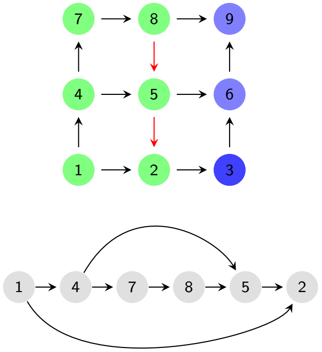

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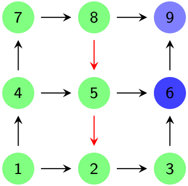

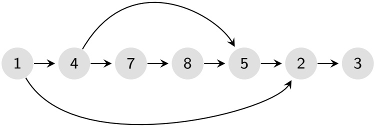

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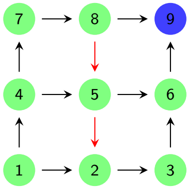

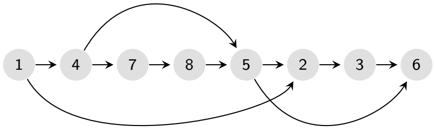

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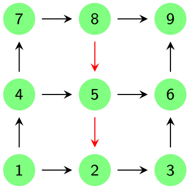
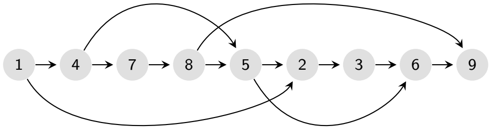
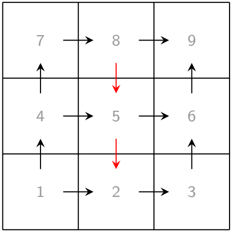




Triangularisation of the coefficient matrix

By rearranging the rows and columns in the order $(1, 4, 7, 8, 5, 2, 3, 6, 9)$, we obtain a triangular V:



	1	4	7	8	5	2	3	6	9
1	*								
4	*	*							
7		*	*						
8			*	*					
5		*		*	*				
2	*				*	*			
3						*	*		
6					*		*	*	
9				*				*	*

Duff and Reid. *An implementation of Tarjans algorithm for block triangularisation of a matrix.* 1978.

Dennis, Martinez and Zhang. *Triangular decomposition methods for solving reducible nonlinear systems.* 1994.

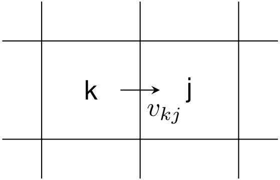
Triangularisation of the coefficient matrix

Again, this can be written in matrix notation

$$\frac{1}{\Delta t}(\tilde{S}^n - \tilde{S}^{n-1}) + L \begin{bmatrix} f(\tilde{S}^n_1) \\ \vdots \\ f(\tilde{S}^n_m) \end{bmatrix} = PQ(\tilde{S}^n).$$

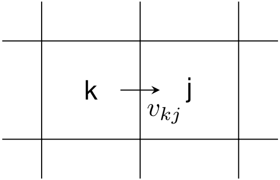
where $\tilde{S} = PS$, $L = PVP^T$ and P is a permutation matrix obtained from a topological ordering of the grid cells.

Implicit First-Order Upwind Scheme 2D



$$\frac{S_k^n - S_k^{n-1}}{\Delta t} - \frac{1}{h}\Big(\sum_j \mathsf{max}(v_{kj},0) f(S_k^n) + \sum_j \mathsf{min}(v_{kj},0) f(S_j^n)\Big) = Q_k(S_k^n).$$

Implicit First-Order Upwind Scheme 2D



Find $u_h \in V_h$ such that

$$\int_K \frac{S_k^n - S_k^{n-1}}{\Delta t} v_h - \int_K f(S_k^n)\,\mathbf{v}\cdot\nabla v_h + \sum_j \int_{\partial K} v_h \hat{f}(S_k^n, S_j^n, v_{kj}) = \int_K Q_k(S_k^n) v_h,$$

For all $v_h \in V_h$.

Here $\hat{f}$ is the upwind flux given by

$$\hat{f}(S_k, S_j, v_{kj}) = f(S_k)\max(v_{kj}, 0) + f(S_j)\min(v_{kj}, 0).$$

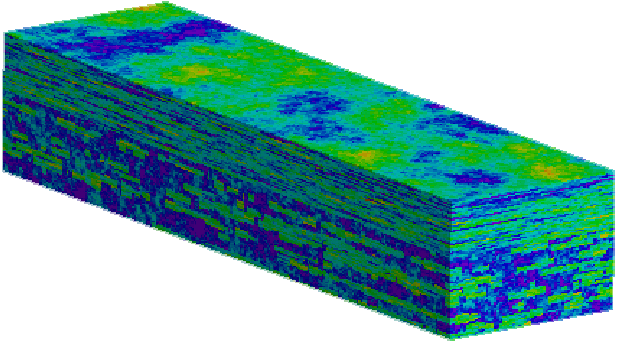
Delineation of Reservoirs

Consider a domain with multiple injectors at positions $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Solve

$$\mathbf{v} \cdot \nabla C_i = Q_i$$

where $Q_i > 0$ for the injector at injector $\mathbf{x}_i$ and zero elsewhere.

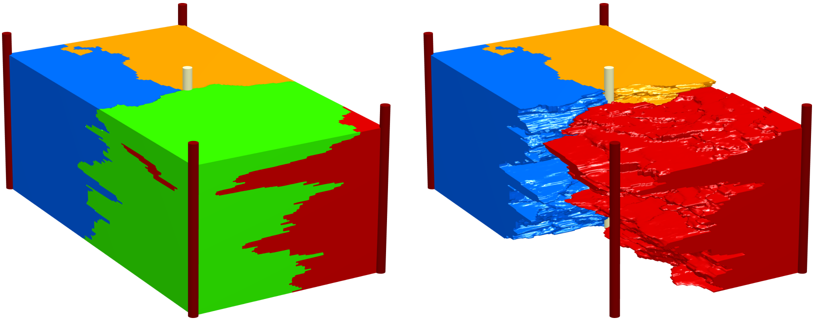
$$C_i(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \text{ on streamline from injector } i, \\ 0 & \text{otherwise.} \end{cases}$$



Porosity of the model 2 of the SPE Comparative Solution Project.

Delineation of Reservoirs Continued

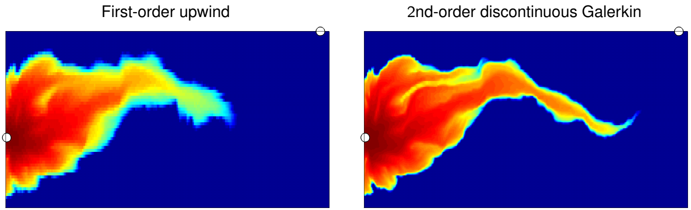
Model 2 of the SPE Comparative Solution Project*.



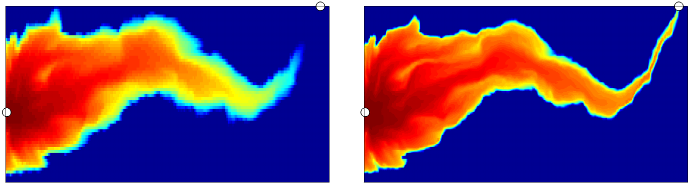
Size: $60 \times 220 \times 85$ (1.122 mill. grid blocks)
Simulation time: a few minutes.

Christie and Blunt *Tenth SPE Comparative Solution Project: A Comparison of Upscaling Techniques*

Twophase Flow in 2D



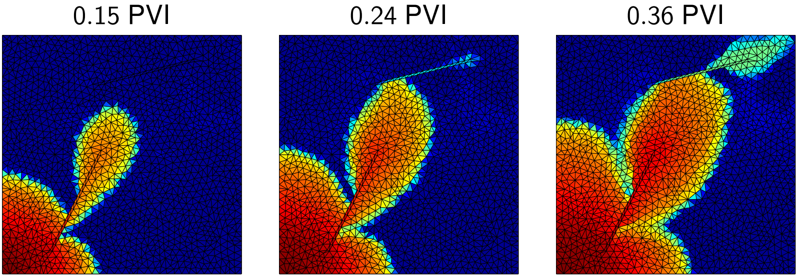
Solution after 0.2 PVI



Solution after 0.3 PVI.

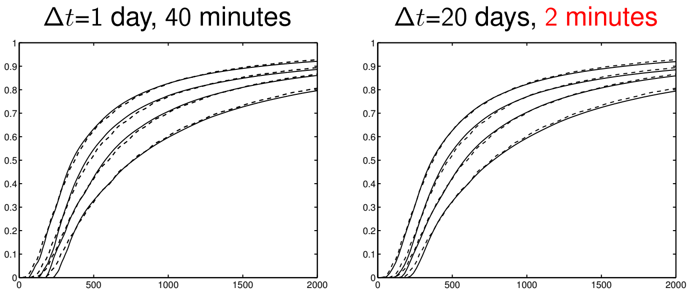
Water flooding in layer 6 of the same model computed with 3 pressure updates.

Twophase Flow in 2D



Water flooding in homogeneous domain with highly permeable fractures computed with the first-order upwind scheme.

Model 2 of the SPE Comparative Solution Project



Watercut curves computed with the first-order upwind scheme (solid line) and with FrontSim (dashed).

Highlights

Why is this a good idea

- Extremely fast solvers: $\mathcal{O}(n)$ operations for n unknowns.
- Local control over Newton iteration.
- Small memory requirements.
- Based on well-known *conservative* discretisation.

Similar to streamline methods in performance: millions of grid cells on desktop computers!