Multiscale Mixed Finite-Element Methods for Simulation of Flow in Highly Heterogeneous Porous Media

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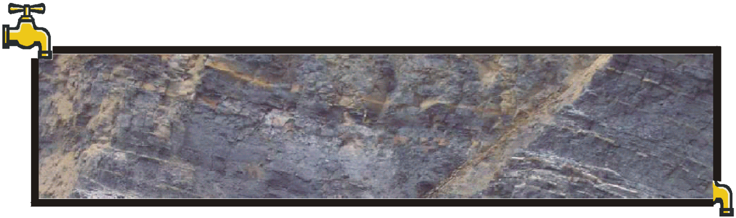


Pressure equation:

$$-\nabla \cdot \mathbf{v} = q, \qquad \mathbf{v} = -\mathbf{K}(\mathbf{x})\lambda(S)\nabla p,$$

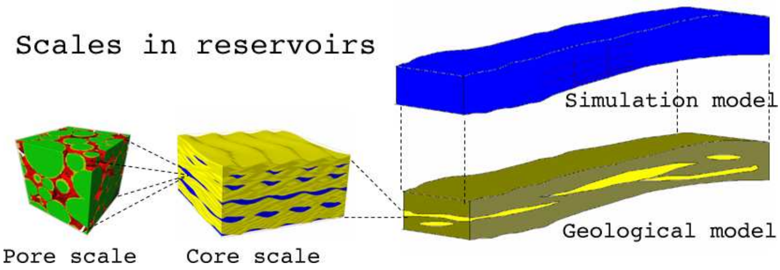
Fluid transport:

$$\phi \partial_t S + \nabla \cdot (\mathbf{v} f(S)) = \varepsilon \nabla \big(\mathbf{D}(S, \mathbf{x}) \nabla S\big)$$

Physical Scales in Porous Media Flow

The scales that impact fluid flow in oil reservoirs range from

- the micrometer scale of pores and pore channels
- via dm–m scale of well bores and laminae sediments
- to sedimentary structures that stretch across entire reservoirs.

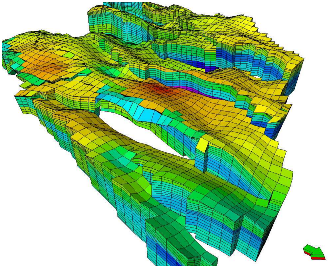


Geomodels consist of geometry and rock parameters (permeability $\mathbf{K}$ and porosity ϕ):

- $\mathbf{K}$ spans many length scales and has multiscale structure

$$\max \mathbf{K} / \min \mathbf{K} \sim 10^3\text{–}10^{10}$$

- Details on all scales impact flow



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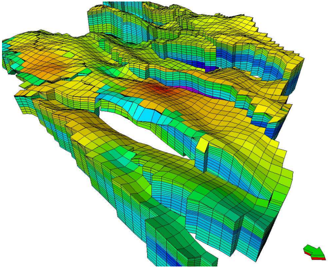


Gap between simulation models and geomodels:

- High-resolution geomodels may have $10^7 - 10^9$ cells
- Conventional simulators are capable of about $10^5 - 10^6$ cells

Traditional solution: upscaling of parameters

Upscaling the Pressure Equation

Upscaling: combine cells to derive coarse grid, determine effective cell properties

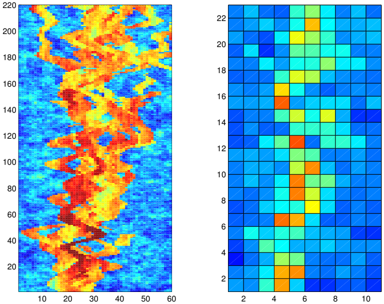
Assume that p satisfies the elliptic PDE:

$$-\nabla\big(K(x)\nabla p\big) = q.$$

Upscaling amounts to finding a new field $K^*(\bar{x})$ on a coarser grid such that

$$-\nabla\big(K^*(\bar{x})\nabla p^*\big) = \bar{q},$$

$$p^* \sim \bar{p}, \qquad \mathbf{v}^* \sim \bar{\mathbf{v}}\ .$$



Here the overbar denotes averaged quantities on a coarse grid.

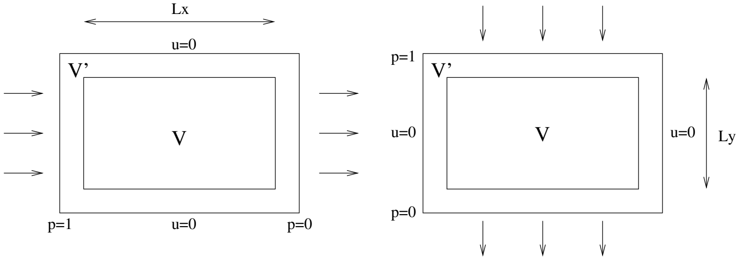
Upscaling the Pressure Equation, cont'd

How do we represent fine-scale heterogeneities on a coarse scale?

- Arithmetic, geometric, harmonic, or power averaging

$$K^* = \left(\frac{1}{|V|} \int_V K(x)^p \, dx \right)^{1/p}$$

- Equivalent permeabilities ($K^*_{xx} = -Q_x L_x / \Delta P_x$)



Vision — Fast Simulation of Geomodels

Vision:

Direct simulation of fluid flow on high-resolution geomodels of highly heterogeneous and fractured porous media in 3D.

Why multiscale methods?

Small-scale variations in the permeability can have a strong impact on large-scale flow and should be resolved properly. Observation:

- the pressure may be well resolved on a coarse grid
- the fluid transport should be solved on the finest scale possible

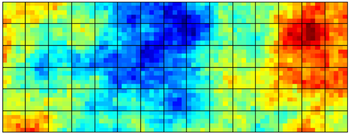
Thus: a multiscale method for the pressure equation should provide velocity fields that can be used to simulate flow on a fine scale

Developing an Alternative to Upscaling

We seek a multiscale methodology that:

- incorporates small-scale effects into the discretisation on a coarse scale
- gives a detailed image of the flow pattern on the fine scale, without having to solve the full fine-scale system
- is robust and flexible with respect to the coarse grid
- is robust and flexible with respect to the fine grid and the fine-grid solver
- is accurate and conservative
- is fast and easy to parallelise

Standard upscaling:



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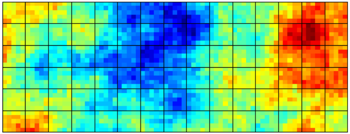

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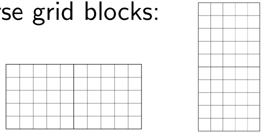
Coarse grid blocks:



From Upscaling to Multiscale Methods

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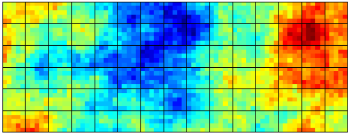

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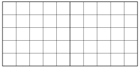

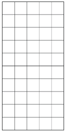

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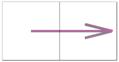
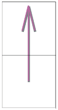
Flow problems:





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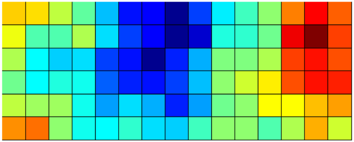

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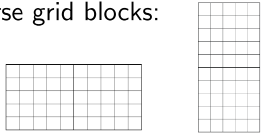

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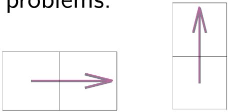

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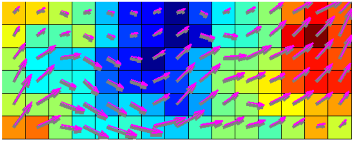

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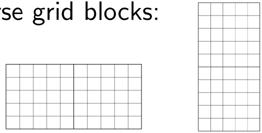

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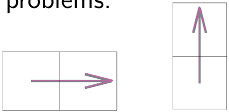

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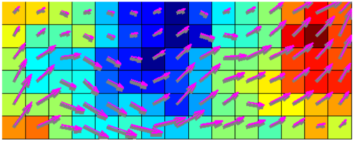

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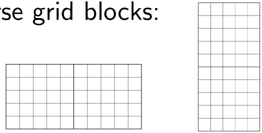

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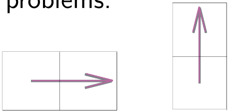
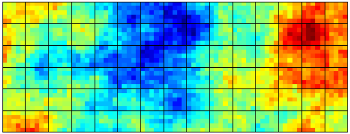


Multiscale method:



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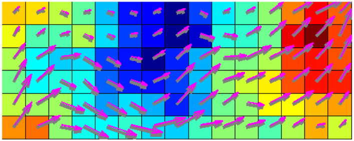

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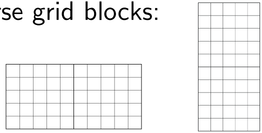

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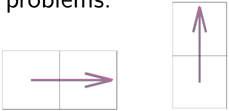

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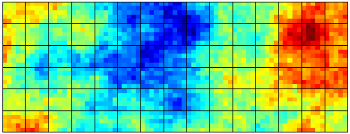

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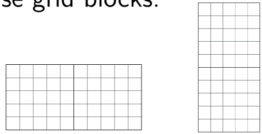

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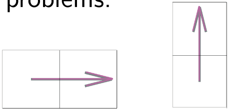

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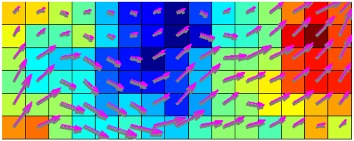

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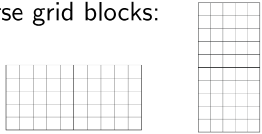

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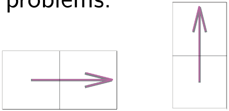

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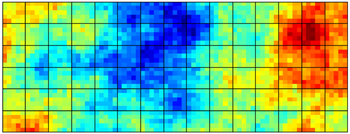

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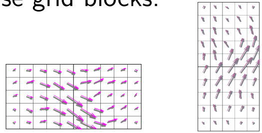

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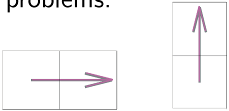

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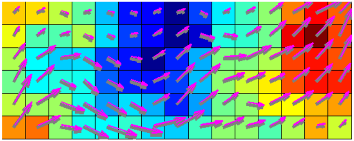

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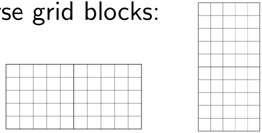

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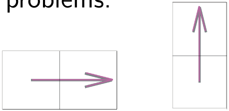

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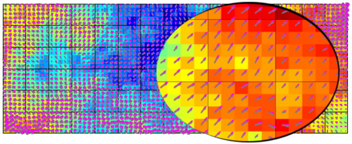

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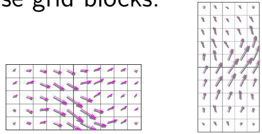

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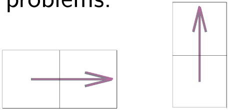


Mixed formulation:

Find $(v, p) \in H_0^{1,\text{div}} \times L^2$ such that

$$\int (\lambda K)^{-1} u \cdot v \, dx - \int p \nabla \cdot u \, dx = 0, \qquad \forall u \in H_0^{1,\text{div}},$$
$$\int \ell \nabla \cdot v \, dx = \int q \ell \, dx, \quad \forall \ell \in L^2.$$

Multiscale discretisation:

Seek solutions in low-dimensional subspaces

$$U^{ms} \subset H_0^{1,\text{div}} \text{ and } V \in L^2,$$

where local fine-scale properties are incorporated into the basis functions.

$$\begin{pmatrix} B & C \\ C^T & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

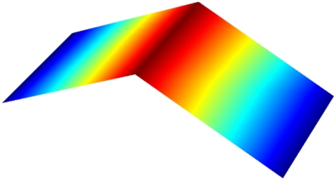
$$b_{ij} = \int_\Omega \psi_i (\lambda K)^{-1} \psi_j \, dx,$$

$$c_{ik} = \int_\Omega \phi_k \nabla \cdot \psi_i \, dx$$

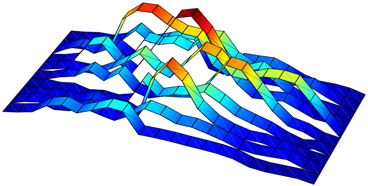
Basis ϕ_k for pressure: equal one in cell k, zero otherwise

Basis ψ_i for velocity:

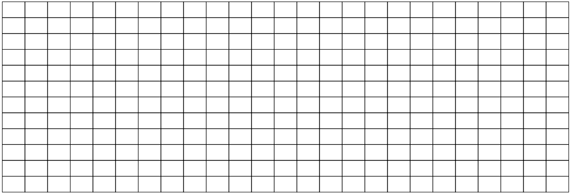
1.order Raviart–Thomas:



Multiscale:



We assume we are given a *fine* grid with permeability and porosity attached to each fine-grid block.



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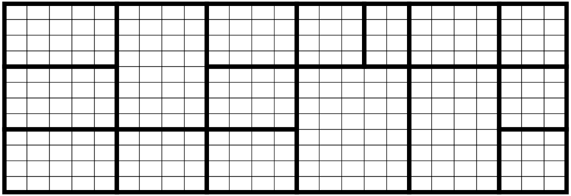

We construct a *coarse* grid, and choose the discretisation spaces V and U^{ms} such that:

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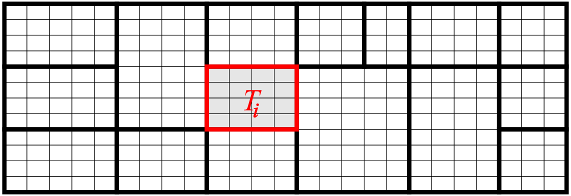

We construct a *coarse* grid, and choose the discretisation spaces V and U^{ms} such that:

- For each coarse block T_i, there is a basis function $\phi_i \in V$.

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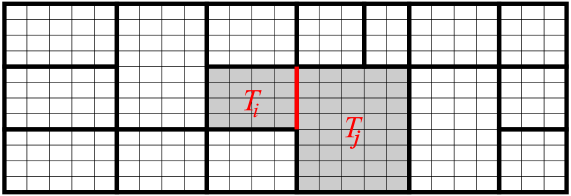


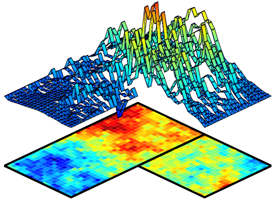
We construct a *coarse* grid, and choose the discretisation spaces V and U^{ms} such that:

- For each coarse block T_i, there is a basis function $\phi_i \in V$.
- For each coarse edge Γ_{ij}, there is a basis function $\psi_{ij} \in U^{ms}$.

For each coarse edge Γ_{ij}, define a basis function

$$\psi_{ij} : T_i \cup T_j \to R^2$$

with unit flux through Γ_{ij} and no flow across $\partial(T_i \cup T_j)$.



We use $\psi_{ij} = -\lambda K \nabla \phi_{ij}$ with

$$\nabla \cdot \psi_{ij} = \begin{cases} w_i(x), & \text{for } x \in T_i, \\ -w_j(x), & \text{for } x \in T_j, \end{cases}$$

with boundary conditions $\psi_{ij} \cdot n = 0$ on $\partial(T_i \cup T_j)$.

Global velocity:

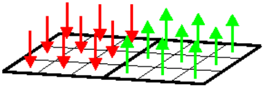
$v = \sum_{ij} v_{ij} \psi_{ij}$, where v_{ij} are (coarse-scale) coefficients.

If T_i contains a source, i.e., $\int_{T_i} q dx \neq 0$, then

$$w_i(x) = \frac{q(x)}{\int_{T_i} q(\xi)\, d\xi}$$

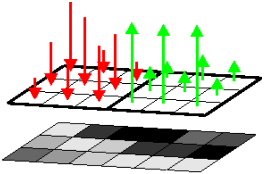
Otherwise we may choose

$$w_i(x) = \frac{1}{|T_i|}$$



or to avoid high flow through low-perm regions

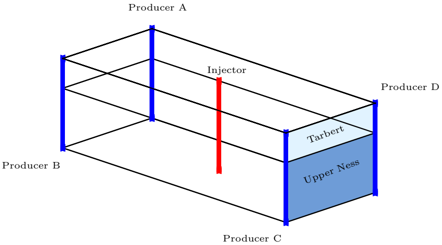
$$w_i(x) = \frac{\text{trace}(K(x))}{\int_{T_i} \text{trace}(K(\xi))\, d\xi}$$

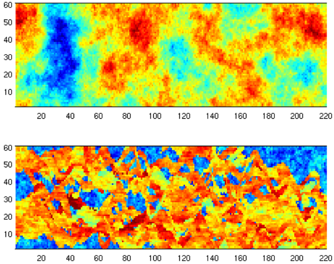


The latter is more accurate - even for strong anisotropy.

Advantage: Accuracy

10th SPE Comparative Solution Project

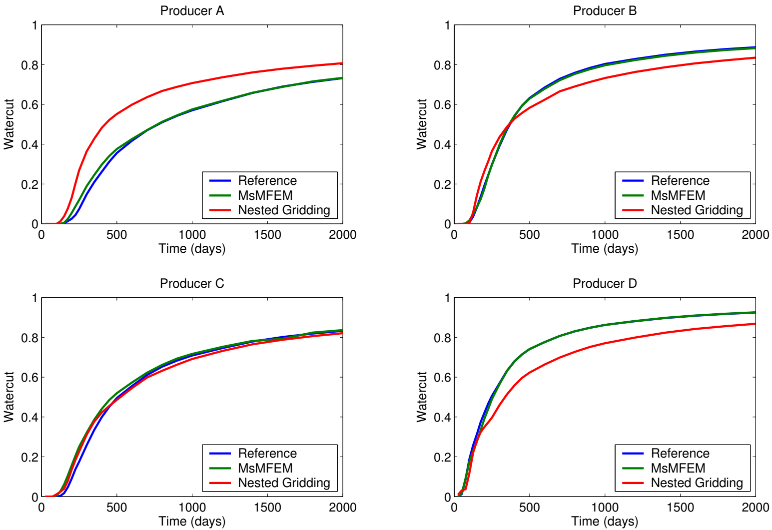




- Geomodel: $60 \times 220 \times 85 \approx 1,1$ million grid cells
- Simulation: 2000 days of production

Advantage: Accuracy

SPE10 Benchmark ($5 \times 11 \times 17$ Coarse Grid)

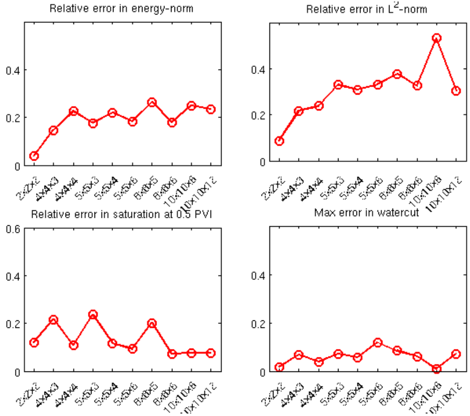


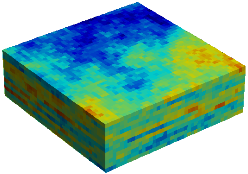
Nested gridding: upscaling + downscaling

Question:

Does refining the coarse grid increase accuracy?

Error-measures for various coarse mesh-sizes.



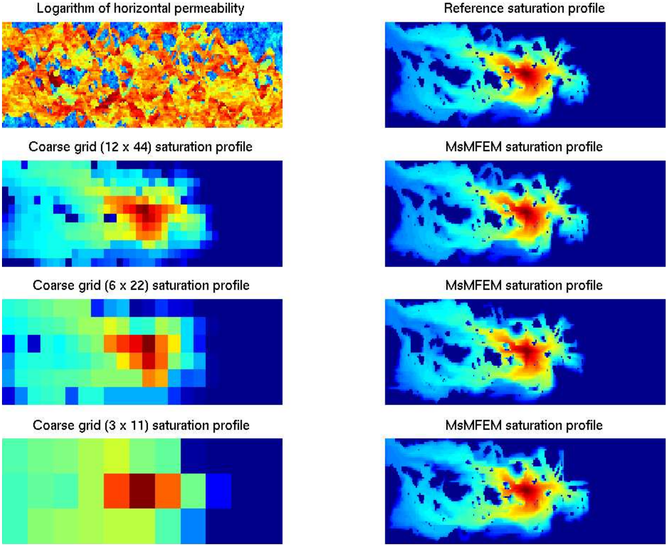


Chen and Hou 2002: error is bounded by

$$\mathcal{O}(H + \sqrt{\epsilon} + \sqrt{\frac{\epsilon}{H}})$$

Advantage: Robustness

SPE10, Layer 85 (60×220 Grid)



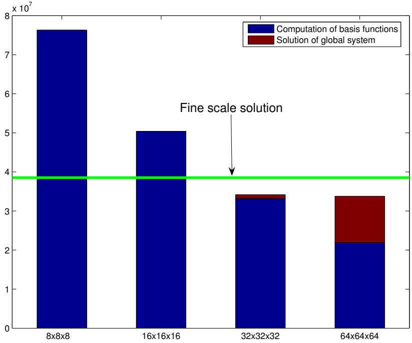
Computational Complexity

Direct solution may be more efficient, so why bother with multiscale?

- Full simulation: $\mathcal{O}(10^2)$ time steps.
- Basis functions need not be recomputed

Also:

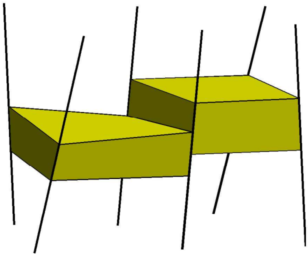
- Possible to solve very large problems
- Easy parallelization

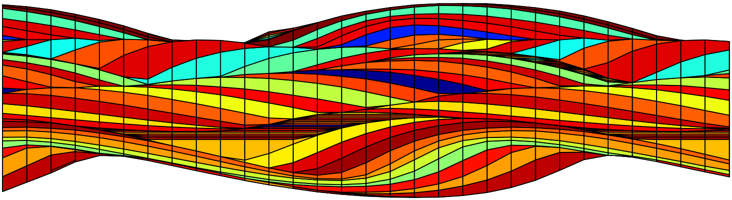


Order-of-magnitude argument: $128 \times 128 \times 128$ grid, linear algebra of complexity $\mathcal{O}(n^{1.2})$.

Specified in terms of:

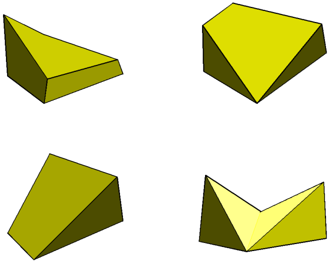
- areal 2D mesh of vertical or inclined pillars
- each volumetric cell is restriced by four pillars
- each cell is defined by eight corner points, two on each pillar



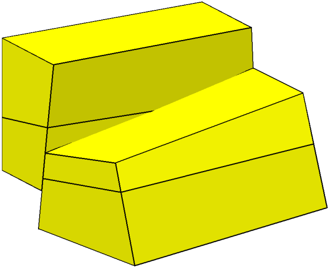


Accurate simulation of industry-standard grid models is challenging!

Skew and deformed grid blocks:



Non-matching cells:



Our approach: mixed finite elements and/or mimetic methods

I. Aavatsmark (after lunch): multipoint finite-volume methods

Discretisation on General Subgrids

- Can use standard mixed FEM for many geometries provided that one has
  - mappings (Piola transforms)
  - reference elements
- Subdivision of corner-point cells into tetrahedra
- Mimetic finite differences (recent work by Brezzi, Lipnikov, Shashkov, Simoncini)

Mimetic Finite Differences

Let u, v be piecewise linear vector functions and $\mathbf{u}$, $\mathbf{v}$ be the corresponding vectors of discrete velocities over faces in the grid, i.e.,

$$\mathbf{v}_k = \frac{1}{|e_k|} \int_{e_k} v(s) \cdot n \, ds$$

Then the block $\mathbf{B}$ in the mixed system satisfies

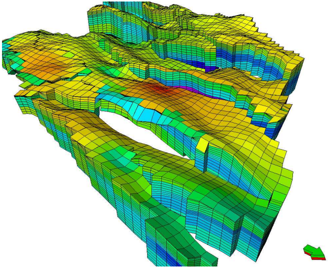
$$\int_\Omega v^T K^{-1} u = \mathbf{v}^T \mathbf{B} \mathbf{u} \qquad \left(= \sum_{E \in \Omega} \mathbf{v}_E^T \mathbf{B}_E \mathbf{u}_E \right)$$

The matrices $\mathbf{B}_E$ define *discrete inner products*

Mimetic idea:

Replace $\mathbf{B}_E$ with some $\mathbf{M}_E$ that mimics some properties of the continuous inner product (SPD, globally bounded, Gauss-Green for linear pressure)

Multiscale Mixed FEM on Corner-Point Grids



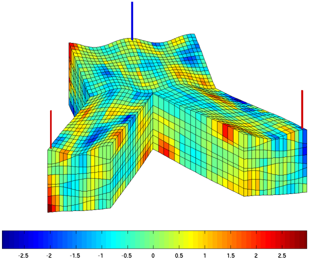
Subdivision strategy:

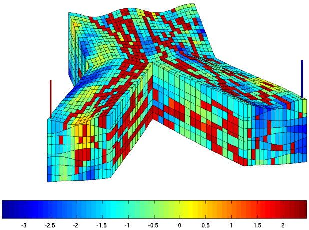
- implicitly assumes each face to be piecewise planar
- must split every non-degenerate cell in six (or five) tetrahedrons

Mimetic strategy:

- either assume faces piecewise planer or curved
- one degree of freedom per moderately curved face
- easy to deal with non-matching faces
- the discrete inner product can be used on the coarse scale in conjunction with any subgrid solver

Accuracy on a 32 × 32 × 8 Model





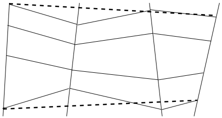
RT0 MFEM Coarse grid	Homogeneous $e(v)$	$e(s)$	Log-normal $e(v)$	$e(s)$	Fluvial $e(v)$	$e(s)$
16 × 16 × 4	0.1233	0.0161	0.1915	0.0495	0.4069	0.2299
8 × 8 × 2	0.1300	0.0206	0.2769	0.1087	0.3845	0.2904
4 × 4 × 1	0.1070	0.0225	0.2135	0.1501	0.3046	0.2420
2 × 2 × 1	0.0112	0.0090	0.1111	0.0724	0.1564	0.0680
Mimetic FDM	$e(v)$	$e(s)$	$e(v)$	$e(s)$	$e(v)$	$e(s)$
16 × 16 × 4	0.1152	0.0193	0.1963	0.0532	0.4143	0.2278
8 × 8 × 2	0.1282	0.0213	0.3174	0.1157	0.4742	0.3607
4 × 4 × 1	0.1070	0.0249	0.2212	0.1582	0.3119	0.2442
2 × 2 × 1	0.0111	0.0103	0.1214	0.0751	0.1589	0.0679

Advantage:Flexibility

Multiscale mixed/mimetic formulation:

coarse grid = union of cells in fine grid

- Given a numerical method that works on the fine grid, the implementation is straightforward.
- One avoids resampling when going from fine to coarse grid, and vice versa

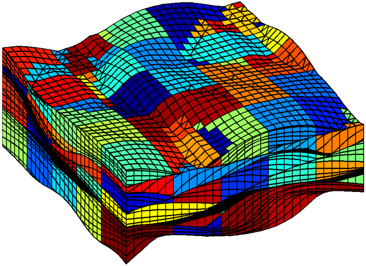


Other formulations:

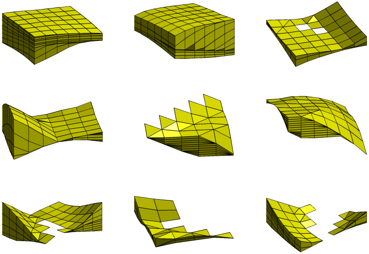
Finite-volume methods: based upon *dual grid* $\longrightarrow$ special cases that complicate the implementation in the presence of faults, local refinements, etc.

Flexibility wrt. Grids

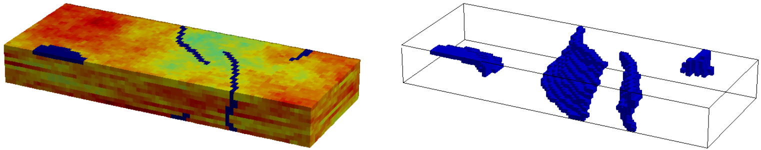
Corner-point grid model:

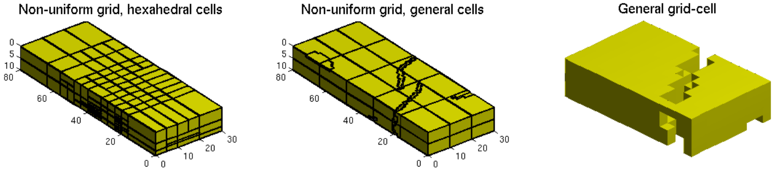


Coarse-grid cells:

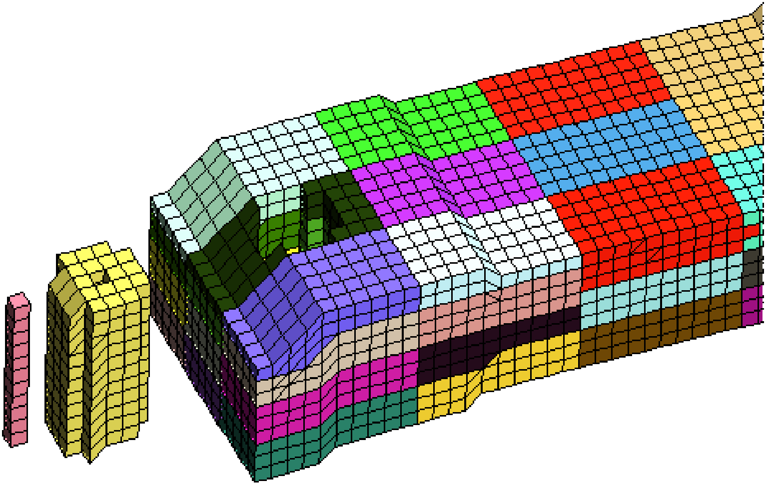


Flexibility wrt. Grids
Around Flow Barriers, Fractures, etc



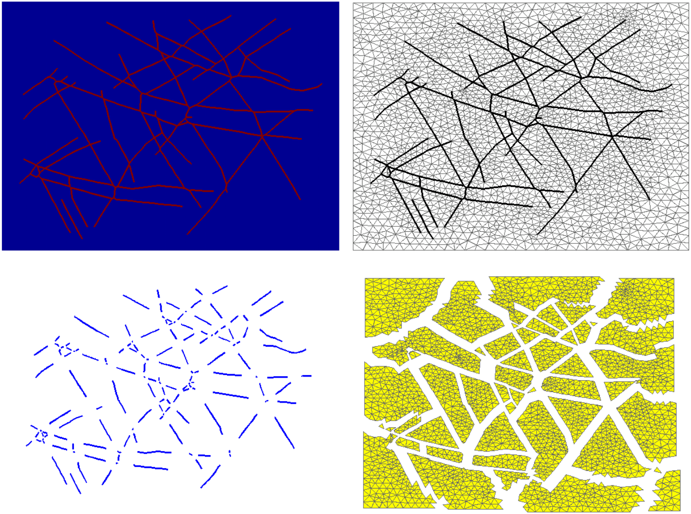


Flexibility wrt. Grids
Around Wells



Future work

Extension to faults and fracture networks


[1]

[1] Grid model courtesy of M. Karimi-Fard, Stanford