Two Mixed Finite-Element Based Multiscale Methods for Elliptic Problems in Porous Media Flow

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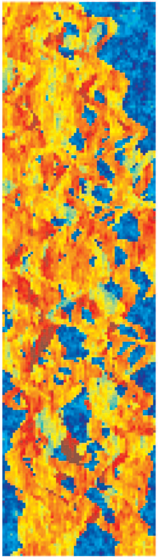
Outline

Multiscale Methods for Elliptic PDEs



- Intended application: Flow (pressure) equation in reservoir simulation.
- Problem: Number of unknowns exceeds capabilities of existing approaches.
- Goal: Include effect of fine-scale features, but only solve a coarse-scale equation.
- Means: Pre-compute fine-scale numerical solutions to local problems.
- Result: Mass-conservative fine-scale approximations.

Model Problem

- Here we focus on capturing the effect of the fine-scale variation in the coefficients.
- To isolate this effect we consider incompressible, isothermal one-phase flow in a closed reservoir,

$$\nabla \cdot \mathbf{u} = q, \qquad \mathbf{u} = -\mathbf{K}\nabla p, \qquad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega,$$

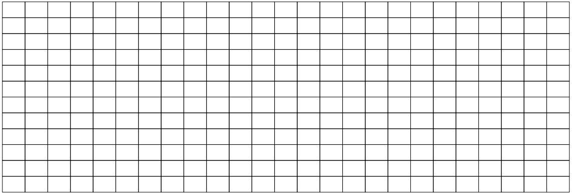
i.e., the variable-coefficient Poisson equation with homogeneous Neumann boundary conditions.

Mixed Formulation

Find $(\mathbf{u},\ p) \in H_0^{1,\text{div}}(\Omega) \times L^2(\Omega)$ such that,

$$
\begin{aligned}
(\mathbf{K}^{-1} \cdot \mathbf{u},\ \mathbf{v}) - (p,\ \nabla \cdot \mathbf{v}) &= 0 && \forall \mathbf{v} \in H_0^{1,\text{div}}(\Omega), \\
(\nabla \cdot \mathbf{u},\ l) &= (q,\ l) && \forall l \in L^2(\Omega).
\end{aligned}
$$

- Standard MFEM: Seek solution in finite-dimensional subspaces, $\mathbf{V}_h \times W_h \subset H_0^{1,\text{div}}(\Omega) \times L^2(\Omega)$.
- Fine-scale approximation spaces constructed from piecewise polynomials on elements.

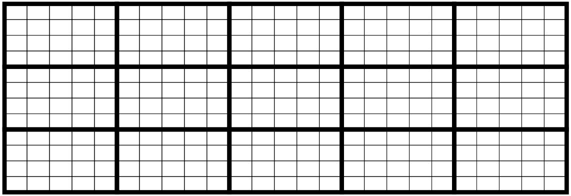


Mixed Formulation

Find $(\mathbf{u},\ p) \in H_0^{1,\text{div}}(\Omega) \times L^2(\Omega)$ such that,

$$\begin{aligned}(\mathbf{K}^{-1}\cdot\mathbf{u},\ \mathbf{v}) - (p,\ \nabla\cdot\mathbf{v}) &= 0 && \forall\mathbf{v} \in H_0^{1,\text{div}}(\Omega),\\ (\nabla\cdot\mathbf{u},\ l) &= (q,\ l) && \forall l \in L^2(\Omega).\end{aligned}$$

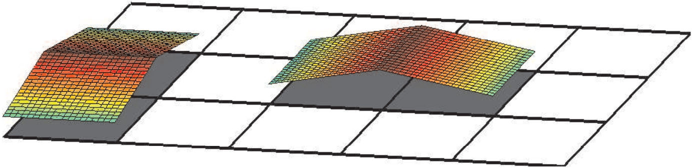
- Multiscale MFEM: Seek solution in finite-dimensional subspaces $\mathbf{V}_{H,h} \times W_{H,h} \subset H_0^{1,\text{div}}(\Omega) \times L^2(\Omega)$.
- Coarse-scale approximation spaces constructed from local numerical solutions.



Two Standard Mixed Finite-Element Methods

Lowest order Raviart-Thomas method (RT0):

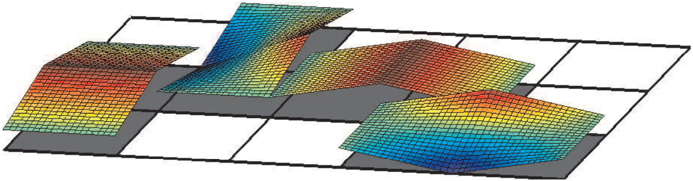
- W_h^{RT0}: Piecewise constants.
- $\mathbf{V}_h^{\text{RT0}}$: Interface normal velocity is constant.



Two Standard Mixed Finite-Element Methods

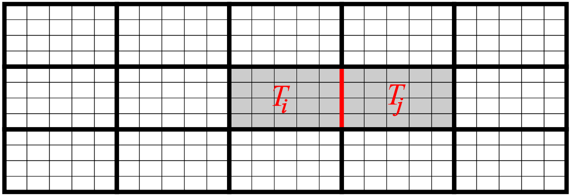
Lowest order Brezzi-Douglas-Marini method (BDM1):

- W_h^{BDM1}: Piecewise constants.
- $\mathbf{V}_h^{\text{BDM1}}$: Interface normal velocity is linear.



In particular: $\mathbf{V}_h^{RT0} \subset \mathbf{V}_h^{BDM1}$.

MsMFEM is a generalization of the lowest order Raviart-Thomas (RT0) method on the coarse mesh:



- For each coarse cell T_i, there is a constant basis function for pressure, $\phi_i \in W_{H,h}$.
- For each coarse edge Γ_{ij}, there is a generalized basis function for velocity, $\psi_{ij} \in \mathbf{V}_{H,h}$.

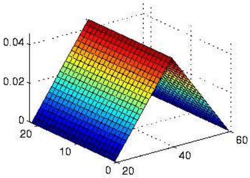
The Multiscale Mixed Finite-Element Method (2)

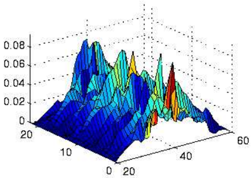
The velocity basis functions ψ_{ij} are constructed by solving,

$$\nabla \cdot \boldsymbol{\psi}_{\mathbf{ij}} = \begin{cases} w_i(x) / \int_{T_i} w_i(\xi)\, \mathrm{d}\xi, & \text{for } x \in T_i, \\ -w_j(x) / \int_{T_j} w_i(\xi)\, \mathrm{d}\xi, & \text{for } x \in T_j, \end{cases}$$

subject to no-flow conditions in each two-element domain, for some weight function $w_i(x)$.

For constant coefficients these basis functions reduce to the standard RT0 velocity basis functions.





The Numerical Subgrid Upscaling Method

Instead of generalizing standard MFEM basis functions, NSUM includes localized subgrid variations in the approximation spaces:

$$W_{H,h} = W_H \bigoplus_{T_i \in \mathcal{T}_H(\Omega)} W_h(T_i) = W_H \oplus W_h,$$

$$\mathbf{V}_{H,h} = \mathbf{V}_H \bigoplus_{T_i \in \mathcal{T}_H(\Omega)} \mathbf{V}_h(T_i) = \mathbf{V}_H \oplus \mathbf{V}_h.$$

- Both the coarse- and fine-scale spaces can be any standard MFEM spaces.
- The most common choices are BDM1 on the coarse scale and RT0 on the fine scale.

The Numerical Subgrid Upscaling Method (2)

Localization, i.e.,

$$\mathbf{v_h} \cdot \mathbf{n} = 0, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h(T_i)$$

allows a decoupling of the fine and coarse scales:

- First, a subgrid correction to each coarse-scale basis function is computed.
- The final solution can then be obtained by solving a single coarse-scale system.

However:

- Localization limits coarse inter-element flow to be determined by the coarse-scale basis only.

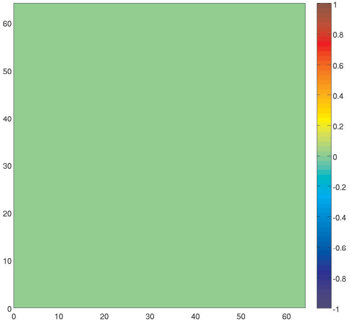
Combining the Two Methods

Starting from the BDM1/RT0 version of NSUM:

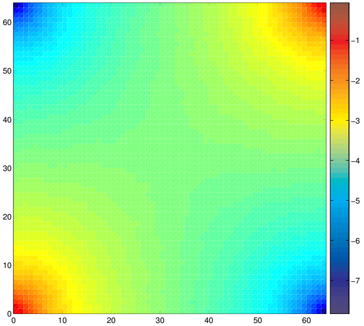
- Remember that $\mathbf{V}^{RT0} \subset \mathbf{V}^{BDM1}$.
- Replace the RT0 part of the NSUM coarse-scale (BDM1) velocity space by the generalized MsMFEM basis $\{\psi_{ij}\}$.
- The NSUM subgrid corrections to the generalized basis functions will be zero.
- Computational cost $\approx$ cost of NSUM.
- Same approach can be used for any MFEM M on the coarse scale, as long as $\mathbf{V}^{RT0} \subset \mathbf{V}^{M}$.

Homogeneous Model

- Homogeneous quarter five-spot.
- Fine mesh: 64×64 cells of unit size.
- Reference solution obtained on a $4\times$ refined mesh.



$\log|\mathbf{K}|$



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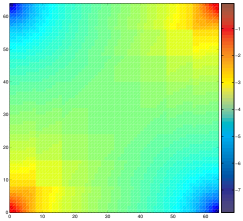
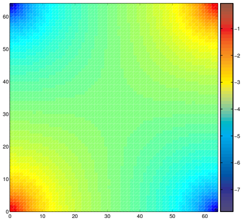
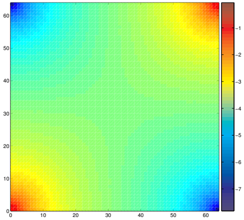
Multiscale velocity solutions on an 8×8 coarse mesh:



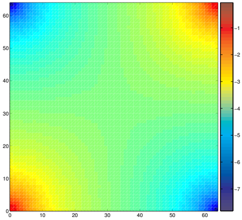
MsMFEM



NSUM



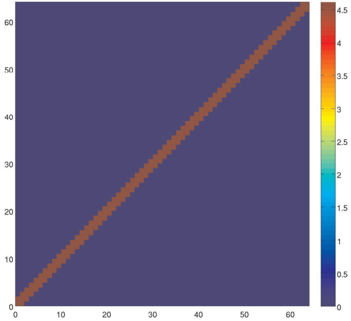
Ms-NSUM



Ms-BDM1

Diagonal Channel

- Diagonal channel, quarter five-spot configuration.
- Fine mesh: 64×64 cells of unit size.
- Reference solution obtained on a $4\times$ refined mesh.



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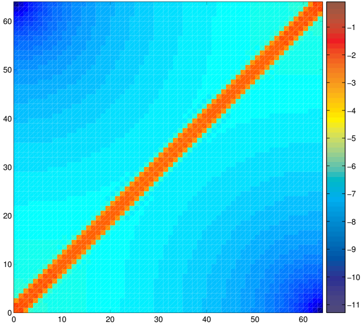

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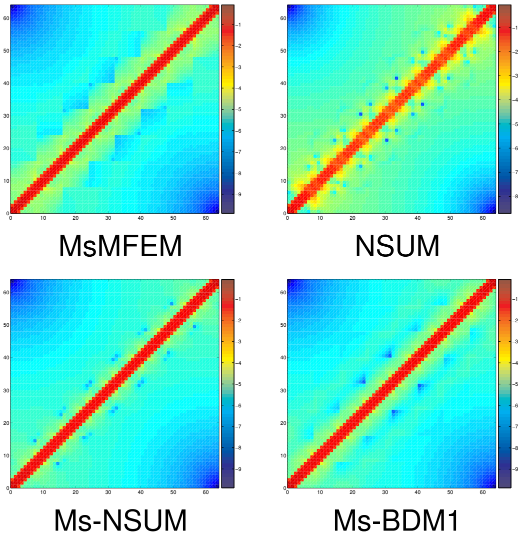
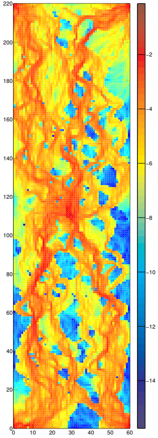
Multiscale velocity solutions on an 8×8 coarse mesh:



Fluvial Model Slice



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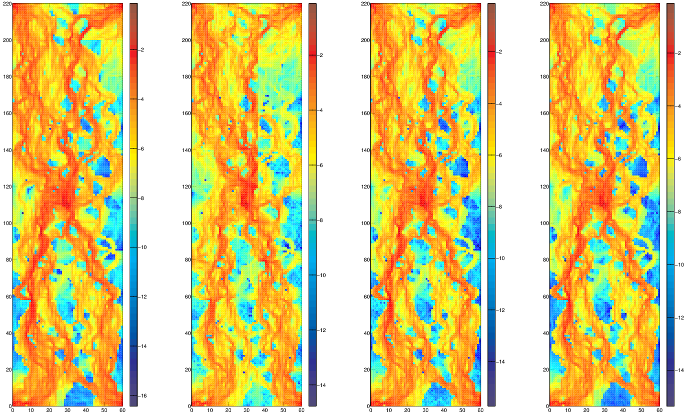
- Slice from Tenth SPE Comparison Project, Model 2.
- Fine mesh: 220×60 cells, here scaled to unit size.
- Reference solution obtained on a $4\times$ refined mesh.

Fluvial Model Slice (2)

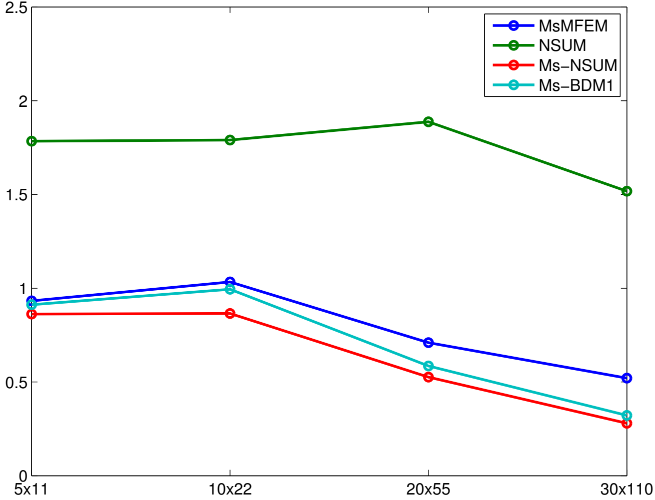
Multiscale velocity solutions on a 5×11 coarse mesh:



MsMFEM NSUM Ms-NSUM Ms-BDM1

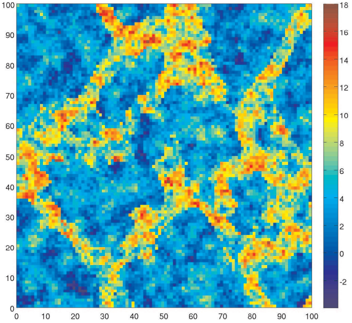
Fluvial Model Slice (3)

Velocity error as a function of coarse-mesh:



Spatiatlly Correlated Log-Normal Permeability

- Spatially correlated log-normal permeability.
- 100 realizations.
- Quarter five-spot configuration.
- Fine mesh: 100×100 cells of unit size.
- Reference solution obtained on a $4\times$ refined mesh.



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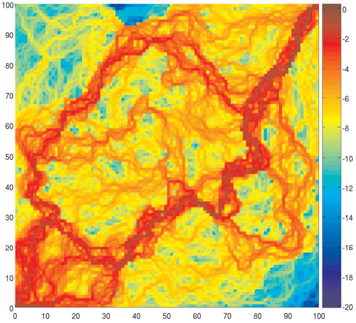

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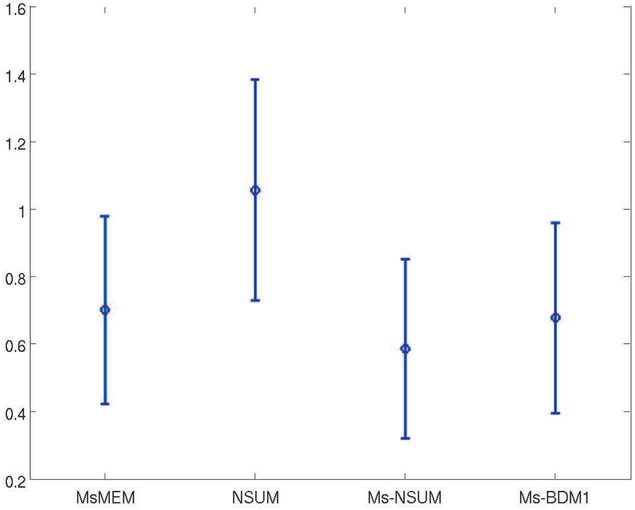
Mean and std.dev. of velocity error (10×10 coarse mesh):



Concluding Remarks

By introducing the MsMFEM generalized basis functions into the NSUM framework, we obtain a family of multiscale finite-element methods that:

- Correctly represents inter-element flow near particular features such as channels.
- Are less sensitive to grid-orientation effects.
- Are easy to generalize to higher order for relatively smooth problems.
- Provide opportunities for adaptive schemes based on accuracy/efficiency trade-offs.