A Multiscale Mixed Finite-Element Solver for Compressible Black-Oil Flow

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Multiscale Pressure Solvers

Two-level methods for equations:

- with a near-elliptic behavior
- with strongly heterogeneous coefficients
- without scale separations

Aim:

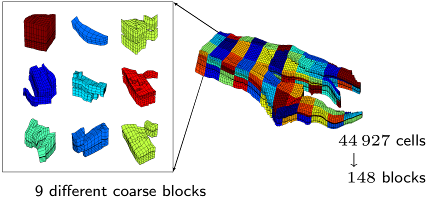
- describe global flow patterns on coarse grid
- accurately account for fine-scale structures

Provide a mechanism to recover approximate fine-scale solutions

The Multiscale Mixed Finite Element (MsMFE) Method

The algorithm in a nutshell

1) Generate coarse grid (automatically)



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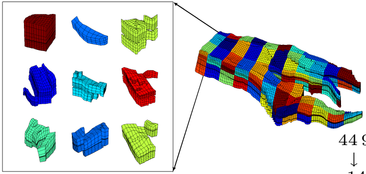
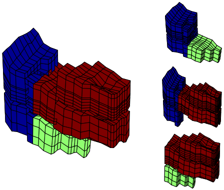


44 927 cells
↓
148 blocks

9 different coarse blocks

2) Detect all adjacent blocks



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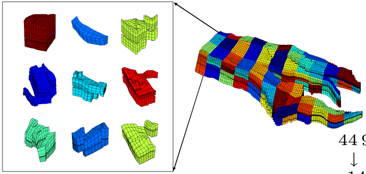

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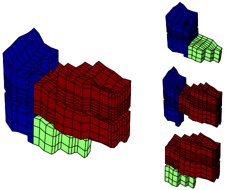
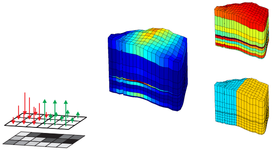


3) Compute basis functions



Solve flow problem for all pairs of blocks

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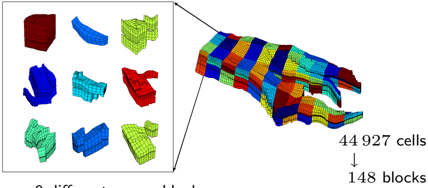

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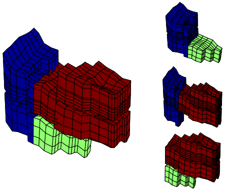

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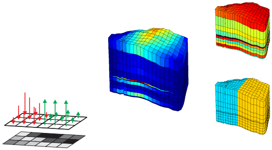
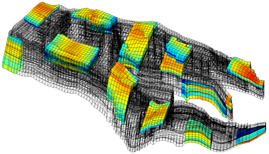


Solve flow problem for all pairs of blocks

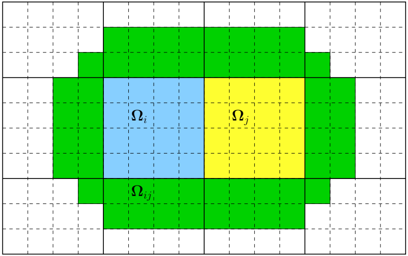
4) Build global solution



Basis functions: building blocks for global solution

The Mixed Finite Element (MsMFE) Method

Computation of multiscale basis functions



Each cell Ω_i: pressure basis ϕ_i
Each face Γ_{ij}: velocity basis ψ_{ij}

$$\vec{\psi}_{ij} = -\lambda \mathbf{K} \nabla \phi_{ij}$$

$$\nabla \cdot \vec{\psi}_{ij} = \begin{cases} w_i(x), & x \in \Omega_i \\ -w_j(x), & x \in \Omega_j \\ 0, & \text{otherwise} \end{cases}$$

The Mixed Finite Element (MsMFE) Method

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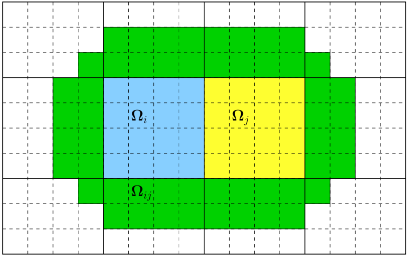

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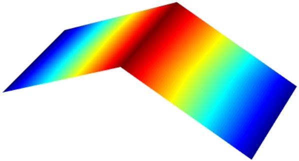
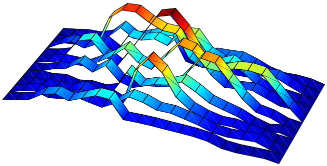
Homogeneous **K**:



Heterogeneous **K**:



The weight function distributes $\nabla \cdot v$ on the coarse blocks:

$$
\begin{aligned}
(\nabla \cdot v)|_{\Omega_i} &= \sum_j \nabla \cdot (v_{ij}\psi_{ij}) = w_i \sum_j v_{ij} \\
&= w_i \int_{\partial\Omega_i} v \cdot n \, ds = w_i \int_{\Omega_i} \nabla \cdot v \, dx
\end{aligned}
$$

Different roles:

Incompressible flow: $\nabla \cdot v = q$

Compressible flow: $\nabla \cdot v = q - c_t \partial_t p - \sum_j c_j v_j \cdot \nabla p$

The Mixed Finite Element (MsMFE) Method

Choice of weight function, $w_i = \theta(x)/\int_{\Omega_i} \theta(x)\,dx$

Incompressible flow:

$$\int_{\Omega_i} q dx = 0, \qquad \theta(x) = \mathrm{trace}(\mathbf{K}(x))$$

$$\int_{\Omega_i} q dx \neq 0, \qquad \theta(x) = q(x)$$

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Compressible flow:

- $\theta \propto q$: compressibility effects concentrated where $q \neq 0$
- $\theta \propto \mathbf{K}$: $\nabla \cdot v$ over/underestimated for high/low $\mathbf{K}$

Another choice motivated by physics:

$$\theta(x) = \phi(x), \qquad \text{Motivation: } c_t \frac{\partial p}{\partial t} \propto \phi$$

The Mixed Finite Element (MsMFE) Method

Key to effiency: reuse of computations

Computational cost consists of:

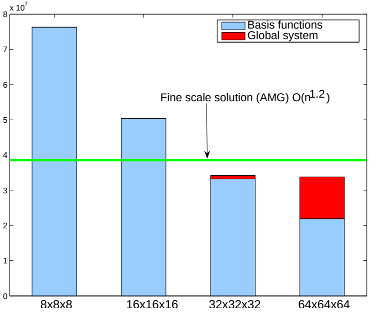
- **basis functions** (fine grid)
- **global problem** (coarse grid)

High efficiency for multiphase flows:

- Elliptic decomposition
- Reuse basis functions
- Easy to parallelize

Example: 128^3 grid

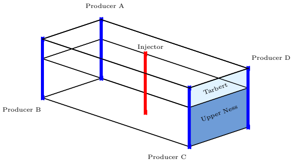
operations versus upscaling factor



The Mixed Finite Element (MsMFE) Method

Recap from 2007 SPE RSS: million-cell models in minutes

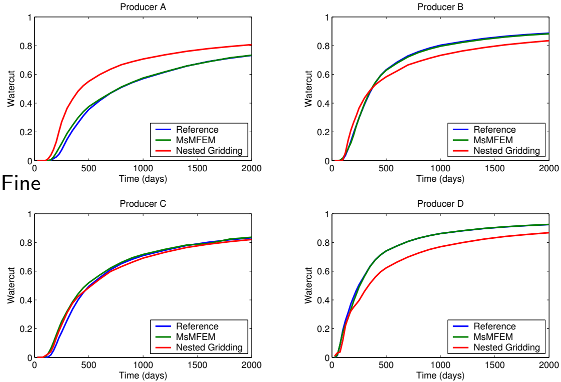
SPE 10, Model 2:



grid: $60 \times 220 \times 85$
Coarse grid: $5 \times 11 \times 17$
2000 days production
25 time steps

multiscale + streamlines:
142 sec on a 2.4 GHz PC

Water-cut curves at the four producers

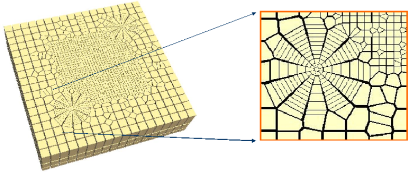


upscaling/downscaling, multiscale, fine grid

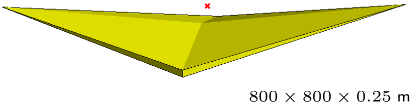
MsMFE for Complex Grids

Challenges posed by grids from real-life models

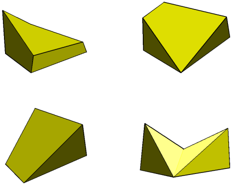
Unstructured grids:



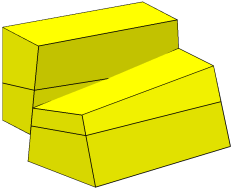
(Very) high aspect ratios:



Skewed and degenerate cells:



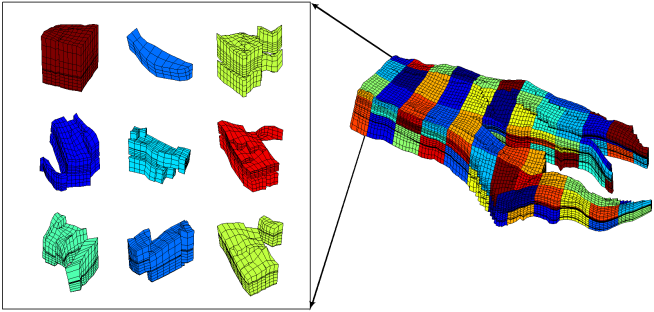
Non-matching cells:



MsMFE for Complex Grids

Applicable to general unstructured grids

Coarse blocks: (arbitrary) connected collection of cells
$\longrightarrow$ fully automated coarsening strategies



Coarse blocks: logically Cartesian in index space

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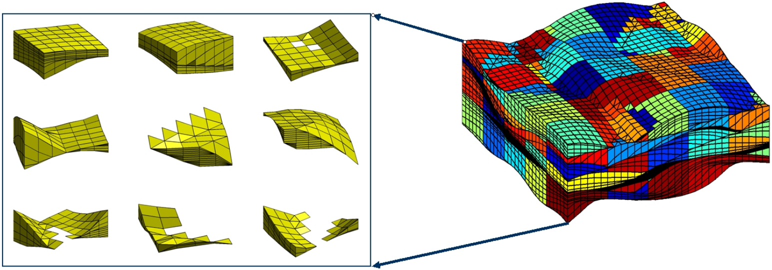

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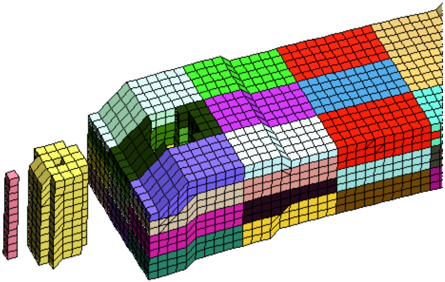
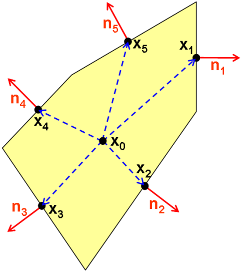


MsMFE for Complex Grids

Fine-grid formulation

Discretization using a mimetic method (Brezzi et al):

$$\boldsymbol{u}_E = \lambda \boldsymbol{T}_E (p_E - \boldsymbol{\pi}_E),$$
$$\boldsymbol{T}_E = |E|^{-1} \boldsymbol{N}_E \mathbf{K}_E \boldsymbol{N}_E^{\mathsf{T}} + \tilde{\boldsymbol{T}}_E$$



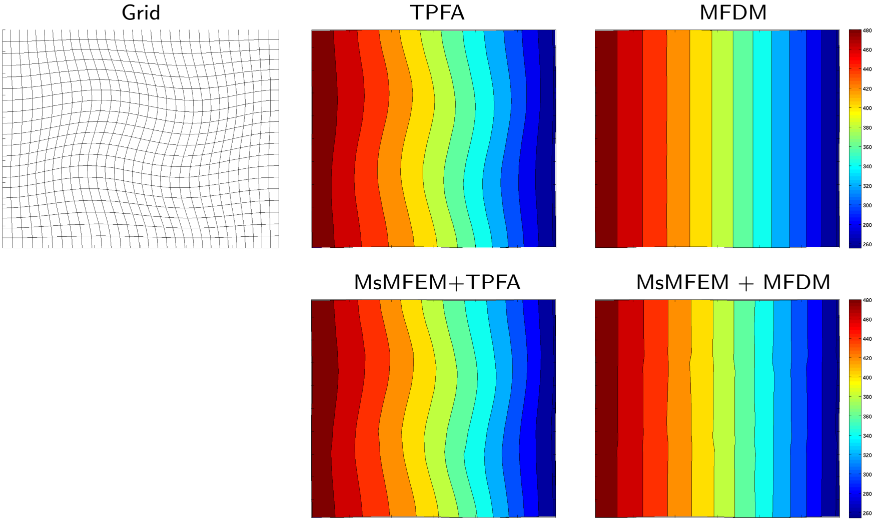
$\boldsymbol{N}_E$: face normals
$\boldsymbol{X}_E$: vector from face to cell centroids
$\tilde{\boldsymbol{T}}_E$: arbitrarily such that $\tilde{\boldsymbol{T}}_E \boldsymbol{X}_E = 0$

Key features:

- Applicable for general polyhedral cells
- Non-conforming grids treated as conforming polyhedral
- Generic implementation for all grid types
- Monotonicity as for MPFA

MsMFE for Complex Grids

Example: single phase, homogeneous **K**, linear pressure drop



Pressure equation:

$$c\frac{\partial p}{\partial t} + \nabla \cdot \vec{u} - \zeta \vec{u} \cdot \mathbf{K}^{-1}\vec{u} = q, \quad \vec{u} = -\mathbf{K}\lambda\nabla p$$

Time-discretization and linearization:

$$c_{\nu-1}\frac{p_{\nu}^{n} - p^{n-1}}{\Delta t} + \nabla \cdot \vec{u}_{\nu}^{n} - \zeta_{\nu-1}^{n}\vec{u}_{\nu-1}^{n} \cdot \mathbf{K}^{-1}\vec{u}_{\nu}^{n} = q$$

Hybrid system:

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{C}^{\mathsf{T}} - \boldsymbol{V}_{\nu-1}^{\mathsf{T}} & \boldsymbol{P}_{\nu-1} & \boldsymbol{0} \\ \boldsymbol{D}^{\mathsf{T}} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\nu} \\ -\boldsymbol{p}_{\nu} \\ \boldsymbol{\pi}_{\nu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{P}_{\nu-1}\boldsymbol{p}^{n-1} + \boldsymbol{q} \\ \boldsymbol{0} \end{bmatrix}$$

MsMFE for Compressible Black-Oil Models

Coarse-grid formulation

$$\begin{bmatrix} \boldsymbol{\Psi}^{\mathsf{T}} \boldsymbol{B} \boldsymbol{\Psi} & \boldsymbol{\Psi}^{\mathsf{T}} \boldsymbol{C} \boldsymbol{\mathcal{I}} & \boldsymbol{\Psi}^{\mathsf{T}} \boldsymbol{D} \boldsymbol{\mathcal{J}} \\ \widetilde{\boldsymbol{C}}^{\mathsf{T}} & \boldsymbol{\mathcal{I}}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{\mathcal{I}} & \boldsymbol{0} \\ \boldsymbol{\mathcal{J}}^{\mathsf{T}} \boldsymbol{D}^{\mathsf{T}} \boldsymbol{\Psi} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ -\boldsymbol{p} \\ \boldsymbol{\pi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\mathcal{I}}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{p}_f^n \\ \boldsymbol{0} \end{bmatrix}$$

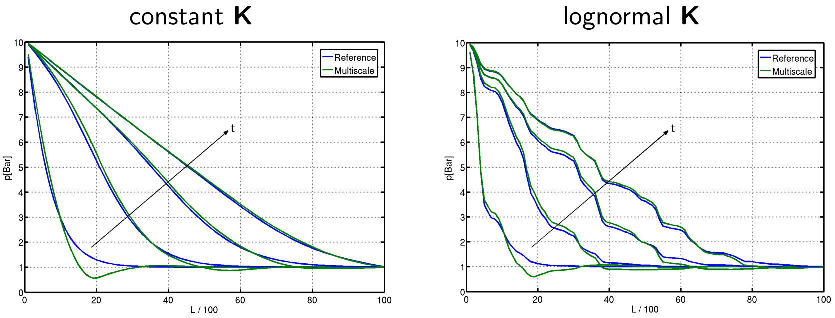
- $\boldsymbol{\Psi}$ – velocity basis functions
- $\boldsymbol{\Phi}$ – pressure basis functions
- $\boldsymbol{\mathcal{I}}$ – prolongation from blocks to cells
- $\boldsymbol{\mathcal{J}}$ – prolongation from block faces to cell faces
- $\widetilde{\boldsymbol{C}} = \boldsymbol{\Psi}^{\mathsf{T}} (\boldsymbol{C} - \boldsymbol{V}) \boldsymbol{\mathcal{I}} - \boldsymbol{D}_\lambda \boldsymbol{\Phi}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{\mathcal{I}}$

New feature: fine-scale pressure

$$\boldsymbol{u}_f \approx \boldsymbol{\Psi} \boldsymbol{u}, \qquad \boldsymbol{p}_f \approx \boldsymbol{\mathcal{I}} \boldsymbol{p} + \boldsymbol{\Phi} \boldsymbol{D}_\lambda \boldsymbol{u}, \quad \boldsymbol{D}_\lambda = \mathrm{diag}(\lambda_i^0 / \lambda_i)$$

MsMFE for Compressible Black-Oil Models

Example 1: tracer transport in gas (Lunati&Jenny 2006)

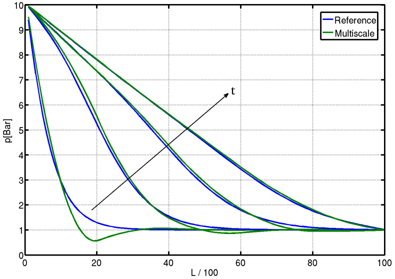


$p(0,t) = 1$ bar, $p(x,0) = 10$ bar, coarse grid: 5 blocks, fine grid: 100 cells

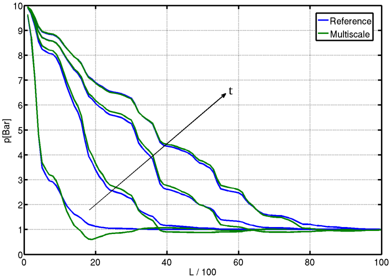
MsMFE for Compressible Black-Oil Models

Example 1: tracer transport in gas (Lunati&Jenny 2006)

constant **K**



lognormal **K**



$p(0,t) = 1$ bar, $p(x,0) = 10$ bar, coarse grid: 5 blocks, fine grid: 100 cells

Remedy: correction functions (Lunati, Jenny et al; Nordbotten)

Approximate residual equation by

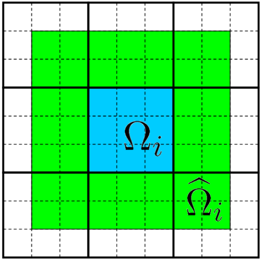
$$\hat{\boldsymbol{u}} = \sum_{\Omega_i \subset \Omega} \hat{\boldsymbol{u}}_i, \quad \hat{\boldsymbol{p}} = \sum_{\Omega_i \subset \Omega} \hat{\boldsymbol{p}}_i,$$

such that $\boldsymbol{u} \approx \boldsymbol{u}_{\text{ms}} + \hat{\boldsymbol{u}}$ and $\boldsymbol{p} \approx \boldsymbol{p}_{\text{ms}} + \hat{\boldsymbol{p}}$.

Local problems:

$(\hat{\boldsymbol{u}}_i, \hat{\boldsymbol{p}}_i)$ solves residual equation locally in $\widehat{\Omega}_i$ such that

- Zero right-hand-side in $\widehat{\Omega}_i \setminus \Omega_i$
- Zero flux BCs on $\partial\widehat{\Omega}_i$



MsMFE for Compressible Black-Oil Models

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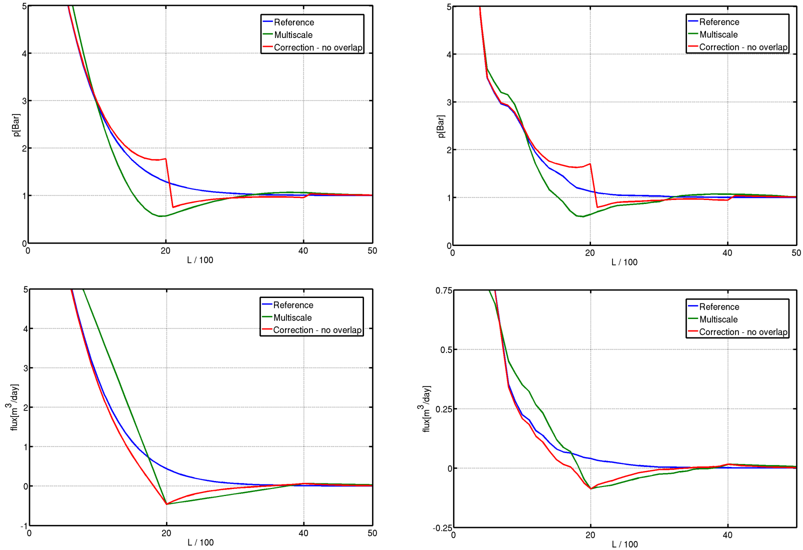
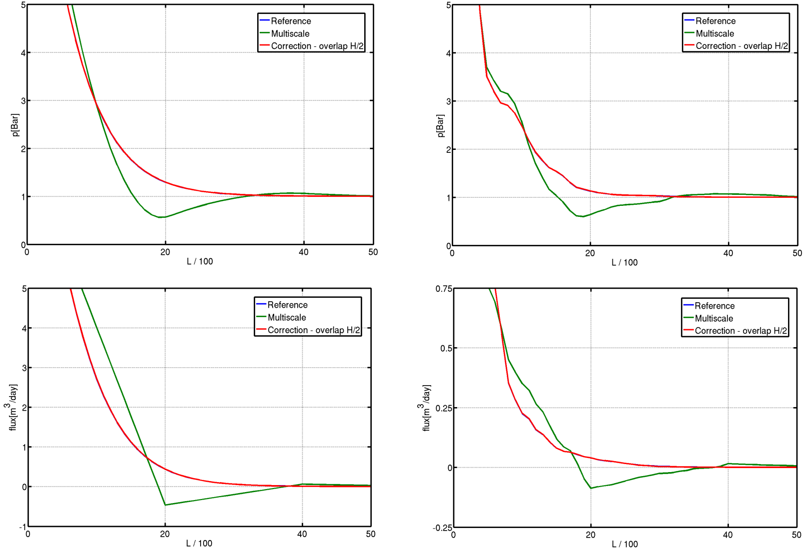
Non-overlapping correction:



pressure

flux

Overlapping $\mathcal{O}(H/2)$ correction:

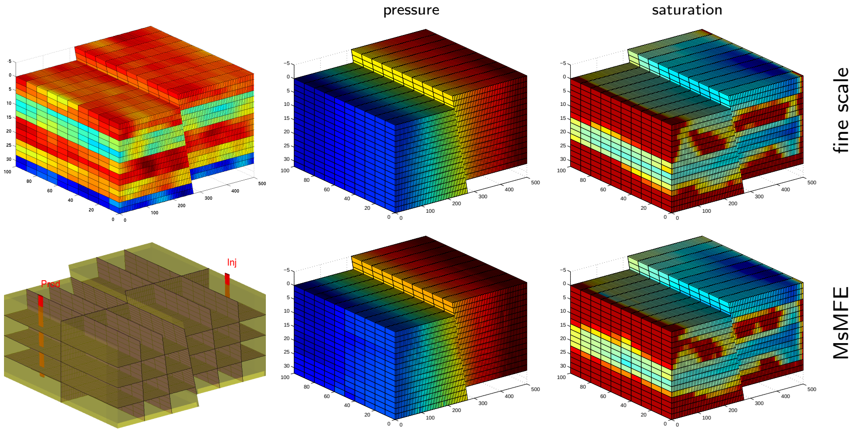


pressure

flux

MsMFE for Compressible Black-Oil Models

Example 2: block with a single fault



Fine grid: $90 \times 10 \times 16$ cells. Coarse grid: $6 \times 2 \times 4$ blocks.

1000 m^3/day water injected into compressible oil at 205 bar (p_{bh} of 200 bar).

Conclusions and Outlook

The MsMFE method:

- is flexible with respect to grids
- allows automated coarsening
- requires correction functions for compressible flow

Future research:

- adaptivity of basis/correction functions
- parallelization
- error estimation (via VMS framework)?