Multiscale mixed FEM on general coarse- and fine grids

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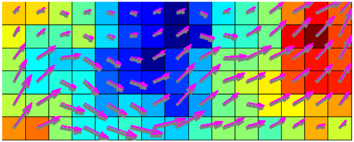
Outline

Standard method:

Upscaled model:



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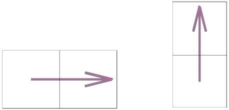
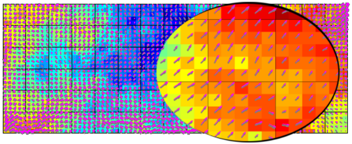
Building blocks:



Two-scale method:

Geomodel:



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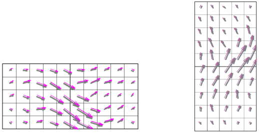

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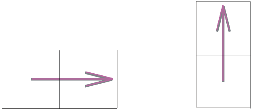


Why multiscale?

Small scale variations in the permeability can have a strong impact on large scale flow and should be resolved properly.

- the pressure may be well resolved on a coarse grid
- the fluid transport should be solved on the finest scale possible

Thus: a multiscale method for the pressure equation should provide velocity fields that can be used to simulate flow on a fine scale.

Model equations

Elliptic pressure equation:

$$v = -\lambda(S)K\nabla p$$
$$\nabla \cdot v = q$$

Hyperbolic saturation equation:

$$\phi \frac{\partial S}{\partial t} + \nabla \cdot (v f(S)) = q_w$$

- Total velocity:

$$v = v_o + v_w$$

- Total mobility:

$$\begin{aligned} \lambda &= \lambda_w(S) + \lambda_o(S) \\ &= k_{rw}(S)/\mu_w + k_{ro}(S)/\mu_o \end{aligned}$$

- Saturation water: S
- Fractional flow water:

$$f(S) = \lambda_w(S)/\lambda(S)$$

Multiscale mixed finite elements

Mixed formulation:

Find $(v,p) \in H_0^{1,\text{div}} \times L^2$ such that

$$\int (\lambda K)^{-1} u \cdot v \, dx - \int p \nabla \cdot u \, dx = 0, \qquad \forall u \in H_0^{1,\text{div}},$$
$$\int l \nabla \cdot v \, dx = \int q l \, dx, \quad \forall l \in L^2.$$

Multiscale discretization:

Seek solutions in low-dimensional subspaces

$$U^{ms} \subset H_0^{1,\text{div}} \text{ and } V \in L^2,$$

where local fine-scale properties are incorporated into the basis functions.

$$\begin{pmatrix} B & C \\ C^T & O \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}$$

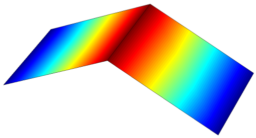
where

$$b_{ij} = \int_\Omega \psi_i^T (\lambda K)^{-1} \psi_j \, dx$$

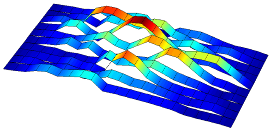
$$c_{ik} = \int_\Omega \phi_k \nabla \cdot \psi_i \, dx$$

Basis for pressure ϕ_k: 1 in cell k, zero otherwise.
Basis for velocity ψ_i:

RT^0



Multiscale



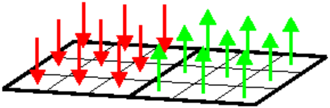
Multiscale basis functions for velocity

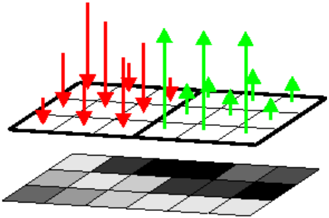
Each basis function ψ is the solution of a local flow-problem over two neighboring cells E_k, E_l: $\psi_{kl} = -\lambda K \nabla \phi_{kl}$ with

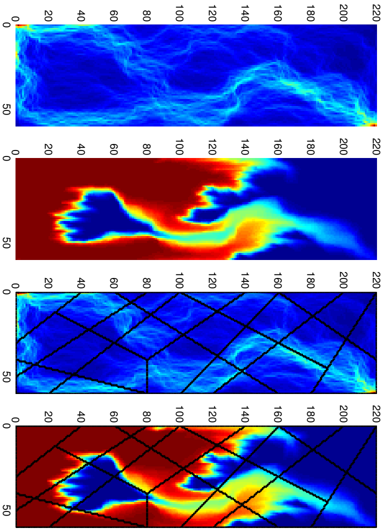
$$\nabla \cdot \psi_{kl} = \begin{cases} w_k(x), & \text{for } x \in E_k \\ -w_l(x), & \text{for } x \in E_l, \end{cases}$$

with BCs $\psi_{kl} \cdot n = 0$ on $\partial(T_i \cup \Gamma_{ij} \cup T_j)$.

Weights w_k, w_l:







Accuracy: flow scenarios match closely fine grid simulations.

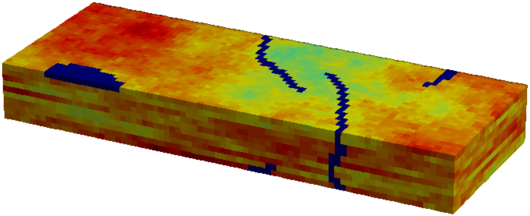
Mass conservation: the method conserves mass on both the coarse and the fine grid.

Efficiency: computation of basis functions can be parallelized, and is done only once (moderate mobilty ratio).

Flexibility: unstructured and irregular grids are handled easily.

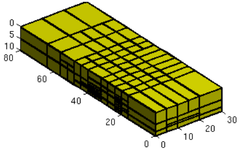
Example: general coarse grid cells

Permeability field:

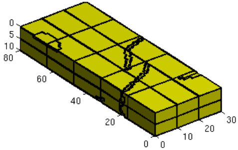


Grids:

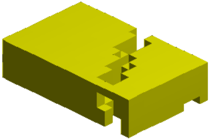
Non-uniform grid, hexahedral cells



Non-uniform grid, general cells

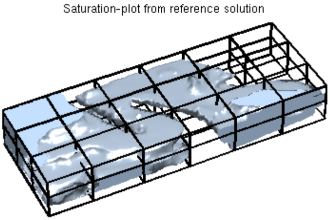


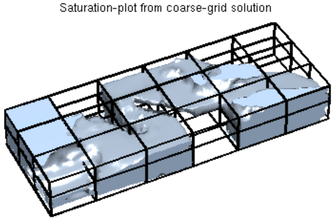
General grid-cell



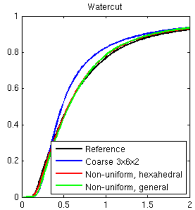
Example: general coarse grids

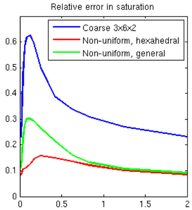
Saturation plots:

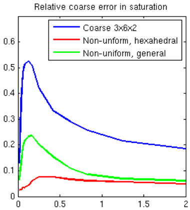




Watercut and saturation errors:







What can be done for general subgrids?

- Can use standard mixed FEM for many geometries. Will need a bunch of mappings (Piola transforms) to a bunch of reference elements.
- Subdivision of elements into tetrahedra (2- or 3-scale).
- Mimetic finite differences (Recent work by Brezzi, Lipnikov, Shashkov, Simoncini).

Mimetic idea

Let u, v be piecewise linear vector functions, and let $\boldsymbol{v}, \boldsymbol{u}$ be the corresponding vectors of the descrete velocities over the faces in our grid, i.e.

$$\boldsymbol{v}_k = \frac{1}{|e_k|} \int_{e_k} v(s) \cdot n \, ds$$

Then the B in the mixed system satisfies

$$\int_\Omega v^T K^{-1} u \, dx = \boldsymbol{v}^T B \boldsymbol{u} \qquad \left(= \sum_{E \in \Omega} \boldsymbol{v}_E B_E \boldsymbol{u}_E \right)$$

The B_E define *discrete inner products*.

Mimetic idea: Exchange B_E with some M_E that *mimics* some properties of the continuous inner product.

Conditions on the discrete inner product (Brezzi et. al.)

Let E be a polyhedron with faces $e_i, i = 1, \dots n_E$, and $\boldsymbol{v}_E, \boldsymbol{u}_E$ be vectors of discrete velocities over the faces e_i.

1. SPD and globally bounded: There exists s_*, S^* such that for every E

$$s_* |E| \boldsymbol{v}_E^T \boldsymbol{v}_E \le \boldsymbol{v}_E^T M_E \boldsymbol{v}_E \le S^* |E| \boldsymbol{v}_E^T \boldsymbol{v}_E$$

2. Gauss-Green for linear pressure: Let p be linear on E, and $\boldsymbol{v}_E$ correspond to $v = K\nabla p$. Then for every $\boldsymbol{u}_E$:

$$\boldsymbol{v}_E^T M_E \boldsymbol{u}_E + \int_E p \sum_{i=1}^{n_E} |e_i| \boldsymbol{u}_{E,i} \, dx = \sum_{i=1}^{n_E} \int_{e_i} p \boldsymbol{u}_{E,i} \, ds$$

$$\int_E (K\nabla q)^T K^{-1} u \, dx + \int_E q \nabla \cdot u \, dx = \int_{\partial E} q u^T n \, dS$$

- Converges for very general polyhedral grids (planar/moderately curved faces).
- Convergence for strongly curved faces requires extra degrees of freedom.

Expressions for the inner product (Brezzi et. al.)

Let origo be the at the centroid of E, and define $(n_E \times d)$-matrices N and R by

$$N(:,i) = \boldsymbol{n}_i^T, \qquad R(:,i) = |e_i|\boldsymbol{c}_i^T,$$

where $\boldsymbol{n}_i$ and $\boldsymbol{c}_i$ are the normal vector and centroid of face e_i respectively.

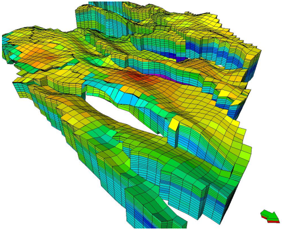
General family of M_E satisfying (1)–(2):

$$M_E = \frac{1}{|E|} R K^{-1} R^T + C U C^T$$

- $n_E \times (n_E - d)$-matrix C spans null space of N^T
- U *any* SPD $(k_E - d) \times (k_E - d)$-matrix.

Similar expression for the *inverse family* for direct use in the hybrid formulation.

Multiscale mixed FEM on corner point grids



Subdivision strategy:

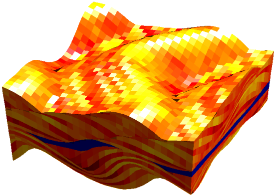
- Implicitly assumes each face is piecewise planar.
- Must split every non-degenerate CP-cell in six tetrahedrons.

Mimetic strategy:

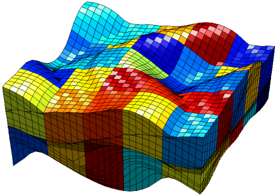
- Either assume faces piecewise planar or curved.
- One degree of freedom per moderately curved CP-face.
- Easy to deal with non-matching faces.
- The discrete inner product can be used on the coarse scale in conjuction with any subgrid solver.

Multiscale mixed FEM on corner point grids

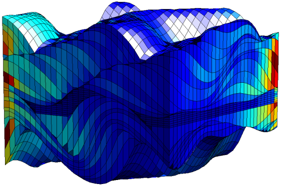
Permeability:



Fine grid/Coarse grid-blocks



Fine scale velocity:



Multiscale velocity: